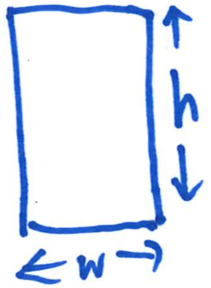
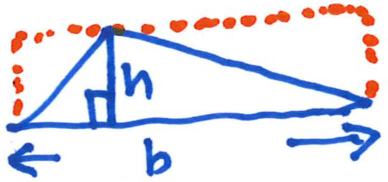
Math 152. Instructor: Michael Monagan.

A1 A2 M1 A3 A4 M2 A5 A6 M3 A7 A8 Final.

5.1 Areas and Distances

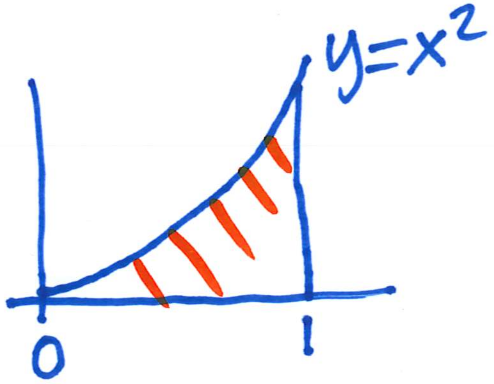


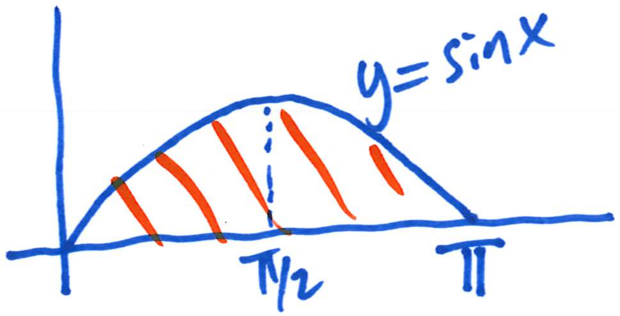
Area $h \cdot w$

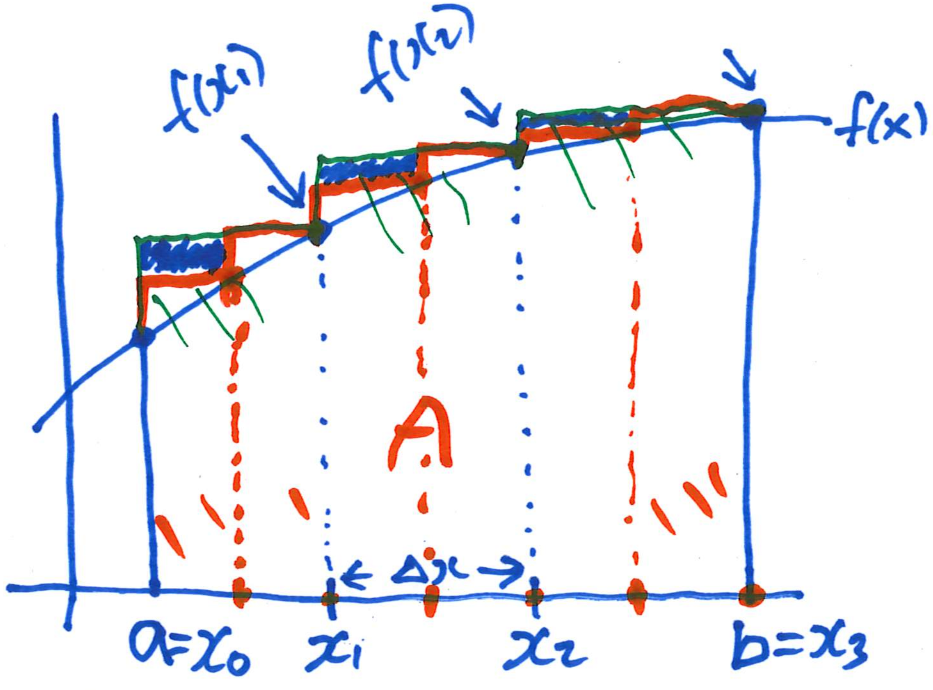


Area $\frac{b \cdot h}{2}$

Area πr^2







Divide $[a,b]$ into n subintervals $[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$ of equal width $\Delta x = \frac{b-a}{n}$ so that $x_i = x_0$

