Zeros of Eisenstein Series

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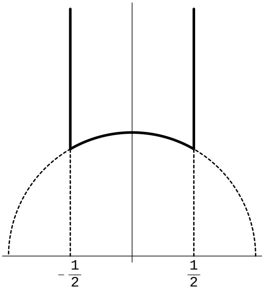
The modular group

$\Gamma := \mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$$

A fundamental domain for this action is

$$\mathcal{F} := \{z \in \mathbb{H} : -1/2 \leq \mathrm{Re}(z) \leq 1/2, |z| \geq 1\}$$



Modular forms on Γ

A modular form of integer weight k for Γ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ with

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z), \qquad z \in \mathbb{H}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

We can extend f to a point at ∞ and write a Fourier series for f at ∞ with $q := e^{2\pi i z}$:

$$f(z) = \sum_{n=0}^{\infty} a(n) q^n$$

Eisenstein series

For even weight $k \geq 4$,

$$G_k(z) := \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d) \neq (0,0)}} (cz + d)^{-k}$$

are modular forms of weight k.
We can normalize so that the constant term in the Fourier expansion at ∞ is 1:

$$E_k(z) := \frac{1}{2\zeta(k)} G_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

More useful for us:

$$E_k(z) := \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d) = 1}} (cz + d)^{-k}$$

Eisenstein series

Eisenstein series can be viewed as arising from the Weierstrass $\wp$ function, which satisfies

$$(\wp'(z))^2 = 4\wp(z)^3 - 60G_4\wp(z) - 140G_6$$

This differential equation can be solved recursively to get

$$\wp(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty}(2k+1)G_{2k+2}z^{2k}$$

Zeros of Eisenstein series

- How many zeros does $E_k(z)$ have in $\mathcal{F}$?

- Where are the zeros located in $\mathcal{F}$?

Number of zeros

Valence Formula: If $f \not\equiv 0$ is a modular form of weight k, let $\nu_\tau(f)$ be the order of f at $\tau \in \mathcal{F}$. Then

$$\nu_\infty(f) + \frac{1}{2}\nu_i(f) + \frac{1}{3}\nu_\omega(f) + \sum_{\tau \in \mathcal{F}, \tau \neq i, \omega} \nu_\tau(f) = \frac{k}{12}$$

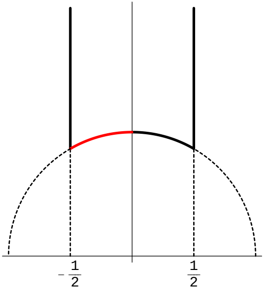
Since E_k is holomorphic on $\mathbb{H} \cup \{\infty\}$, it has $\approx k/12$ zeros in $\mathcal{F}$, counted with multiplicity.

Location of zeros

Theorem (F.K.C. Rankin, Swinnerton-Dyer)

For all even $k \geq 4$, the zeros of $E_k(z)$ are all located on the arc

$$A := \{z = e^{i\theta} : \pi/2 \leq \theta \leq 2\pi/3\}$$



Outline of RSD result

- Write $k = 12n + s$, $s \in \{4, 6, 8, 10, 0, 14\}$
- Sufficient to show $E_k(e^{i\theta})$ has at least n zeros in $(\pi/2, 2\pi/3)$
- Define $F_k(\theta) := e^{ik\theta/2} E_k(e^{i\theta}) = \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} (ce^{i\theta/2} + de^{-i\theta/2})^{-k}$
- Separate four terms with $c^2 + d^2 = 1$. Using Euler's formula,

$$F_k(\theta) = 2\cos(k\theta/2) + R$$

where R is the remaining sum over terms with $c^2 + d^2 > 1$

Outline of RSD result

- Bound $|R| < 2$ on $[\pi/2, 2\pi/3]$
- The number of zeros of $F_k(\theta)$ (hence $E_k(e^{i\theta})$), is at least the number of zeros of $2\cos(k\theta/2)$ on this interval, by an Intermediate Value Theorem argument.
- The zeros of $2\cos(k\theta/2)$ can be easily counted. There are n of them.

