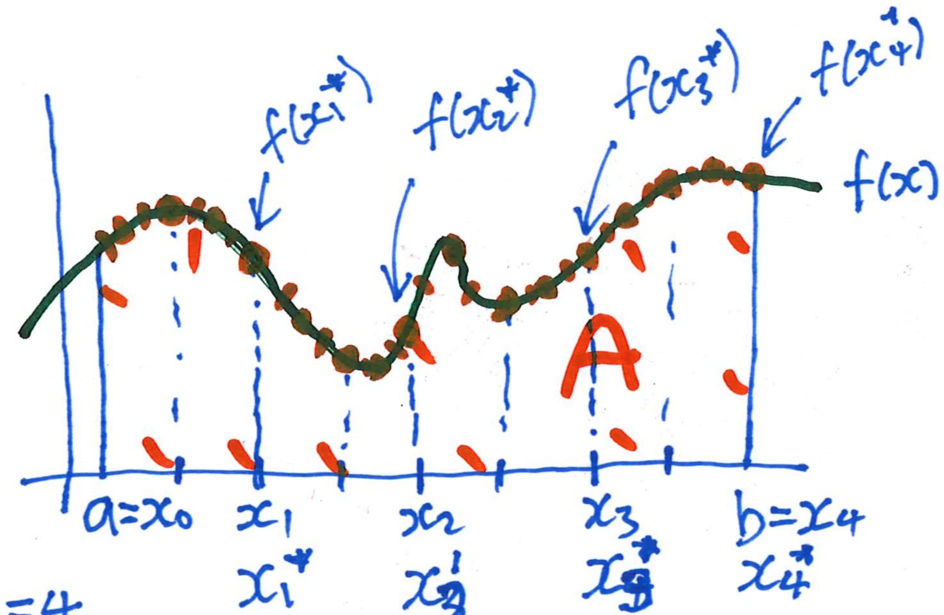
6·5 Average of a Function

Given n numbers $y_1, y_2, \ldots, y_n$ their average is $(y_1+y_2+\cdots+y_n)/n$.
What's the average f_{AVE} of a function [continuous] on $[a,b]$?



Divide $[a,b]$ into n subintervals $[x_0,x_1], [x_1,x_2], \ldots, [x_{n-1},x_n]$ of equal width $\Delta x = (b-a)/n$. Pick x_i^* in $[x_{i-1},x_i]$ for $1 \le i \le n$

Then $\displaystyle\sum_{i=1}^{n} f(x_i^*)/n \approx f_{AVE}$

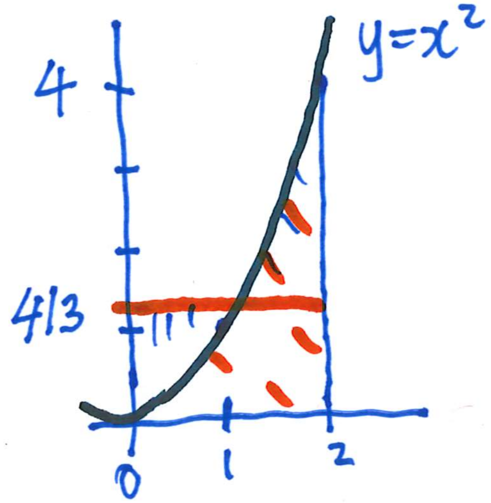
$n=4$
$n=8$
$n=16$

Define $$f_{AVE} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)/n = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{f(x_i^*)}{(b-a)} \underbrace{\frac{(b-a)}{n}}_{=\Delta x}$$

$$= \frac{1}{b-a} \lim_{n\to\infty} \boxed{\sum_{i=1}^{n} f(x_i^*)\cdot\Delta x} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Note If $f(x) \ge 0$ on $[a,b]$ then $f_{AVE} = \dfrac{A}{b-a}$.

