# Exact computation of the Bifurcation point $B_4$ of the logistic map and the Bailey-Broadhurst conjectures.

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**Abstract**

We compute explicitly the irreducible polynomial corresponding to the bifurcation point $B_4$ of the logistic map. This settles in the positive two recent conjectures by Bailey and Broadhurst.

## 1 Explicit computation of bifurcation points of the logistic map

The determination of the bifurcation points of the logistic map:

$$x_{n+1} = f(x_n) = \mu x_n(1 - x_n),\ \mu \in (0, 4]$$

is specified by the following system of equations:

$$\begin{array}{l} x_2 = \mu\, x_1\,(1 - x_1) \\ x_3 = \mu\, x_2\,(1 - x_2) \\ \dots \\ x_n = \mu\, x_{n-1}\,(1 - x_{n-1}) \\ x_1 = \mu\, x_n\,(1 - x_n) \end{array} . \tag{1}$$

together with the condition

$$\left| \prod_{i=1}^{n} \mu\,(1 - 2x_i) \right| = 1 \tag{2}$$

expressing the fact that the bifurcation point is the right endpoint of the interval for $\mu$ for which the period n solution is stable. Equation (2) can in turn be replaced by the two equations:

$$\prod_{i=1}^{n} \mu\,(1 - 2x_i) = 1,\ or\ \prod_{i=1}^{n} \mu\,(1 - 2x_i) = -1. \tag{3}$$

Equations (1) together with one of equations (3) constitute a system of nonlinear polynomial equations specifying fully the successive bifurcation points of the logistic map.

## 2 Gröbner Bases

The theory of Gröbner bases has been the cornerstone of a reformulation of parts of Commutative Algebra and Algebraic Geometry from a computational point of view. The theory was developed by B. Buchberger, see [Becker & Weispfenning, 1993] for a systematic exposition. Historically, one of the early applications of Gröbner bases was a new method for solving systems of polynomial equations. In the case of systems of linear equations, the Gröbner method or resolution is equivalent to the classical Gauss triangulation of the system. The Gröbner method applies to non-linear systems of equations as well. The starting point of the theory is to define an order of the unknowns which induces an order of the monomials. A Gröbner basis is computed with respect to this order. A particularly useful order is the lexicographical order. A Gröbner basis computed with respect to a lexicographical order of the unknowns is called a lexicographical Gröbner basis. When the system of polynomial equations has a finite number of solutions, the lexicographical Gröbner basis computation provides us with a concise and useful form of the solutions of the system. In particular, it is guaranteed that the lexicographical Gröbner basis will contain a polynomial in one of the unknowns alone. This univariate polynomial is then used to find all the solutions of the system. The existence of this univariate polynomial is not accidental and can be explained in terms of the Elimination Theorem. To make precise this part of the theory of Gröbner bases theory that we use in the rest of the paper we give an outline of the Elimination Theorem following [Cox et al. 1997].

Given an ideal $I$ generated by a finite set of polynomials $f_1, \dots, f_m$ in the polynomials ring in $n$ variables $K[x_1, \dots, x_n]$, the *kth elimination ideal* denoted by $I_k$ is the ideal of the polynomial ring in $n-k$ variables $K[x_{k+1}, \dots, x_n]$ defined by

$$I_k = I \cap K[x_{k+1}, \dots, x_n].$$

The ideal $I_k$ contains all consequences of the equations $f_1 = \dots = f_m = 0$ which eliminate the first $k$ variables $x_1, \dots, x_k$. In the above definition $k$ ranges from 0 to $n-1$ with $I_0$ being equal to the original ideal $I$. Constructing the sequence of ideals $I_1, \dots, I_{n-1}$ is a systematic way of eliminating variables. The last ideal in this sequence, namely $I_{n-1}$, encodes all consequences of the equations $f_1 = \dots = f_m = 0$ which eliminate all variables except the last variable, namely $x_n$. The Elimination Theorem says that a lexicographical Gröbner basis computation gives automatically a basis of all the ideals $I_1, \dots, I_{n-1}$.

**Elimination Theorem** Let $G$ be a Gröbner basis of an ideal $I$ with respect to a lexicographical ordering with the $x_n$ being the smallest variable. Then for every $0 \le k \le n-1$ the set

$$G_k = G \cap K[x_{k+1}, \dots, x_n]$$

is a Gröbner basis of the kth elimination ideal $I_k$.