Approx. A by n rectangles

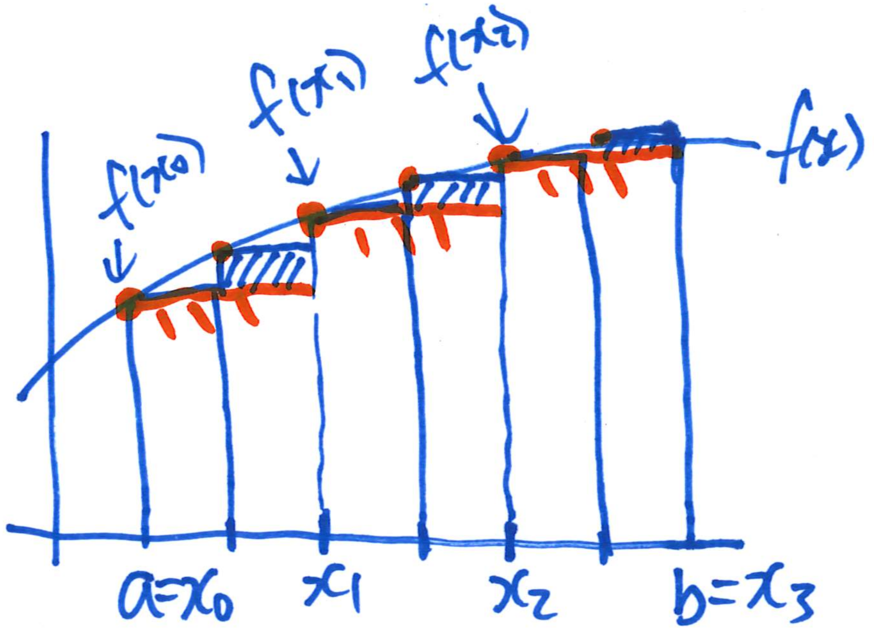
$$R_n = \Delta x f(x_1) + \Delta x \cdot f(x_2) + \cdots + \Delta x f(x_n)$$
$$= \Delta x \left[f(x_1) + f(x_2) + \cdots + f(x_n) \right]$$

$n=3$

$n=6$

$n=12$

$$\text{Area } A = \lim_{n\to\infty} R_n$$



Alternatively we could use the "left-endpoints" of the sub-intervals $x_0, x_1, \ldots, x_{n-1}$.

Let

$$L_n = \Delta x\, f(x_0) + \Delta x\, f(x_1) + \cdots + \Delta x\, f(x_{n-1})$$

$$= \Delta x \sum_{i=0}^{n-1} f(x_i).$$

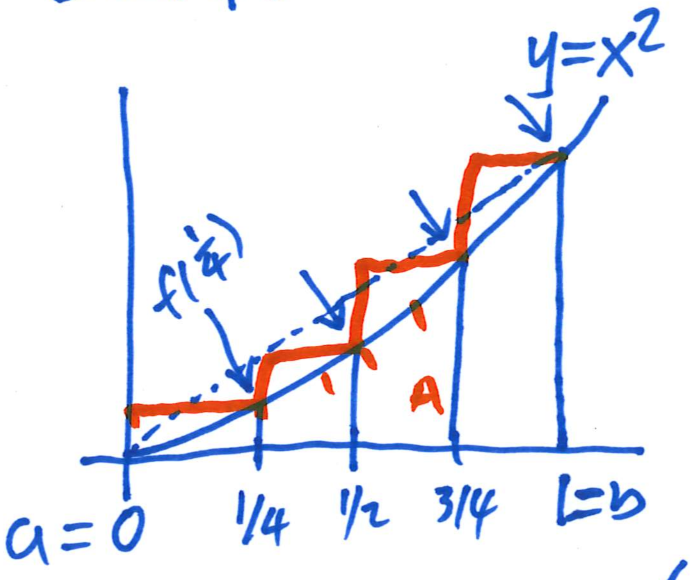
$$\text{Area} = \lim_{n\to\infty} L_n$$

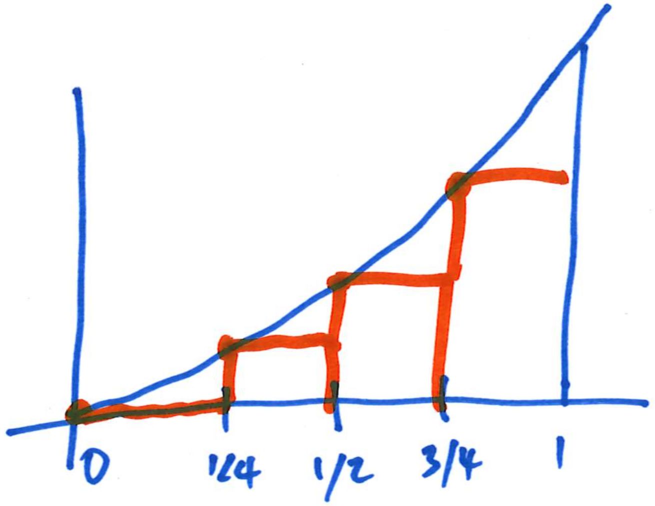
$n=3$

$n=6$

Notice $L_n < A < R_n$ because $f(x)$ is increasing on $[a,b]$.

