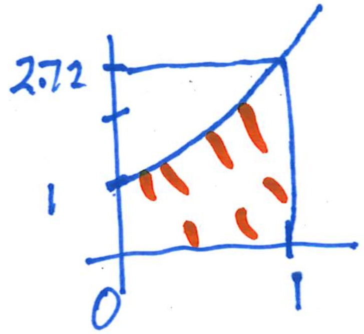
7.7 Approximate Integration

Assignment #4 posted.
Due Monday after Reading Week

Consider $\int_0^1 e^{x^2}\,dx$



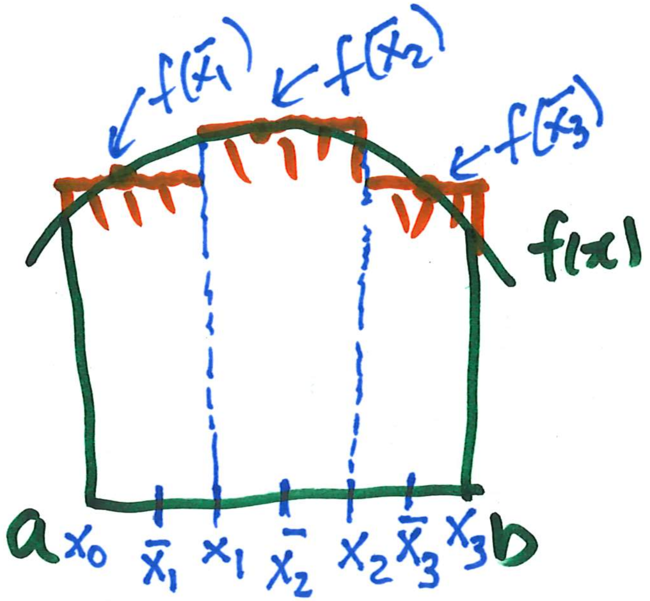
$e^0 = 1$
$e^1 = 2.72$

Unfortunately e^{x^2} has no "elementary" antiderivative so we can't use the FTC.
Other functions with no elementary antiderivative include

$\frac{e^x}{x}$, $\frac{\sin x}{x}$, $\frac{1}{\ln x}$, $e^x \cdot \ln x$, $\ln(\ln x)$, $\sin x^2$ etc.

Three simple ways of approximating $\int_a^b f(x)\,dx$ are the midpoint rule M_n and the trapezoidal rule T_n and Simpson's rule S_n.

M_n



$n=3$

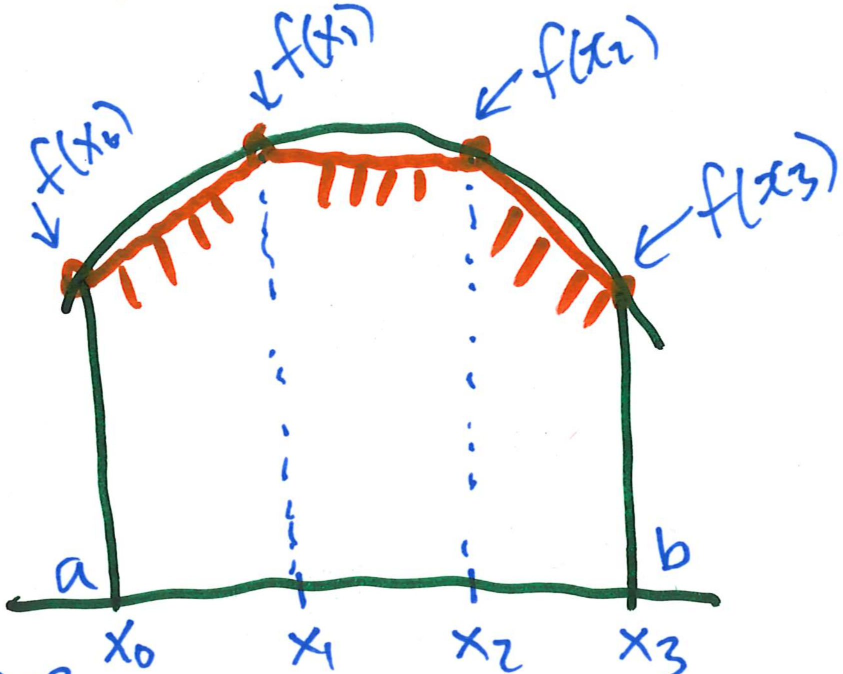
Divide $[a,b]$ into n subintervals $[x_0,x_1],[x_1,x_2],\dots,[x_{n-1},x_n]$ of width $\Delta x = \frac{b-a}{n}$.

Pick $\bar{x}_i$ = midpoint of $[x_{i-1},x_i]$ for $1 \le i \le n$.

