7.4 Integration of Rational Functions by Partial Fractions.

$2\cdot x^{-2}$

$$\int \left(\frac{2}{x^2} - \frac{1}{x} + \frac{1}{x+1}\right) dx = \frac{2(x+1) - x(x+1) + x^2}{x^2(x+1)} = \int \frac{x+2}{x^2(x+1) = x^3+x^2}\, dx$$

$$-2x^{-1} \quad -\ln|x| + \ln|x+1| + C$$

Partial Fraction Decomposition.

rational function of x

Step ① $\int \frac{N(x)}{D(x)}\,dx$ e.g. $\int \frac{2x^3}{x^2-1}\,dx$

If $\deg(N) \geq \deg(D)$ then divide N by D to get $\frac{N}{D} = Q + \frac{R}{D}$

quotient

remainder

$2x \leftarrow Q$

$D = x^2 - 1 \;\big)\; 2x^3 + 1 = N$

$-2x \cdot D \qquad -(2x^3 - 2x)$

$= \qquad 0 + 2x + 1 \leftarrow R.$

$$\int \frac{2x^3+1}{x^2-1}\,dx = \int 2x + \frac{2x+1}{x^2-1}\,dx$$

$$= \int 2x\,dx + \int \frac{2x+1}{x^2-1}\,dx$$

x^2 ✓ ?

Step ② $\boxed{\deg(N) < \deg(D).}$ Factor D

$$x^2 - 1 = (x-1)(x+1).$$

$$D = x^3+1 = (x+1)(x^2-x+1)$$

$$x^3 - x^2 + x - 1 = (x-1)(x^2+1).$$

Polynomials with real coefficients can be factored into a product of linear or <u>irreducible quadratic factors</u> with real coefficients.

To factor ax^2+bx+c use the quadratic formula to calculate the roots

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \alpha, \beta$$ then $ax^2+bx+c = a(x-\alpha)(x-\beta).$

E.g. $x^3 - 3x^2 + 2x = x(x^2-3x+2)$

$$= x(x-2)(x-1)$$

ax^2+bx+c

$1 \cdot x^2 - 3x + 2$

$$x = +3 \pm \sqrt{9-8} \,/\, 2$$

$$= \frac{3 \pm 1}{2} = 2, 1$$

$$x^3 + x = x(\underline{x^2+1})$$

does not factor

$$x^2+1=0$$
$$x^2 = -1$$
$$x = \pm\sqrt{-1}$$

Step 3 Partial Fraction Decomposition

CASE 1 The denominator D has different linear factors.

$$\frac{2x+1}{[(x+2)](x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} \left[+ \frac{C}{x+2}\right]$$

To solve for A and B first clear the denominators.

$$\boxed{2x+1 = A(x+1) + B(x-1)} = (A+B)x + (A-B)\cdot 1$$

Method ① Equate coefficients in $x^0, x^1, x^2, \ldots$

$$\begin{array}{llcll} x^1 & 2 & = & \overset{=3/2}{A}+B & \Rightarrow B = 1/2 \\ x^0 & +\ 1 & = & A-B & \\ \hline & 3 & = & 2A & \Rightarrow A = \frac{3}{2} \end{array}$$

We have

$$\int \frac{2x+1}{(x-1)(x+1)} = \int \frac{3/2}{x-1} + \int \frac{\frac{1}{2}}{x+1}$$

$$= \frac{3}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| + C.$$

Method ② Evaluate at $x=-1$, $x=+1$.

$x=-1$ $\quad -1 = 0 + B(-2) \Rightarrow B = 1/2$

$x=1$ $\quad 3 = 2A + 0 \Rightarrow A = 3/2$

CASE II $\quad D$ has a quadratic factor

$$\frac{N}{(x\ )(\quad\ )(ax^2+bx+c)} = \frac{c}{(x\ )} + \frac{\quad}{(\quad\ )} + \frac{Ax+B}{ax^2+bx+c}$$

E.g. $\quad \dfrac{2x^2+x+2}{x^3+x} = \dfrac{2x^2+x+2}{x(x^2+1)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1} = \dfrac{2}{x} + \dfrac{1}{x^2+1}$

$\Rightarrow \quad 2x^2+x+2 = A(x^2+1) + (Bx+C)x$

$x=0 \qquad 2 = A\cdot 1 + 0 \Rightarrow A=2.$

$x^2 \qquad 2 = (\overset{=2}{A}+B) \Rightarrow B=0$

$x^1 \qquad 1 = 0 + C \Rightarrow C=1.$

$$\int\left(\frac{2}{x} + \frac{1}{x^2+1}\right)dx = 2\ln|x| + \tan^{-1}x + C. \qquad \int \frac{1}{x^2+1}\,dx = \tan^{-1}x$$

CASE III Denominator D has repeated factors.

$$\frac{N}{(x-1)^{3}(x^2+1)^2 \cdot x} = \frac{A}{(x-1)^1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{Dx+E}{(x^2+1)} + \frac{Fx+G}{(x^2+1)^2} + \frac{H}{x}$$

E.g.
$$\frac{1}{x(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Dx+E}{(x^2+1)^2} + \frac{C}{x} = \frac{-x}{x^2+1} + \frac{-x}{(x^2+1)^2} + \frac{1}{x}$$

$$1 = x(x^2+1)(Ax+B) + \underline{x(Dx+E)} + C(x^4+2x^2+1)$$
$$= (Ax^4+Bx^3+Ax^2+Bx) + (Dx^2+Ex) + Cx^4 + 2Cx^2 + C.$$

x^4 $\quad 0 = A + C^{=1} \Rightarrow A = -1$

x^3 $\quad 0 = B$

x^2 $\quad 0 = A^{=-1} + D + 2C^{=1}. \Rightarrow D = -1.$

x^1 $\quad 0 = B + E \Rightarrow E = 0$

x^0 $\quad 1 = C$

$$\int \left(\frac{-x}{x^2+1} - \frac{x}{(x^2+1)^2} + \frac{1}{x} \right) dx$$
$\downarrow u = x^2+1$ $\quad \downarrow u = x^2+1$ $\quad \downarrow$ $\ln|x|$

$$\int \frac{-x}{x^2+1}\,dx = \int \frac{-x}{u} \cdot \frac{dx}{2x} = \int -\frac{1}{2u}\,du = -\frac{1}{2}\ln|u|$$

$u = x^2+1 \quad \frac{du}{dx} = 2x \quad dx = du/2x$

$$= -\frac{1}{2}\ln|1+x^2|$$
$$= -\tfrac{1}{2}\ln(1+x^2)$$