Example





$n=4 \quad \Delta x = \frac{1-0}{4} = \frac{1}{4}$

$$R_4 = \frac{1}{4}\left(f(\tfrac{1}{4}) + f(\tfrac{1}{2}) + f(\tfrac{3}{4}) + f(1)\right)$$

$$= \frac{1}{4}\left((\tfrac{1}{4})^2 + (\tfrac{1}{2})^2 + (\tfrac{3}{4})^2 + 1^2\right)$$

$$= \frac{1}{4} \quad \frac{1+4+9+16}{16} = \frac{30}{64} = 0.46875$$

$$L_4 = \frac{1}{4}\left(f(0) + f(\tfrac{1}{4}) + f(\tfrac{1}{2}) + f(\tfrac{3}{4})\right)$$

$$= \frac{1}{4} \quad \frac{0+1+4+9}{16} = \frac{14}{64} = .21875$$

$$.21875 < A < .46875$$

$R_{1000} = .33383$

$L_{1000} = .33283.$

$A = \frac{1}{3}$??

Recall: $1+2+3+4+\cdots+n = \frac{n(n+1)}{2}$

$1^2+2^2+3^2+4^2+\cdots+n^2 = \frac{\underline{n}(\underline{n}+1)(\underline{2n}+1)}{6} = \frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}.$

$\boxed{f(x)=x^2}$ on $[a,b]=[0,1]$. $\Delta x = \frac{1-0}{n} = \frac{1}{n}$ $x_i = 0 + i\cdot\frac{1}{n}$

$x_1 = \frac{1}{\underline{n}}$ $x_2 = \frac{2}{\underline{n}}$ $\cdots$

$$R_n = \frac{1}{n}\left(f\left(\tfrac{1}{n}\right)+f\left(\tfrac{2}{n}\right)+\cdots+f\left(\tfrac{n}{n}\right)\right)$$

$$= \frac{1}{n}\left(\left(\tfrac{1}{n}\right)^2+\left(\tfrac{2}{n}\right)^2+\cdots+\left(\tfrac{n}{n}\right)^2\right)$$

$$= \frac{1}{n}\left(\frac{1^2+2^2+\cdots+n^2}{n^2}\right) = \frac{1}{n^3}\left(1^2+2^2+\cdots+n^2\right)$$

$$= \frac{1}{n^3}\left[\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}\right] = \frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}$$

Area $\lim_{n\to\infty} R_n = \lim_{n\to\infty}\left(\frac{1}{3}+\underset{\substack{\downarrow\\0}}{\underline{\frac{1}{2n}}}+\underset{\substack{\downarrow\\0}}{\frac{1}{6n^2}}\right) = \frac{1}{3}$!!

Exercise. Show that $\lim_{n\to\infty} L_n = \frac{1}{3}$.

Derivatives

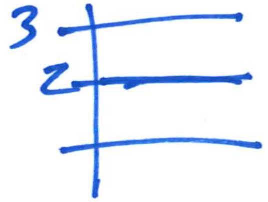
$f(x)$ →	$f'(x)$
$x^2 + 5x$	$\underline{2}x^1 + 5\cdot 1$
$\sin x + \cos x$	$\cos x + (-\sin x)$
$x^2 \cdot \ln x$	$2x \ln x + \frac{1}{x}\cdot x^2$
$\sin(\underbrace{2x}_{}) \overset{u(x)}{}$	$\cos(2x)\cdot 2$
5	0
e^x	e^x
$\sqrt{x} = x^{\frac{1}{2}}$	$\frac{1}{2}\cdot x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

$$\frac{u(x)}{v(x)} = u(x)\cdot v(x)^{-1}$$

Antiderivatives

$f'(x)$ →	$f(x)$
$2x$	$x^2 + C$
$3 + \frac{1}{x}$	$3x + \ln x + C$
$\cos x$	$\sin x + C$
e^{-x}	$-e^{\cancel{-x}} + C$
0	$2,\ C$

any real number



The antiderivative of $f(x)$ is not unique.

Theorem 1. (4.9) If $f'(x) = g'(x)$ then $f(x) = g(x) + C$

The constant of $\int$.