MACM 401/MATH 701/MATH 819 Assignment 5, Spring 2007.

Michael Monagan

This assignment is to be handed in by Tuesday March 27th at the beginning of class. For problems involving Maple calculations and Maple programming, please submit a printout of a Maple worksheet. Late Penalty: -20% for each day late.

Question 1: Factorization in $\mathbb{Z}[x]$ (30 marks)

Factor the following polynomials in $\mathbb{Z}[x]$.

$$p_1 = x^{10} - 6\,x^4 + 3\,x^2 + 13$$

$$p_2 = 8\,x^7 + 12\,x^6 + 22\,x^5 + 25\,x^4 + 84\,x^3 + 110\,x^2 + 54\,x + 9$$

$$p_3 = 9\,x^7 + 6\,x^6 - 12\,x^5 + 14\,x^4 + 15\,x^3 + 2\,x^2 - 3\,x + 14$$

$$p_4 = x^{11} + 2\,x^{10} + 3\,x^9 - 10\,x^8 - x^7 - 2\,x^6 + 16\,x^4 + 26\,x^3 + 4\,x^2 + 51\,x - 170$$

First compute the square free factorization for each polynomial. Use the Maple command `gcd(...)` to do this.

Now factor each non-linear square-free factor as follows. Use the Maple command `Factor(...) mod p` to factor the square-free factors over $\mathbb{Z}_p$ modulo the primes $p = 13, 17, 19$. From this information, determine whether each polynomial is irreducible over $\mathbb{Z}$ or not. If not irreducible, try to discover what the irreducible factors are by considering combinations of the modular factors and Chinese remaindering (if necessary) and trial division over $\mathbb{Z}$.

Using Chinese remaindering here is not inefficient in general. Why? Thus for the polynomial p_4, use Hensel lifting instead. That is, using a suitable prime of your choice from $17, 19, 23$, Hensel lift each factor mod p to determine the irreducible factorization of p_4 over $\mathbb{Z}$.

Question 2: Factorization in $\mathbb{Z}_p[x]$ (30 marks)

(a) Factor the following polynomials over $\mathbb{Z}_{11}$ using the Cantor-Zassenhaus algorithm.

$$a_1 = x^4 + 8\,x^2 + 6\,x + 8,$$

$$a_2 = x^6 + 3\,x^5 - x^4 + 2\,x^3 - 3\,x + 3,$$

$$a_3 = x^8 + x^7 + x^6 + 2\,x^4 + 5\,x^3 + 2\,x^2 + 8.$$

Use Maple to do all polynomial arithmetic, that is, you can use the `Gcd(...) mod p` and `Powmod(...) mod p` commands.

(b) Now compute the square-roots of the integers $a = 3, 5, 7$ in the integers modulo p, if they exist, for $p = 10^{20} + 129 = 100000000000000000129$ via factoring the polynomial $x^2 - a \bmod p$ using the Cantor-Zassenhaus algorithm. Show your working.

Question 3: A linear x-adic Newton iteration (20 marks).

Let p be an odd prime and let $a(x) = a_0 + a_1 x + ... + a_n x^n \in \mathbb{Z}_p[x]$ with $a_0 \neq 0$ and $a_n \neq 0$. Suppose $\sqrt{a_0} = \pm u_0 \bmod p$. Design an x-adic Newton iteration algorithm that given u_0, determines if $u = \sqrt{a(x)} \in \mathbb{Z}_p[x]$ and if so computes u. Let

$$u = u_0 + u_1 + ... + u_{k-1} x^{k-1} + ... + u_{n-1} x^{n-1}.$$

Derive the update formula for u_k. Show your working.

Now implement your algorithm in Maple and test it on the two polynomials $a_1(x)$ and $a_2(x)$ below using $p = 101$ and $u_0 = +5$. Please print out the sequence of values of $u_0, u_1, u_2, ...$ that your program computes. Note, one of the polynomials has a sqrt in $\mathbb{Z}_p[x]$, the other does not.

$$a_1 = 81\,x^6 + 16\,x^5 + 24\,x^4 + 89\,x^3 + 72\,x^2 + 41\,x + 25$$

$$a_2 = 81\,x^6 + 46\,x^5 + 34\,x^4 + 19\,x^3 + 72\,x^2 + 41\,x + 25$$

Question 4: Cost of the linear p-adic Newton iteration (20 marks)

Let $a \in \mathbb{Z}$ and $u = \sqrt{a}$. Suppose $u \in \mathbb{Z}$. The linear P-adic Newton iteration for computing u from $u \bmod p$ that we gave in class is based on the following linear p-adic update formula:

$$u_k = -\frac{\phi_p(f(u^k)/p^k)}{f'(u_0)} \bmod p.$$

where $f(u) = a - u^2$. A direct coding of this update formula for the $\sqrt{}$ problem $\mathbb{Z}$ led to the code below where I've modified the algorithm to stop if the error $e < 0$ instead of using a bound B.

```
ZSQRT := proc(a,u0,p) local U,pk,k,e,uk,i;
    u := mods(u0,p);
    i := modp(1/(2*u0),p);
    pk := p;
    for k do
        e := a - u^2;
        if e = 0 then return(u); fi;
        if e < 0 then return(FAIL) fi;
        uk :=  mods( iquo(e,pk)*i, p );
        u := u + uk*pk;
        pk := p*pk;
    od;
end:
```

The running time of the algorithm is dominated by the squaring of `u` in `a := a - u^2` and the long division of `u` by `pk` in `iquo(e,pk)`. Assume the input a is of length n base p digits. At the beginning of iteration k, $u = u^{(k)} = u_0 + u_1 p + ... + u_{k-1} p^{k-1}$ is an integer of length at most k base p digits. Thus squaring `u` costs $\mathrm{O}(k^2)$ (assuming classical integer arithmetic). In the division of `e` by $\mathtt{pk} = p^k$, `e` will be an integer of length n base p digits. Assuming classical integer long division is used, this division costs $O((n-k+1)k)$. Since the loop will run $k = 1, 2, ..., n/2$ for the $\sqrt{}$ problem the total cost of the algorithm is dominated by $\sum_{k=1}^{n/2} k^2 + (n-k+1)k \in O(n^3)$.

Redesign the algorithm so that the overall time complexity is $O(n^2)$ assuming classical integer arithmetic. Prove that your algorithm is $O(n^2)$. Now implement your algorithm in Maple and verify that it works correctly and that the running time is $\mathrm{O}(n^2)$. Use the prime $p = 9973$.

Hint 1: $e = a - u^2 = a - {u^{(k)}}^2 = a - \left(u^{(k-1)} + u_{k-1}p^{k-1}\right)^2 = (a - {u^{(k-1)}}^2) - 2u^{k-1}u_{k-1}p^{k-1} - u_{k-1}^2 p^{2k-2}$. Notice that $a - {u^{(k-1)}}^2$ is the error that was computed in the previous iteration.
Hint 2: We showed that the algorithm for computing the p-adic representation of an integer is $\mathrm{O}(n^2)$. Notice that it does not divide by p^k, rather, it divides by p each time round the loop.