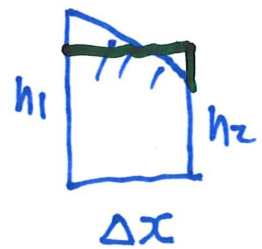
The midpoint rule

$$M_n = \Delta x\left(f(\bar{x}_1)+f(\bar{x}_2)+\dots+f(\bar{x}_n)\right).$$

T_n



$n=3$



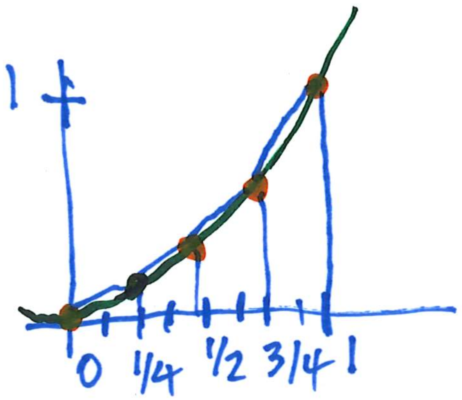
Area $= \frac{h_2+h_1}{2}\cdot\Delta x$

The trapezoidal rule

$$T_n = \Delta x\left[\frac{f(x_0)+f(x_1)}{2}+\frac{f(x_1)+f(x_2)}{2}+\dots+\frac{f(x_{n-1})+f(x_n)}{2}\right]$$

$$T_n = \frac{\Delta x}{2}\left[1\cdot f(x_0)+2f(x_1)+2f(x_2)+\dots+2f(x_{n-1})+1\cdot f(x_n)\right]$$

Example. Calculate T_4 and M_4 for $\int_0^1 x^2\,dx = \frac{1}{3} = .333$.



$n=4 \quad \Delta x = \frac{1}{4}$

$$T_4 = \frac{1/4}{2}\left(f(0)+2f(\tfrac{1}{4})+2f(\tfrac{1}{2})+2f(3/4)+f(1)\right)$$

$$= 1/8\left[0+2\cdot\frac{1}{16}+2\cdot\frac{1}{4}+2\cdot\frac{9}{16}+1\right]$$

$$= 1/8\left[\frac{0+2+8+18+16}{16} = \frac{44}{16} = \frac{11}{4}\right] = \frac{11}{32} = .34375$$

$$M_4 = \frac{1}{4}\left[f(\tfrac{1}{8})+f(\tfrac{3}{8})+f(\tfrac{5}{8})+f(\tfrac{7}{8})\right] = \frac{21}{64} = .328125$$

How accurate are T_4 & M_4?

$$1/3 - T_4 = 1/3 - \frac{11}{32} = -\frac{1}{96} = -.0104$$

$$1/3 - M_4 = 1/3 - \frac{21}{64} = +\frac{1}{192} = .0052$$

So M_4 is twice as accurate.

Error Bounds for T_n and M_n.

Let $E_T = \int_a^b f(x)\,dx - T_n$ = error in T_n.

$E_M = \int_a^b f(x)\,dx - M_n$ = " " M_n.

If $\boxed{K \geq \max_{a \leq x \leq b} f''(x)}$ the $\boxed{|E_T| \leq \frac{K(b-a)^3}{12n^2}}$ and $|E_M| \leq \frac{K(b-a)^3}{24n^2}$

Example. $a=0$, $b=1$, $n=4$, $f(x)=x^2$, $f'(x)=2x$, $f''(x)=2$. $\Rightarrow K=2$

$$|E_T| \leq \frac{2\cdot(1-0)^3}{12\cdot 4^2} = \frac{2}{12\cdot 16} = \frac{1}{12\cdot 8} = \frac{1}{96}$$

How big must n be so that $|E_T| \leq .0001$?