Example 1. Calculate the average of $y=x^2$ on $[0,2]$

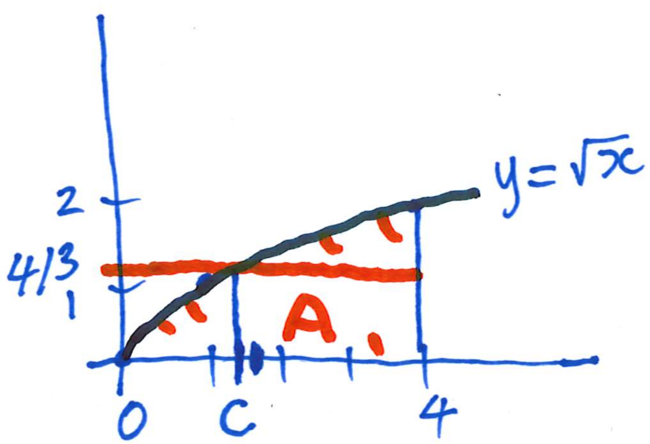


$$f_{AVE} = \frac{1}{2-0}\int_0^2 x^2\,dx = \frac{1}{2}\left[\frac{1}{3}x^3\right]_0^2 = \frac{1}{2}\left(\frac{8}{3}-0\right) = \frac{4}{3}$$

The Mean Value Theorem for Integrals.

If $f(x)$ is continuous on $[a,b]$ then $\exists$ (there exists) some constant c in $[a,b]$ such that $f(c) = f_{AVE}$

Example 2. For $f(x)=\sqrt{x}$ on $[0,4]$ find c such that $f(c)=f_{AVE}$.

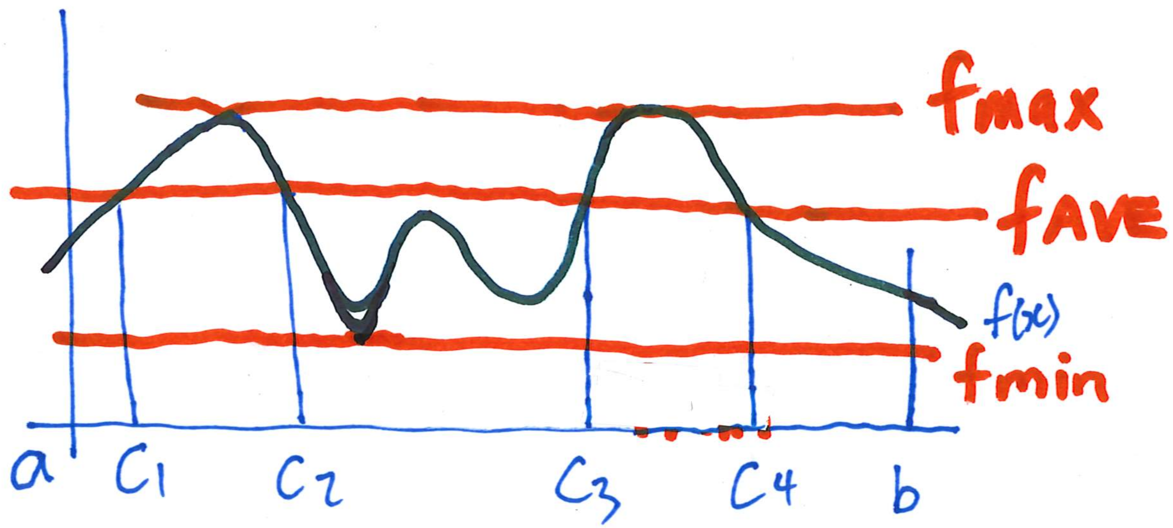


$$A = \int_0^4 \sqrt{x}\,dx = \int_0^4 x^{1/2}\,dx = \left[\frac{2}{3}x^{3/2}\right]_0^4 = \frac{2\cdot 2^3}{3} - 0 = \frac{16}{3}$$