When the system of polynomial equations corresponding to the ideal $I$ has only a finite number of solutions, the lexicographical Gröbner basis computation gives a way to compute a basis of the last ideal $I_{n-1}$, which will be comprised by a polynomial in one variable, namely $x_n$.

Algorithms for computing Gröbner bases have been implemented in many commercial, academic and stand-alone programs. We performed all Gröbner computations in this paper using Magma, a very efficient symbolic algebra program developed by a team led by J. J. Cannon at the University of Sydney, Australia. Magma provides a mathematically rigorous environment for computations in algebra, number theory, geometry and combinatorics. See [Cannon & Playoust, 1996] for an introduction to Magma.

# 3 The 3rd bifurcation point of the logistic map

We start with the following system:

$$\begin{array}{ll} x_2 - \mu\, x_1\, (1 - x_1) & = 0 \\ x_3 - \mu\, x_2\, (1 - x_2) & = 0 \\ x_4 - \mu\, x_3\, (1 - x_3) & = 0 \\ x_1 - \mu\, x_4\, (1 - x_4) & = 0 \\ \mu^4\, (1 - 2\, x_1)\, (1 - 2\, x_2)\, (1 - 2\, x_3)\, (1 - 2\, x_4) + 1 & = 0 \end{array} .$$

The reduced lexicographical Gröbner basis of this system has 7 elements and is computed in Magma in less than 2 seconds. The univariate polynomial in $\mu$ is of degree 24 and factorizes in 4 factors:

$$(\mu^4 + 1)(\mu^4 - 8\,\mu^3 + 24\,\mu^2 - 32\,\mu + 17)(\mu^4 - 4\,\mu^3 - 4\,\mu^2 + 16\,\mu + 17)$$

$$(\mu^{12} - 12\,\mu^{11} + 48\,\mu^{10} - 40\,\mu^9 - 193\,\mu^8 + 392\,\mu^7 + 44\,\mu^6 + 8\,\mu^5 - 977\,\mu^4 - 604\,\mu^3 + 2108\,\mu^2 + 4913)$$

A rigorous symbolic real root isolation analysis shows that the first three polynomials do not have any real roots and that the degree 12 polynomial has 4 real roots contained in the intervals $\left[\frac{7}{2}, \frac{15}{4}\right], \left[\frac{15}{4}, 4\right]$, $\left[-\frac{7}{4}, -\frac{3}{2}\right], \left[-2, -\frac{7}{4}\right]$. Using the first interval we compute the value of the constant $B_3$ (third bifurcation point of the logistic map) with 50 digits of accuracy:

$$B_3 = 3.5440903595519228536159659866048045405830998454446.$$

This degree 12 polynomial has also been computed in [Bailey & Broadhurst, 2001] using integer relations detection algorithms.

## 4 The 4th bifurcation point of the logistic map, first Bailey-Broadhurst conjecture

In [Bailey & Broadhurst, 2001] the authors conjecture that $B_4$ might satisfy a 240−degree polynomial. In this section we prove this conjecture using the technique of the previous paragraph.

We start with the following system:

$$\begin{aligned}
&x_2 - \mu\, x_1\, (1 - x_1) && = 0\\
&x_3 - \mu\, x_2\, (1 - x_2) && = 0\\
&x_4 - \mu\, x_3\, (1 - x_3) && = 0\\
&x_5 - \mu\, x_4\, (1 - x_4) && = 0\\
&x_6 - \mu\, x_5\, (1 - x_5) && = 0\\
&x_7 - \mu\, x_6\, (1 - x_6) && = 0\\
&x_8 - \mu\, x_7\, (1 - x_7) && = 0\\
&x_1 - \mu\, x_8\, (1 - x_8) && = 0\\
&\mu^8\,(1 - 2\,x_1)\,(1 - 2\,x_2)\,(1 - 2\,x_3)\,(1 - 2\,x_4)\,(1 - 2\,x_5)\,(1 - 2\,x_6)\,(1 - 2\,x_7)\,(1 - 2\,x_8) + 1 && = 0
\end{aligned}$$

The reduced lexicographical Gröbner basis of this system has 12 elements and is computed in Magma in approximately $5\frac{1}{2}$ hours. The univariate polynomial in $\mu$ is of degree 288 and factorizes in 5 factors:

$(\mu^8 + 1)\ (\mu^8 - 8\,\mu^7 + 8\,\mu^6 + 64\,\mu^5 - 80\,\mu^4 - 256\,\mu^3 + 128\,\mu^2 + 512\,\mu + 257)$

