Lec21B Hermite and Horrowitz

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Hermites Method 11.3 [on p 487].

① Compute the square-free factorization of $Q = q_1 q_2^2 \cdots q_k^k$. $\gcd(q_i, q_j) = 1$, $\gcd(q_i, q_i') = 1$.
If $k=1$ then stop. Th 11.6 $\Rightarrow \int P/Q$ is logarithmic.

Let $Q = q_k^k \cdot T$ where $T = \underline{q_1 q_2^2 \cdots q_{k-1}^{k-1}}$

② Solve $\sigma(T q_k') + \tau q_k = P$ for $\sigma, \tau \in K[x]$ with $\sigma = 0$ or $\deg \sigma < q_k$.

Note $\gcd(T q_k', q_k) = 1$.

$$\int \frac{P}{Q} = \int \frac{\sigma T q_k' + \tau q_k}{Q} = \int \frac{\sigma q_k'}{q_k^k} + \frac{\tau}{T \cdot q_k^{k-1}}$$

$$= \frac{1/(1-k)}{q_k^{k-1}} \cdot \sigma - \int \frac{1/(1-k) \cdot \sigma'}{q_k^{k-1}} + \int \frac{\tau}{T q_k^{k-1}} = \int \frac{\tau - \sigma'/(1-k) \cdot T}{T \cdot q_k^{k-1}}$$

$$\int \frac{P}{Q} = \frac{\sigma/(1-k)}{q_k^{k-1}} + \int \frac{\tau - \sigma'/(1-k) T}{q_1 q_2^2 \cdots q_{k-1}^{k-1} \cdot q_k^{k-1}} = Q/q_k$$

apply same method.

③ Recursively $\int$ until the denominator is square-free.

E.g. $\int \frac{x^2}{(x-1)^3 (x^2-2)}$. $P = x^2$, $T = x^2 - 2$, $q_k = x - 1$, $k = 3$, $q_k' = 1$.

Solve $\sigma \cdot (x^2-2) \cdot 1 + \tau \cdot \underline{(x-1)} = x^2 - 2$ for σ, τ with $\deg \sigma < 1$.
I get $\sigma = -1$, $\tau = 2x + 2$

$$\int f(x)\,dx = \frac{-1/(1-3)}{(x-1)^2} + \int \frac{\overset{\tau}{2x+2} - \overset{\sigma'}{0}}{(x-1)^2 (x^2-2)} \quad \begin{matrix} = P \\ = Q \end{matrix}$$

Let $T = x^2 - 2$, $k = 2$, $q_k = x - 1$, $q_k' = 1$, $P = 2x + 2$.
Solve $\sigma \cdot (x^2-2) \cdot 1 + \tau \underline{(x-1)} = 2x + 2$ for σ, τ with $\deg \sigma < 1$.
I get $\sigma = -4$, $\tau = 4x + 6$.

$$\int f(x)\,dx = \frac{1}{2(x-1)^2} - \frac{\overset{\sigma/(1-k)}{4/(1-2)}}{\underset{q_k^{k-1}}{(x-1)^1}} + \int \frac{4x + 6 - 0}{(x^2-2) \cdot (x-1)}$$

↑ logarithmic part.

$\qquad 2(x-1) \qquad (x-1)' \qquad \qquad (x^2-2)\cdot(x-1)$

q_k^{k-1} $\qquad\qquad\qquad\uparrow$ logarithmic part.

The rational part is $\frac{1}{2(x-1)^2}+\frac{4}{x-1}$.

Horrowitz's method 11.4

Th 11.6

(1) $\int \frac{P}{Q} = \frac{A}{B} + \int \frac{C}{D}$ where
$Q = q_1 q_2^2 \cdots q_k^k$
$B = q_2 q_3^2 \cdots q_k^{k-1} = \gcd(Q,Q')$.
$D = q_1 q_2 \cdots q_k = Q/B$.

① Step Compute $B=\gcd(Q,Q')$ and $D = Q/B$. $\Rightarrow \boxed{Q=BD}$

If $B=1$ stop. $A=0$ and $C=P$ and $\int \frac{P}{Q}$ is logarithmic.

$(1)'$ $\quad \frac{P}{Q} = \frac{A'}{B} - \frac{B'A}{B^2} + \frac{C}{D}$ (2)

$(2)\times Q \Rightarrow P = A'D - \frac{B'AD}{B} + C\cdot B$ (3).

Claim $B \mid B'D$ $\qquad B = q_2 q_3^2 \cdots q_k^{k-1}$ $\qquad B' = q_3 q_4^2 \cdots q_k^{k-2}\cdot\Delta$

$D = q_1 q_2 \cdots q_k$ $\qquad B'D = q_1 q_2 q_3^2 \cdots q_k^{k-1}\cdot\Delta$

Let $H = (B'D)/B \in K[x]$.

Let $P = A'D - AH + CB$ (4).

Th 11.6 $\deg(A) < \deg(B)$ $\quad \deg(C) < \deg(D)$.

Let $A = \sum_{i=0}^{\deg B-1} a_i x^i$ and $C = \sum_{i=0}^{\deg D-1} c_i x^i$ $\qquad Q=BD$

Let $n = \deg B + \deg D = \deg Q$.

Step ③. Equate coefficients in x^i in (4). and solve the $n\times n$ linear system for a_i and c_i.

Cost $O(n^3)$. Hermite's method can be done $O(n^2)$.
arithmetic ops in K.

Example. $\int \frac{x^2 = P}{(x-1)^3(x^2-2) = Q}$ $\qquad P = x^2$, $Q = (x-1)^3(x^2-2)$. $\qquad B = (x-1)^2 = \gcd(Q,Q')$, $D = (x-1)(x^2-2) = Q/B$.

$\int \frac{P}{Q} = \frac{A}{(x-1)^2} + \int \frac{C}{(x-1)(x^2-2)}$ $\qquad \deg A < 2$, $\deg B < 3$.

$$\int \frac{P}{Q} = \frac{A}{\underbrace{(x-1)^2}_{B}} + \int \frac{C}{\underbrace{(x-1)(x^2-2)}_{D}}$$

deg $A < 2$
deg $B < 3$.

Th 11.6 says $\deg A < \deg B = 2$ and $\deg C < \deg D = 3$ so

$A = a_0 + a_1 x \qquad C = c_0 + c_1 x + c_2 x^2 \qquad B' = 2(x-1)$

$P = A'D - AH + CB \qquad H = B'D/B = \frac{2(x-1)(x^2-2)(x-1)}{(x-1)^2} = 2x^2 - 4.$

$$1 \cdot x^2 = a_1(x-1)(x^2-2) - (a_0 + a_1 x)(2x^2-4) + (c_0 + c_1 x + c_2 x^2)(x^2 - 2x + 1)$$

$[x^4] \quad 0 = c_2$

$[x^3] \quad 0 = a_1 - 2a_1 + c_1 - 2c_2$

$[x^2] \quad 1 = -a_1 - 2a_0 + c_0 - 2c_1 + c_2$

$[x] \quad 0 = -2a_1 + 4a_1 - 2c_0 + c_1$

$[x^0] \quad 0 = +2a_1 + 4a_0 + c_0$

Maple

$a_0 = -7/2$
$a_1 = 4$
$c_0 = 6$
$c_1 = 4$
$c_2 = 0$

$$\int \frac{x^2}{(x-1)^3(x^2-2)} = \frac{-7/2 + 4x}{(x-1)^2} + \int \frac{6+4x}{(x-1)(x^2-2)}$$

rational part.