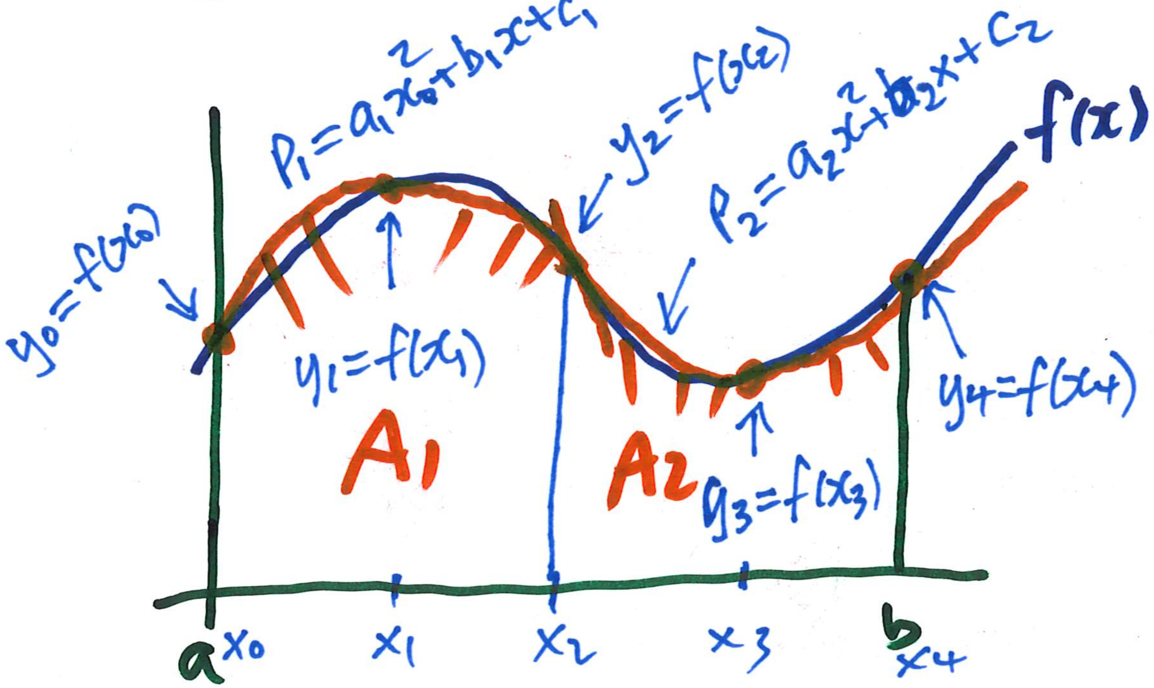
$$|E_T| \leq \boxed{\frac{2\cdot 1^3}{12\cdot n^2} \leq .0001} \Rightarrow \frac{2\cdot 10^4}{12} \leq n^2 \Rightarrow n \geq \sqrt{\frac{10^4}{6}} = \frac{100}{\sqrt{6}} = 40.8$$

Solve for n.

$\Rightarrow n=41.$

Simpson's Rule

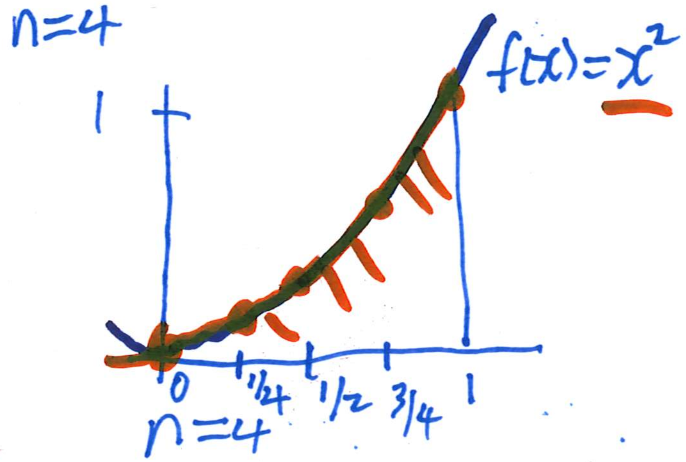
$$\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}\left[1\cdot f(x_0) + 4 f(x_1) + 2 f(x_2) + 4 f(x_3) + 2 f(x_4) + \cdots + 4 f(x_{n-1}) + f(x_n)\right]$$



$$A_1 = \text{Area under } P_1 = \frac{\Delta x}{3}(y_0 + 4y_1 + y_2)$$

$$A_2 = \text{Area under } P_2 = \frac{\Delta x}{3}(y_2 + 4y_3 + y_4)$$

$$S_4 = A_1 + A_2 = \frac{\Delta x}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\right]$$



$$S_4 = \frac{1/4}{3}\left[0 + 4\cdot\frac{1}{16} + 2\cdot\frac{1}{4} + 4\cdot\frac{9}{16} + 1\cdot 1^2\right]$$

$= 1/3$ exact area because $f(x) = x^2$.

Error Bound for Simpson's rule S_n.

Let $E_S = \int_a^b f(x)\,dx - S_n$ be the error of S_n.

If $K \geqslant \max\limits_{a \le x \le b} f''''(x)$ then $|E_S| \leqslant \dfrac{K \cdot (b-a)^5}{180 \cdot n^4}$.

Example. $f(x) = 1/x = x^{-1}$, $f'(x) = -x^{-2}$, $f''(x) = 2x^{-3}$; $f'''(x) = -6x^{-4}$; $f''''(x) = 24x^{-5}$
$= 24/x^5$

$[a,b] = [1,2]$ The $\max\limits_{1 \le x \le 2} \dfrac{24}{x^5} = 24$ at $x=1$, so $K=24$.

What value of n should I take so that $|E_S| \leqslant .0001$?

$$|E_S| \leq \frac{24 \cdot (2-1)^5}{180 \cdot n^4} \leq .0001 \Rightarrow \frac{24 \cdot 10^4}{180} \leq n^4 \Rightarrow n \geqslant \sqrt[4]{\frac{24 \cdot 10^4}{180}} = 6.04.$$

$\Rightarrow n \geqslant 8$. (n must be even for S_n)

If we double n the error in T_n, M_n drops by a factor of 4
and " " " S_n " " " " " 16.

If $n \to 2n$, $n^4 \to (2n)^4 = 16n^4$