$(\mu^8 - 16\,\mu^7 + 112\,\mu^6 - 448\,\mu^5 + 1120\,\mu^4 - 1792\,\mu^3 + 1792\,\mu^2 - 1024\,\mu + 257)$

$(\mu^{24} - 24\,\mu^{23} + 240\,\mu^{22} - 1232\,\mu^{21} + 2880\,\mu^{20} + 1536\,\mu^{19} - 25920\,\mu^{18} + 50688\,\mu^{17} + 10243\,\mu^{16} - 129072\,\mu^{15} + 41264\,\mu^{14} + 31872\,\mu^{13} + 643728\,\mu^{12} - 848192\,\mu^{11} - 930112\,\mu^{10} + 1191680\,\mu^9 - 248573\,\mu^8 + 4168680\,\mu^7 - 3429312\,\mu^6 - 2117680\,\mu^5 - 3993776\,\mu^4 - 4324800\,\mu^3 + 16711168\,\mu^2 + 16974593) P_{240}(\mu)$

where $P_{240}(\mu)$ denotes a polynomial with integer coefficients given in Appendix 1. As before, a rigorous symbolic real root isolation analysis shows that the first four polynomials do not have any real roots and that the degree 240 polynomial has 32 real roots contained in the intervals

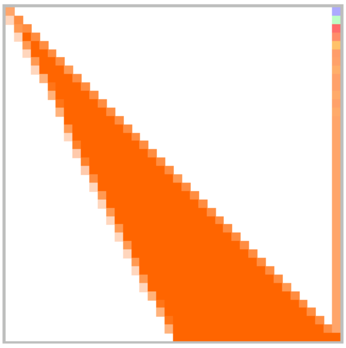
$[\frac{7}{2}, \frac{29}{8}], [\frac{29}{8}, \frac{15}{4}], [\frac{15}{4}, \frac{61}{16}], [\frac{61}{16}, \frac{31}{8}], [\frac{31}{8}, \frac{125}{32}], [\frac{125}{32}, \frac{251}{64}], [\frac{251}{64}, \frac{63}{16}], [\frac{63}{16}, \frac{253}{64}], [\frac{253}{64}, \frac{127}{32}], [\frac{127}{32}, \frac{509}{128}], [\frac{509}{128}, \frac{255}{64}]$, $[\frac{255}{64}, \frac{511}{128}], [\frac{511}{128}, \frac{1023}{256}], [\frac{1023}{256}, \frac{2047}{512}], [\frac{2047}{512}, \frac{4095}{1024}], [\frac{4095}{1024}, 4], [-\frac{13}{8}, -\frac{3}{2}], [-\frac{7}{4}, -\frac{13}{8}], [-\frac{29}{16}, -\frac{7}{4}], [-\frac{15}{8}, -\frac{29}{16}]$, $[-\frac{61}{32}, -\frac{15}{8}], [-\frac{123}{64}, -\frac{61}{32}], [-\frac{31}{16}, -\frac{123}{64}], [-\frac{125}{64}, -\frac{31}{16}], [-\frac{63}{32}, -\frac{125}{64}], [-\frac{253}{128}, -\frac{63}{32}], [-\frac{127}{64}, -\frac{253}{128}], [-\frac{255}{128}, -\frac{127}{64}]$, $[-\frac{511}{256}, -\frac{255}{128}], [-\frac{1023}{512}, -\frac{511}{256}], [-\frac{2047}{1024}, -\frac{1023}{512}], [-2, -\frac{2047}{1024}]$

Using the first interval the value of the constant $B_4$ (fourth bifurcation point of the logistic map) can be computed as before with as many digits of accuracy as desired. In particular, we have, with 50 digits of accuracy:

$$B_4 = 3.5644072660954325977735575865289824506577347383 79.$$

## 5 The 4th bifurcation point of the logistic map, second Bailey-Broadhurst conjecture

In [Bailey & Broadhurst, 2001] the authors also conjecture that $-B_4(B_4 - 2)$ might satisfy a 120−degree polynomial. The exact computation of the minimal polynomial $P_{240}(\mu)$ for $B_4$ in the previous paragraph renders the verification of this conjecture possible. The first minus sign is unimportant, therefore we put $\alpha = B_4(B_4 - 2)$ and proceed to compute an integer relation between the powers $1, \alpha, \alpha^2, \ldots, \alpha^{120}$. The approach we use here bares some similarities with the elegant application of the LLL algorithm in the exact evaluation of a difficult definite integral, illustrated in [Borwein & Corless, 1999]. First we normalize each of the powers $1, \alpha, \alpha^2, \ldots, \alpha^{120}$ in the field defined by the algebraic number $B_4$. We view these normalized expressions as univariate polynomials in $B_4$. Then we form a square $121 \times 121$ matrix $M$ whose columns represent the coefficients of the monomials from 0 up to 120 of each expression. This matrix exhibits an interesting structure as can be seen in the figure below produced with the matrix browser of Maple 7 (white space represents zero elements).



