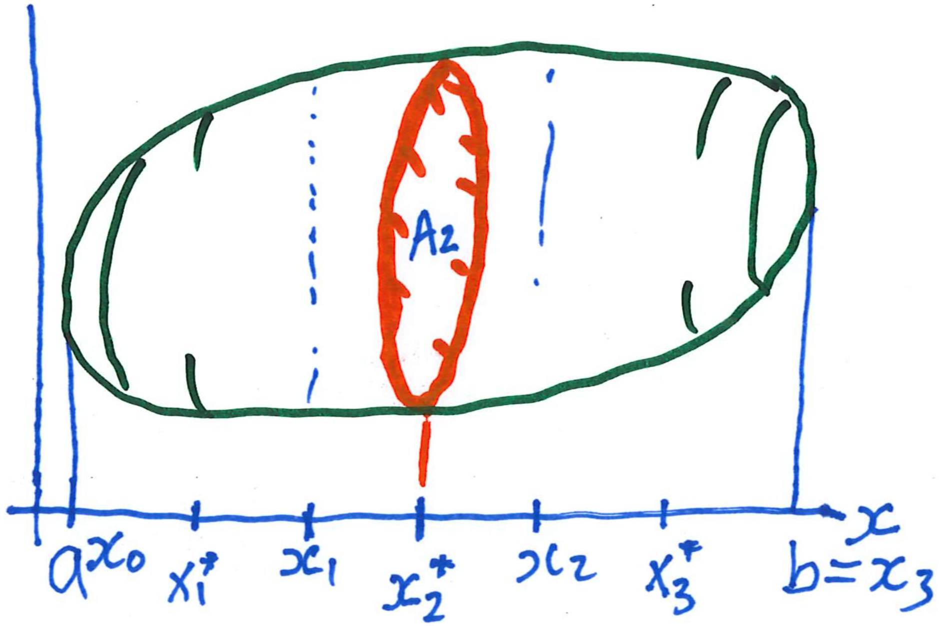
6.2 Volumes

Assignment #2 due Monday. Midterm 1 next Friday.

Let S be a solid. Calculate V the volume of S. Divide $[a,b]$ into n subintervals $[x_0,x_1], [x_1,x_2], \dots, [x_{n-1},x_n]$ of equal width $\Delta x = (b-a)/n$. Pick x_i^* on $[x_{i-1}, x_i]$.

Let $A_i(x_i^*)$ be the area of S at x_i^*.

Let $S_i = \Delta x \cdot A_i(x_i^*)$ = volume of slab i.



$$V \approx \sum_{i=1}^{n} \Delta x \cdot A(x_i^*)$$

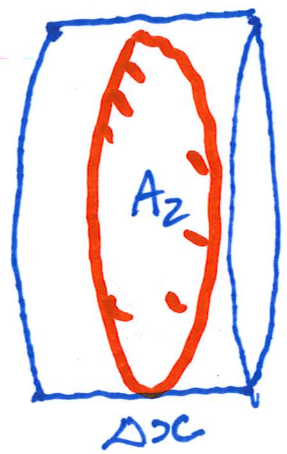
Def. $$V = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x\, A(x_i^*)$$ ← Riemann Sum

$$= \int_a^b A(x)\cdot dx$$

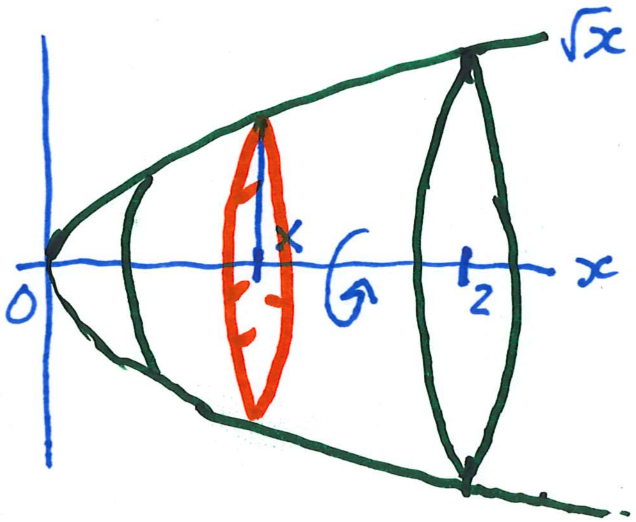
$S_2 =$

Volume $=$

$A_2 \cdot \Delta x$



Example 1. Calculate the volume of the solid formed by rotating $y=\sqrt{x}$ around the x axis on $[0,2]$