Interlacing of zeros

Theorem (Nozaki, '08)

Any zero of $E_k(e^{i\theta})$ *lies between two consecutive zeros of* $E_{k+12}(e^{i\theta})$ *on* $\pi/2 < \theta < 2\pi/3$.

Related results on Γ

- Kohnen (2004) derives a closed formula for the precise locations of the zeros of $E_k(z)$ in terms of the Fourier coefficients

- Gekeler (2001) cites computational evidence that the polynomials

$$\varphi_k(x) = \prod_{\substack{j(z) \text{ where } E_k(z)=0 \\ j(z)\neq 0,1728}} (x - j(z))$$

are irreducible with full Galois group S_d where d is the degree of $\varphi_k(x)$.

Related results on Γ

- Duke and Jenkins (2008) prove that the zeros of the weight k weakly holomorphic modular forms $f_{k,m}(z) = q^{-m} + O(q^{n+1})$ (with $k = 12n + s$ as before) lie on the unit circle. They use a circle method argument to bound the error term.

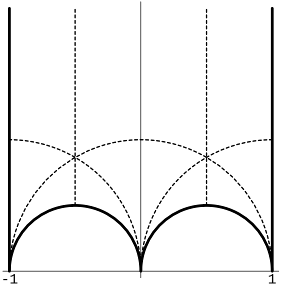
Congruence subgroup $\Gamma(2)$

$$\Gamma(2) := \left\{ \gamma \in \Gamma : \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod 2 \right\}$$

$\Gamma(2)$ has genus 0 and three inequivalent cusps: 0, 1, ∞.

The fundamental domain for $\Gamma(2)$ is

$$D := \{ z \in \mathbb{H} : -1 \leq \mathrm{Re}(z) \leq 1, |z - 1/2| \geq 1/2, |z + 1/2| \geq 1/2 \}$$



Some odd weight Eisenstein series on $\Gamma(2)$

Let $\chi(d) := \left(\frac{-1}{d}\right)$. For $k \geq 1$ let

$$E_{2k+1,\chi}(z) := \frac{1}{2} \sum_{\substack{(c,d)\equiv(0,1) \pmod 2 \\ (c,d)=1}} \chi(d)(cz+d)^{-(2k+1)}$$

Then

$$E_{2k+1,\chi}\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^{-(2k+1)} E_{2k+1,\chi}(z)$$

for all $z \in \mathbb{H}$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(2)$, and is holomorphic at the cusps.

$E_{2k+1,\chi}$ is a modular form of weight $2k+1$ for $\Gamma(2)$ with character χ.

Some odd weight Eisenstein series on $\Gamma(2)$

Zeros of $E_{2k+1,\chi}$ can be studied using the classical Jacobi elliptic function $cn(u)$, a doubly periodic generalization of $\cos u$ which satisfies

$$\frac{\kappa K}{2\pi} cn\left(\frac{2Ku}{\pi}\right) = \frac{\sqrt{q}\cos u}{1+q} + \frac{\sqrt{q^3}\cos 3u}{1+q^3} + \frac{\sqrt{q^5}\cos 5u}{1+q^5} + \cdots$$

where $K = \frac{\pi}{2}\Theta_3^2(2z)$ and $\kappa = \frac{\Theta_2^2(2z)}{\Theta_3^2(2z)}$, with

$$\Theta_2(z) := q^{1/8}\sum_{n\in\mathbb{Z}} q^{n(n+1)/2},\ \Theta_3(z) := \sum_{n\in\mathbb{Z}} q^{n^2/2}$$