Structure of the matrix $M$ in the second Bailey-Broadhurst conjecture

The nullspace of the matrix $M$ is of dimension one and the elements of the (integer) nullvector are the coefficients of the sought integer relation between the powers $1, \alpha, \alpha^2, \ldots, \alpha^{120}$. This is the degree 120 polynomial predicted by the second Bailey-Broadhurst conjecture and is given in Appendix 2. It has exactly 16 real solutions, the smallest of which is in the interval $[4, 6]$ which corresponds to the root $B_4(B_4 - 2)$.

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# Appendix I

*The irreducible minimal polynomial of degree 240 for $B_4$*

$\mu^{240} - 240\,\mu^{239} + 28320\,\mu^{238} - 2189600\,\mu^{237} + 124721280\,\mu^{236} - 5579447808\,\mu^{235} + 204063005440\,\mu^{234} - 6271714126080\,\mu^{233} + 165223981209598\,\mu^{232}$
$-3786945344736816\,\mu^{231} + 76385543221895504\,\mu^{230} - 1368154557110579520\,\mu^{229} + 21914631182296499808\,\mu^{228} - 315666059953185178112\,\mu^{227}$
$+4106696882124628886784\,\mu^{226} - 48409339997421562499584\,\mu^{225} + 518198609701528827740659\,\mu^{224} - 5043269939314578066953376\,\mu^{223}$
$+44629222469177904878921728\,\mu^{222} - 358671670040555853434334784\,\mu^{221} + 2610315752396503220450576064\,\mu^{220} - 17112035343895988165727813376\,\mu^{219}$
$+100121349203491858725042799232\,\mu^{218} - 514382626111226062631941949696\,\mu^{217} + 2248204256027555312133008360732\,\mu^{216}$
$-7759101716927912197481099298720\,\mu^{215} + 16045417780529789648063907170976\,\mu^{214} + 28604821750936354876751683585792\,\mu^{213}$
$-509080998718648954810080159633472\,\mu^{212} + 3078075862151054482824146898172928\,\mu^{211} - 12176156559010773070353552745081344\,\mu^{210}$
$+29193728991972239444905216533852160\,\mu^{209} + 6462663357024645389238458304834637\,\mu^{208} - 472358689220110676912309339494538896\,\mu^{207}$
$+2644630496322582906161720106057350560\,\mu^{206} - 8330087241955510128939618930138195808\,\mu^{205} + 10303986318370029521167989659457723712\,\mu^{204}$
$+50874754097134984445100292476469892864\,\mu^{203} - 379329688529136367760968594396607488640\,\mu^{202} + 1251742509389330267345821292686588112384\,\mu^{201}$
$-1659013492868057129290293445877153057206\,\mu^{200} - 5816873584595148210121301393145317368272\,\mu^{199}$
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$-3608452679263832668085540625463067601018368\,\mu^{194} + 7514186445751108550235209313048399658720256\,\mu^{193}$
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$+203600180942337924880965108100052542609297664\,\mu^{190} - 205968514407770882384364899801931420709361280\,\mu^{189}$
$-731595099993879314561640689391640347250715520\,\mu^{188} + 3565122258593080888160499491093335043149039104\,\mu^{187}$
$-5953495306252298313556597641619612274278291712\,\mu^{186} - 4180068706754781083026282946009064148312342016\,\mu^{185}$
$+44930571918037368087751795095107672843739536088\,\mu^{184} - 95821598583508076578204664507774839858179627840\,\mu^{183}$
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$-1034657586025461238008746814127383939216086288512\,\mu^{180} + 499924802876929085545662302971794480267412373504\,\mu^{179}$
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$+10504942168136510741571357055164111379962539068558788999094039408\,\mu^{102}$
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$-20513387826312660680148179268373081436343260383906780284635796960\,\mu^{100}$
$+3401184480274363504701525184774321247255772597105767642599535462 4\,\mu^{99}$
$+14344828506749730675757676220056095402215364037923229835949185792\,\mu^{98}$
$+36395760253033333985953201700666668604418271322930170154928189952\,\mu^{97}$
$-73497837257536521144634271676814684885674827023380134639673372183\,\mu^{96}$
$-92302000915939920638429215104769820213534254442528258134820624224\,\mu^{95}$
$+89981506438276901669391906036384208953510872152215380047752635904\,\mu^{94}$
$+112204770730469282564580072858611880026543570108329333652018299456\,\mu^{93}$
$+124173806847692733590224372389731881885405603561130494749828008768\,\mu^{92}$