A(x) is the area a circle of radius $\sqrt{x}$.

$$A(x) = \pi r^2 = \pi(\sqrt{x})^2 = \pi x.$$

The volume V is given by

$$V = \int_a^b A(x)\,dx = \int_0^2 \pi x\,dx = \left[\pi \cdot \tfrac{1}{2}x^2\right]_0^2$$

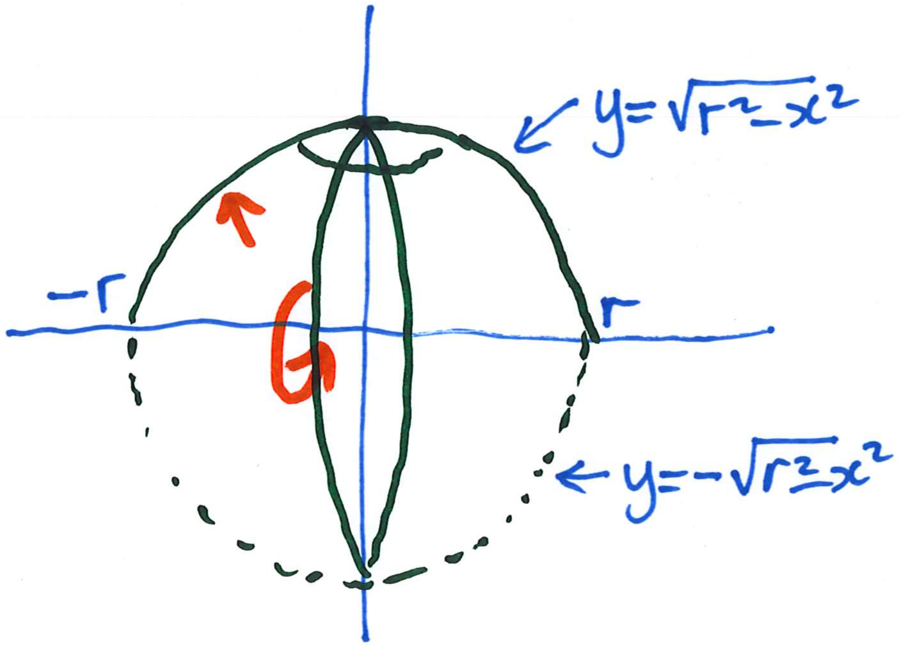
$$= \underline{2\pi - 0.}$$

This is called a "volume of revolution".

If $f(x) \geq 0$ on $[a,b]$ is rotated about the x axis the volume

$$\boxed{V = \int_a^b \pi f(x)^2\,dx}.$$

Example 2.



Equation for a circle of radius r

$$x^2+y^2=r^2$$

$$\Rightarrow\ y^2=r^2-x^2 \Rightarrow y=\pm\sqrt{r^2-x^2}$$

The volume of the sphere is

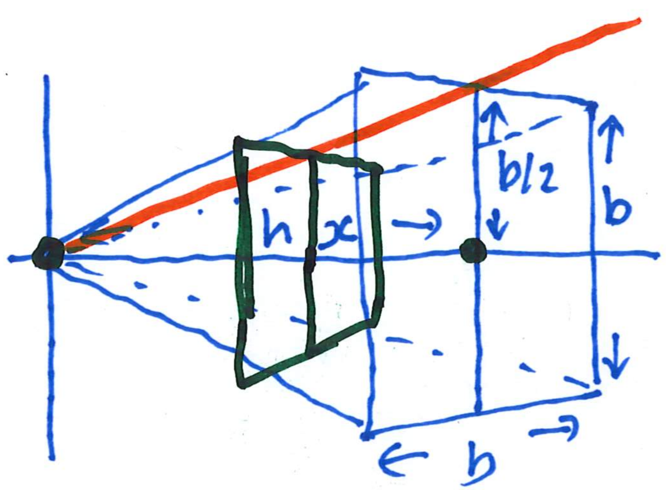
$$V=\int_a^b \underbrace{f(x)^2\cdot\pi}_{A(x)} = \int_{-r}^{r}\pi\cdot y^2\,dx = \int_{-r}^{r}\pi(r^2-x^2)\,dx$$

$$=\left[\pi\left(r^2x-\tfrac{1}{3}x^3\right)\right]_{-r}^{r}$$

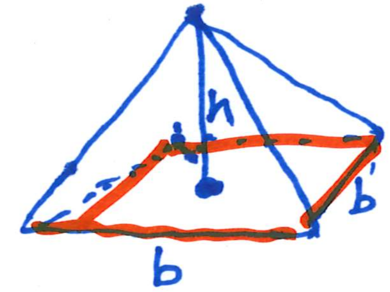
$$=\pi\left(\underbrace{r^3-\tfrac{1}{3}r^3}\right)-\pi\left(-\underbrace{r^3+\tfrac{1}{3}r^3}\right)$$

$$=\pi\,\tfrac{2}{3}r^3 \quad - -\tfrac{2}{3}r^3\pi = \tfrac{4}{3}\pi r^3$$

Example 3   Find the volume of a pyramid of height h and base length b.