Some odd weight Eisenstein series on $\Gamma(2)$

Using the Taylor series for $\cos u$, we have

$$\frac{\kappa K}{2\pi} cn\left(\frac{2Ku}{\pi}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left(\sum_{r=1}^{\infty} \frac{(2r-1)^{2k} q^{(2r-1)/2}}{1+q^{2r-1}}\right) u^{2k}$$

and solving for $cn(u)$, we have

$$cn(u) = \sum_{k=0}^{\infty} \frac{i e_{2k} \pi^{2k+1}}{2^{2k+1}(2k)!} \frac{G_{2k+1}(2z)}{K^{2k+1}\kappa} u^{2k}$$

where

$$G_{2k+1}(z) := E_{2k+1,\chi} \mid_{\gamma_0} (z) = \frac{4(-i)^{2k+1}}{e_{2k}} \sum_{r=1}^{\infty} \frac{(2r-1)^{2k} q^{(2r-1)/4}}{1+q^{(2r-1)/2}}$$

is the Fourier expansion of $E_{2k+1,\chi}$ at the cusp 1.

Another view of $cn(u)$

An alternative description of $cn(u)$ is found by solving the differential equation

$$\left(\frac{\mathrm{d}\, cn(u)}{\mathrm{d}\, u}\right)^2 = (1 - cn^2(u))\left(1 - \lambda + \lambda cn^2(u)\right)$$

to get

$$cn(u) = 1 - \frac{u^2}{2!} + (1 + 4\lambda)\frac{u^4}{4!} - (1 + 44\lambda + 16\lambda^2)\frac{u^6}{6!} + \cdots$$

Here $\lambda(z) := \frac{\Theta_2^4(z)}{\Theta_3^4(z)}$ generates the field of meromorphic modular functions for $\Gamma(2)$ (analogous to $j(z)$ for Γ). Note that $\lambda(2z) = \kappa^2$.

Zeros of $G_{2k+1}(z)$

Equating coefficients in our two expressions for $cn(u)$ we see that

$$\frac{(-1)^k i\pi^{2k+1} e_{2k}}{2^{2k+1}} \cdot \frac{G_{2k+1}(2z)}{K^{2k+1}\kappa} = \frac{(-1)^k e_{2k} G_{2k+1}(2z)}{\Theta_3^{4k}(2z) G_1(2z)}$$

is a polynomial $p_{2k+1}(\lambda)$ of degree $k-1$, having the same zeros as G_{2k+1} in D.

This is expected, since this quotient is a modular function that is holomorphic on $\mathbb{H}$, but now we can easily compute the zeros!

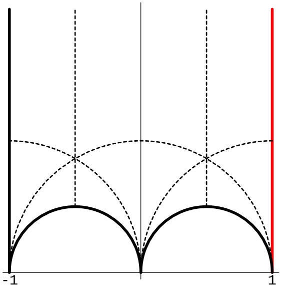
Computational results

k	$p_{2k+1}(\lambda)$	Zeros
2	$1+4\lambda$	-0.25
3	$1+44\lambda+16\lambda^2$	$-0.0229, -2.7271$
4	$1+408\lambda+912\lambda^2+64\lambda^3$	$-0.0025, -0.4598, -13.778$
5	$1+3688\lambda+30764\lambda^2+15808\lambda^3+256\lambda^4$	$-.00027, -.1280, -1.8792, -59.7425$

- For $2k+1 \leq 51$ the λ-values of the zeros of G_{2k+1} are real and lie in $(-\infty, 0)$.
- The λ-zeros of G_{2k-1} interlace with the λ-zeros of G_{2k+1}.
- The polynomials $p_{2k+1}(\lambda)$ are irreducible with Galois group S_{k-1} for $k \leq 9$.

Location of zeros of G_{2k+1}

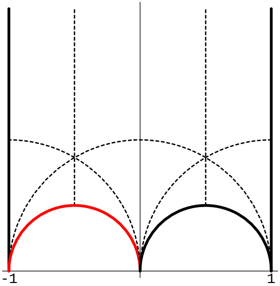
In the fundamental domain D for $\Gamma(2)$, $\lambda(z) \in (-\infty, 0)$ precisely on $\mathrm{Re}(z) = 1$.