$$f_{AVE} = A/4-0 = \frac{4}{3}$$

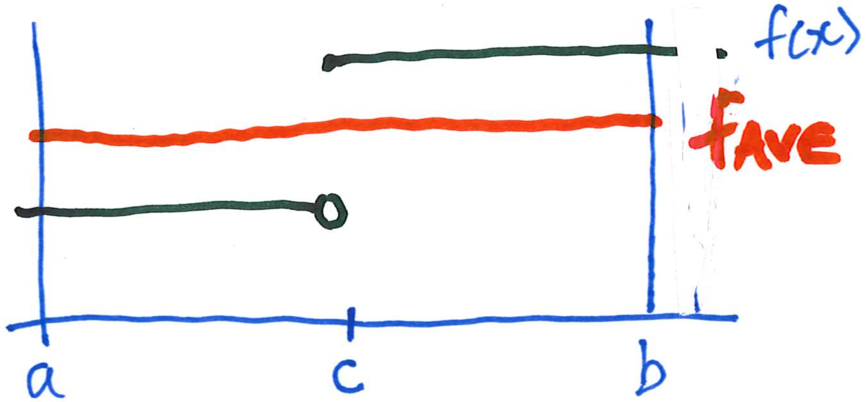
Solve $f(c) = \frac{4}{3} = \sqrt{c} \Rightarrow \frac{16}{9} = c = 1.7\dot{7}$

Proof.



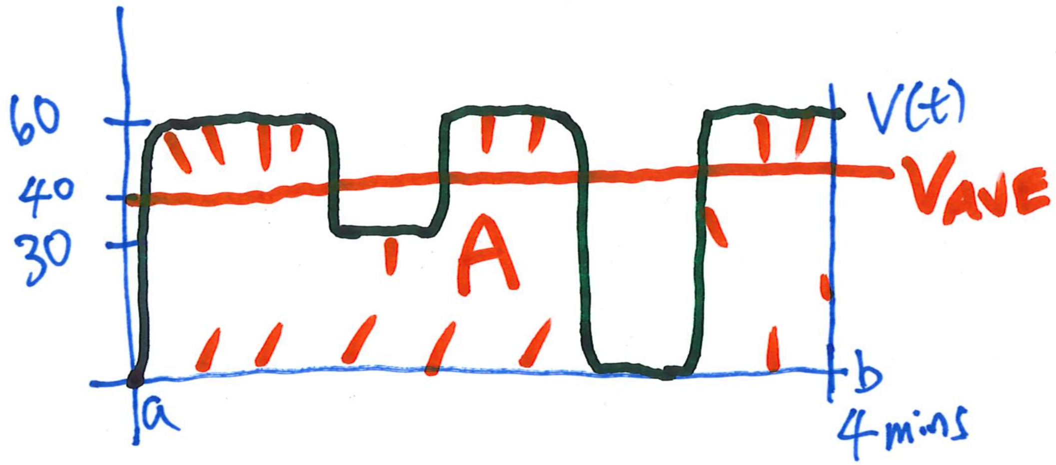
$f_{min} \leq f_{AVE} \leq f_{max}$

f(x) is continuous means we can get from fmin to fmax so we must go through fave



illustrates why f(x) must be continuous on [a,b]

Suppose a car travels at velocity $v(t) \geq 0$ on $[a,b]$.
Let D be the distance travelled on $[a,b]$



Let A be the area under $v(t)$ on $[a,b]$.
Why is $D = A$?

Let $d(t)$ be the distance travelled at time t.

① Using $d'(t) = v(t)$.

$$A = \int_a^b v(t)\,dt = \int_a^b d'(t) \overset{\text{By FTC(2)}}{=} d(b) - d(a) = D.$$

② Using $D = V_{AVE}(b-a)$ $= \left(\cancel{\frac{1}{b-a}} \int_a^b v(t)\,dt\right) \cdot \cancel{(b-a)} = \int_a^b v(t)\,dt = A$

The FTC (1) says: If $f(x)$ is continuous on $[a,b]$ then

If $g(x) = \int_a^x f(t)\,dt$ then $g'(x) = f(x)$.

TRAP. If $h(x) = \int_a^{x^2} f(t)\,dt$ is $h'(x) = f(x^2)$? <u>No</u>

FTC(2) $h(x) = F(\overset{u}{\underline{x^2}}) - F(a)$ where $F'(x) = f(x)$.

$$h'(x) = F'(x^2)\cdot 2x - 0 =$$

$$= f(x^2)\cdot \underline{2x}$$