$$y = mx + c$$
$$= \frac{b/2}{h}x$$
$$= \frac{1}{2}\frac{b}{h}x$$

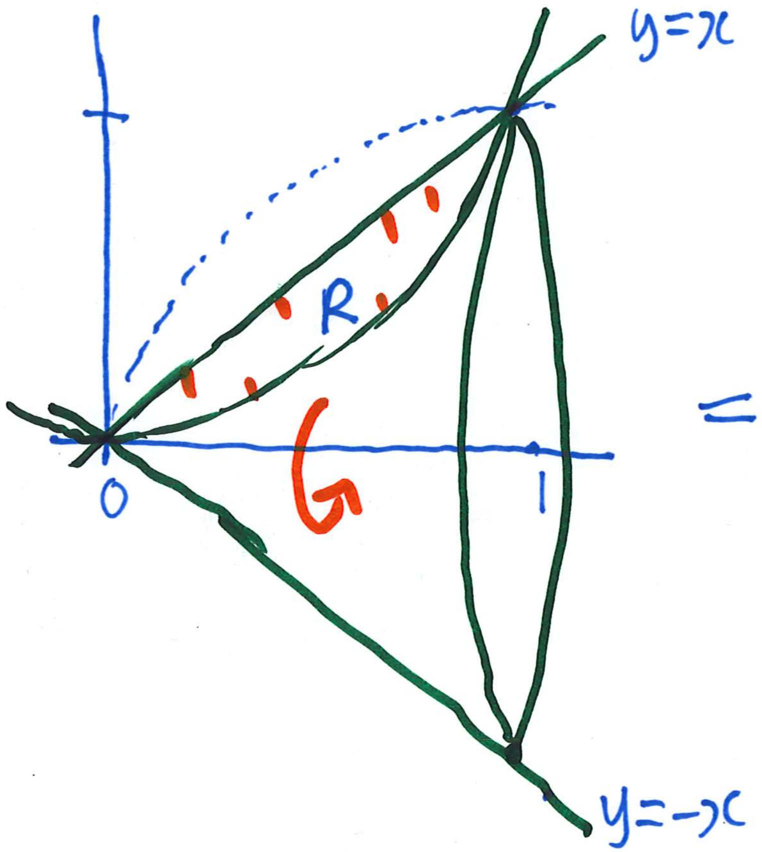


$$V = \int_a^b A(x)\,dx = \int_0^h \frac{b^2}{h^2}x^2\,dx$$
$$= \left[\frac{b^2}{h^2}\frac{1}{3}x^3\right]_0^h = \frac{b^2}{h^2}\cdot\frac{1}{3}h^3 = \frac{1}{3}b^2h.$$

$A(x) :=$ a square of width $2\cdot y = \frac{b}{h}x$

$$\text{Area} = \left(\frac{b}{h}x\right)^2 = \frac{b^2}{h^2}\cdot x^2$$

Example 4. Find the volume of the region R bounded by $y=x$, $y=x^2$, $x=0$, $x=1$ rotated about the x axis.