Location of zeros of G_{2k+1}

In order to use an RSD-type argument, we move the line $\mathrm{Re}(z) = 1$ to the arc $|z + 1/2| = 1/2$ using the transformation $z \to -1/z$. Hence we consider instead

$$G_{2k+1}|_{\gamma_1}(z) := G_{2k+1}(z)|_{2k+1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$



Location of zeros of G_{2k+1}

Theorem (GLST,'10)

For $k \geq 1$, at least 90% of the zeros of $G_{2k+1}(z)$ have real λ-values in the range $(-\infty, 0]$.

Location of zeros of G_{2k+1}

Recalling

$$E_{2k+1,\chi}(z) := \frac{1}{2} \sum_{(c,d)\equiv(0,1)\ (2)} \chi(d)(cz+d)^{-(2k+1)}$$

we define

$$S_k(\alpha,\beta)(z) := \sum_{(c,d)\equiv(\alpha,\beta)\ (4)} \frac{1}{(cz+d)^k}.$$

Then

$$E_{2k+1,\chi}(z) = \frac{1}{2}\left(S_{2k+1}(0,1) + S_{2k+1}(2,1) - S_{2k+1}(0,3) - S_{2k+1}(2,3)\right).$$

Location of zeros of G_{2k+1}

Restricting to $z_\theta = 1/2e^{i\theta} - 1/2$, we can show

$$G_{2k+1} \mid_{\gamma_1} (z_\theta) = \left(S(0,3) + S(1,0) + S(2,1) + S(3,2)\right)(e^{i\theta}).$$

We balance the exponents as in RSD:

$$\begin{aligned} F_{2k+1}(z_\theta) &:= (e^{i\theta/2})^{2k+1} G_{2k+1}|_{\gamma_1}(z_\theta) \\ &= \left(\widetilde{S}_{2k+1}(0,3) + \widetilde{S}_{2k+1}(1,0) + \widetilde{S}_{2k+1}(2,1) + \widetilde{S}_{2k+1}(3,2)\right)(\theta) \end{aligned}$$

where

$$\widetilde{S}_k(\alpha,\beta)(\theta) := \sum_{\substack{(c,d)\equiv(\alpha,\beta)\ (4) \\ (c,d)=1}} (ce^{i\theta/2} + de^{-i\theta/2})^{-k}$$

Location of zeros of G_{2k+1}

$$F_{2k+1}(z_\theta) = \left(\widetilde{S}_{2k+1}(0,3) + \widetilde{S}_{2k+1}(1,0) + \widetilde{S}_{2k+1}(2,1) + \widetilde{S}_{2k+1}(3,2)\right)(\theta)$$

As in RSD, we extract the two terms with $c^2 + d^2 = 1$ to create our main term:

$$F_{2k+1}(z_\theta) = -2i \sin\left(\frac{\theta(2k+1)}{2}\right) + R_{2k+1}(z_\theta)$$

Our goal is to show $|R_{2k+1}(z_\theta)| < 2$.

Bounding the error term

We now collect terms satisfying $c^2 + d^2 = N$ for fixed $N > 1$.

- The terms with $N \leq 100$ are dealt with carefully
- The terms with $N > 100$ can be easily bounded using an appropriate integral

Bounding the error term

For each ordered pair of nonnegative integers (a, b) with a odd and b even, define

$$P(a,b)(\theta) = \sum_{(|c|,|d|)=(a,b) \text{ or } (b,a)} (ce^{i\theta/2} + de^{-i\theta/2})^{-2k-1}$$

where the sum is over those terms appearing in $F_{2k+1}(z_\theta)$.
For example,

$$P(3,0) = \left((-3e^{i\theta/2})^{-2k-1} + (3e^{-i\theta/2})^{-2k-1}\right) = \frac{2i}{3^{2k+1}} \sin\left(\frac{\theta(2k+1)}{2}\right)$$

Due to symmetry, each $P(a, b)$ is purely imaginary.

Bounding the error term

We assume that $2k+1 > 51$. When $b = 0$, we have the cases,

$$(a, b) \in \{(3, 0), (5, 0), (7, 0), (9, 0)\}.$$

Then

$$|P(a, b)(\theta)| = \left| \frac{2}{a^{2k+1}} \sin\left(\frac{(2k+1)\theta}{2}\right) \right| \leq \frac{2}{a^{51}}.$$

The contribution from these terms is negligible.

Bounding the error term

When $b \neq 0$ we have

$$|P(a,b)| \leq \frac{2}{((a-b)^2 + 4ab\sin^2(\theta/2))^k} + \frac{2}{((a-b)^2 + 4ab\cos^2(\theta/2))^k}.$$

- For fixed θ, this is worst when a and b are both small and $a - b = 1$.
- When $a - b = 1$ and θ approaches 0 or π, we have problems.
- We must bound $\sin^2(\theta/2)$ and $\cos^2(\theta/2)$ away from 0.

Bounding the error term

$$|P(a,b)| \le \frac{2}{((a-b)^2 + 4ab\sin^2(\theta/2))^k} + \frac{2}{((a-b)^2 + 4ab\cos^2(\theta/2))^k}.$$

Note that $\max(\cos^2(\theta/2), \sin^2(\theta/2)) \ge 1/2$.

We will require a bound $\min(\cos^2(\theta/2), \sin^2(\theta/2)) > \alpha^2 > 0$.

Then we have

$$|P(a,b)| \le \frac{2}{((a-b)^2 + 4ab\alpha^2)^k} + \frac{2}{((a-b)^2 + 2ab)^k}.$$

Bounding the error term

When $c^2 + d^2 = N > 100$ and $|\cos\theta| < \beta$, we can bound the terms by

$$|ce^{i\theta/2} + de^{-i\theta/2}|^2 = c^2 + 2cd\cos\theta + d^2 \geq (1-\beta)(c^2 + d^2)$$

The number of terms with $c^2 + d^2 = N$ is at most $2(2\sqrt{N} + 1) \leq 5\sqrt{N}$, so

$$|R(z_\theta)| < E_{100}(\alpha) + \sum_{N=101}^{\infty} 5N^{1/2}((1-\beta)N)^{-k-1/2}.$$

Bounding the sum with an integral yields

$$|R(z_\theta)| < E_{100}(\alpha) + (1-\beta)^{-k-\frac{1}{2}} \left(\frac{5}{k-1} \cdot 100^{-k+1} \right).$$

Bounding the error term

$$|R(z_\theta)| < E_{100}(\alpha) + (1-\beta)^{-k-\frac{1}{2}} \left(\frac{5}{k-1} \cdot 100^{-k+1} \right) < 2$$

Balancing α and β to maximize the range of θ with error less than 2 yields a range of $(0.05\pi, 0.95\pi)$.

Wrapping up the proof

Now we count the values θ at which $2\sin\left(\frac{\theta(2k+1)}{2}\right) = \pm 2$ and apply the Intermediate Value Theorem to

$$iF_{2k+1}(z_\theta) = 2\sin\left(\frac{\theta(2k+1)}{2}\right) + iR(z_\theta).$$

We see that $G_{2k+1}|_{\gamma_1}(z_\theta)$ must have at least one zero in each interval

$$\left[\frac{A\pi}{2k+1}, \frac{(A+2)\pi}{2k+1}\right] \subset (0.05\pi, 0.95\pi),$$

where $A > 0$ is odd.

Hence we are assured that 90% of the zeros of G_{2k+1} have real, negative λ-values.

Other groups

Fricke groups:

$$\Gamma_0(p) \cup \Gamma_0(p) W_p, \quad W_p := \begin{pmatrix} 0 & -1/\sqrt{p} \\ \sqrt{p} & 0 \end{pmatrix}$$

- Miezaki, Nozaki and Shigezumi (2007) For $p = 2, 3$, all zeros located on arc of fundamental domain
- Shigezumi (2007) partial result towards cases $p = 5, 7$
- Hahn (2007) has more general result about Fuchsian groups of the first kind

Summary

- The RSD method is pushed to its limits as we near the cusps
- The connection to $cn(u)$ could provide a different method which would give us all the zeros
- No similar results are known for groups of nonzero genus