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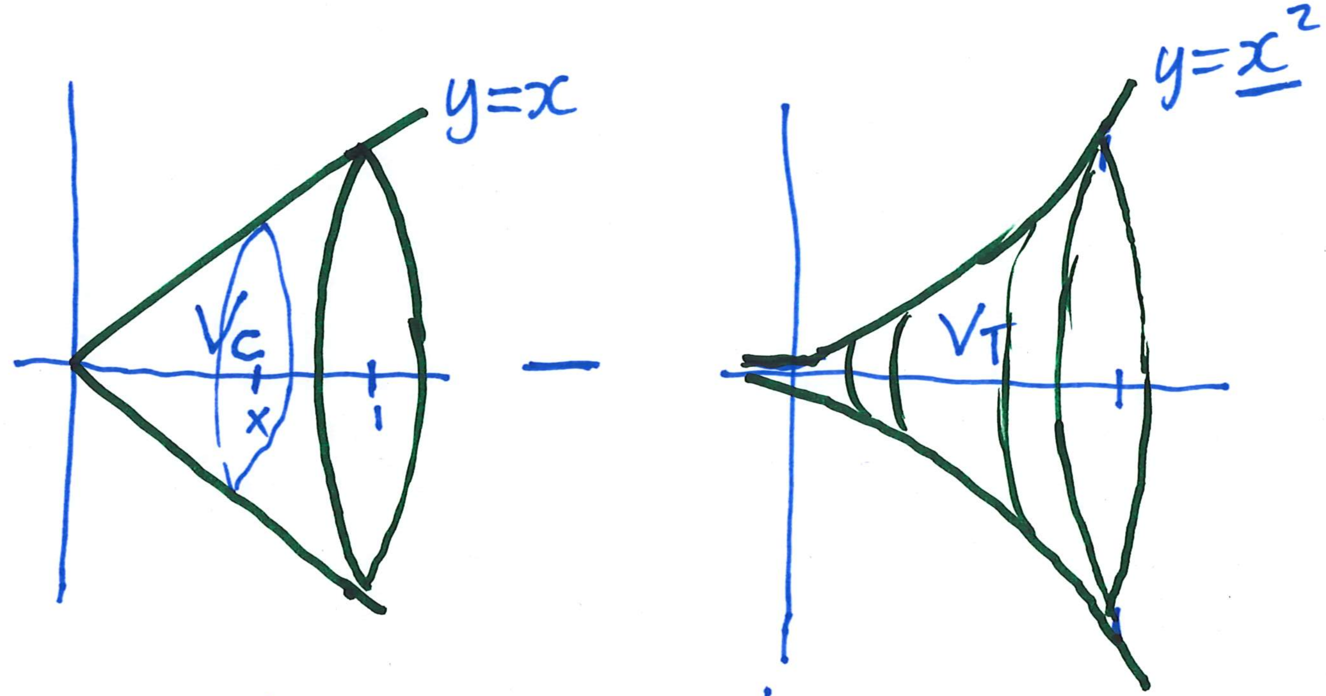
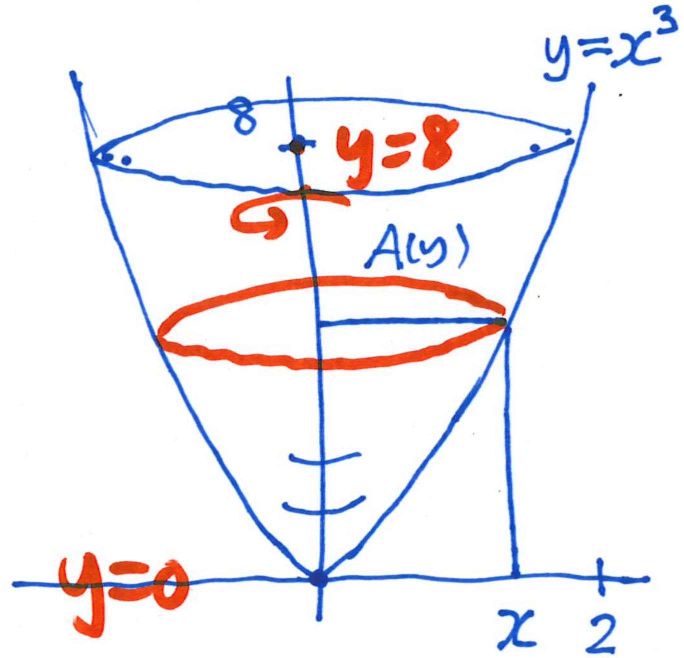


$$= \int_0^1 \pi x^2\,dx - \int_0^1 \pi (x^2)^2\,dx = \cdots = \pi\left(\frac{1}{3}-\frac{1}{5}\right)$$

circle of radius x

circle of radius x^2

Example 5. Find the volume of the solid obtained by rotating $y = x^3$ about the y axis for y on $[0, 8]$.



$$\int_a^b A(x)\,dx$$

$$\int_a^b A(y)\,dy = \int_a^b \pi x^2\,dy = \int_0^8 \pi \left(y^{\frac{1}{3}}\right)^2 dy$$

$$= \cdots = \frac{96}{5}\pi.$$

$$A(y) = \pi \cdot x^2$$

$$y = x^3 \Rightarrow \boxed{x = \sqrt[3]{y} = y^{1/3}}$$