$$
\begin{aligned}
&-3134928763338650448130086342327798606219703214356942962742023150008\,\mu^{91}\\
&-2282916306270251412322731680079253385249284296618238931483017888000\,\mu^{90}\\
&+3163390333206723790163765814834083654676927550226405586841330731522\,\mu^{89}\\
&+1446572507582966231047403446598074969951367559758093779382244881000\,\mu^{88}\\
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&-3802509088231296949811280197483708920918402099707792164532626091520\,\mu^{85}\\
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&+2774233328432857575932575287325589478342621400816966199809695303680\,\mu^{83}\\
&+1579852502840070973295836061014073953664550306822845035277312008704\,\mu^{82}\\
&+1247421641568653062505834891157512214059282576421713999618711756800\,\mu^{81}\\
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&+8941072787669852586887466522178033684499447017218972077933865113856\,\mu^{75}\\
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&-8036267659205355223504773013627338808713147245385716718051878869094 4\,\mu^{67}\\
&+1932262038991483153948376970689661560084184898705609454851585717094 4\,\mu^{66}\\
&+7630758158698320744465748034899449052485097058725955335767875568332 8\,\mu^{65}\\
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&-17566078382436995726193325036625502711312368139467357214707323921395 2\,\mu^{60}\\
&+11532402944524615598291088345543519084843517249886197179472409447372 8\,\mu^{59}\\
&+36079507313755545754141213520454884951657647624634738720373674756480 0\,\mu^{58}\\
&+32872624959503094583383332217835415924597224397169827160948318769612 8\,\mu^{57}\\
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&-50405407881435057418294750166948882960692155634937186816244909176812 8\,\mu^{55}\\
&+47277426209508318364488448594415414690839547633813298757610398923936 0\,\mu^{54}\\
&-33603719766085465103798163899141709035936397032934544781566020524646 4\,\mu^{53}\\
&+78556764359104822028299888060183862616905079571168079740775430791040\,\mu^{52}\\
&+7784048891685408281322231942809015405318971513771893262923818666393 6\,\mu^{51}\\
&+54639866491654522558361793465271405825302784788596677601065259506688 0\,\mu^{50}\\
&-80130870961038620364094600488083304978010491737912952913280611071385 6\,\mu^{49}\\
&-153683303902839747718193388847209479278850172034330413744194974235880 1\,\mu^{48}\\
&+166423141158172638197037216742384585123443884105791121149006330137476 8\,\mu^{47}\\
&-291669556929654812751088420090408749572771806150609515661225561384105 6\,\mu^{46}\\
&+175489669467336718790205251768515860337528403603047682415040870070876 8\,\mu^{45}\\
&+445951563787121662858621974583015716708348637128928605454064876064409 6\,\mu^{44}\\
&+154826296182275849412940768704795753457568291701160419944160544012390 4\,\mu^{43}\\
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&-515960964407722991354725723693001485622626015582966670658424129768780 8\,\mu^{41}\\
&+382233396058554387687250400563945365074557690854590831388068945172052 2\,\mu^{40}\\
&-341586971938784457578725387120578261441046186596505643194477376287630 4\,\mu^{39}\\
&+365682820628052508221576874957656056812384407575915372573604287570750 4\,\mu^{38}\\
&+578486972293958255960794204470472511545173657035808906689815279055238 4\,\mu^{37}\\
&+1322760405007512626473315304233557496652153482585488621554678820734646 4\,\mu^{36}\\
&-1419601577660311248640996780934243351513796048176486342250579891977881 6\,\mu^{35}\\
&-1417228009621969111282866435722564029848974935880475613411466472430054 4\,\mu^{34}\\
&+529604726675873301672997169151497395908385763028744111740179324272179 2\,\mu^{33}\\
&-2026570159881703241751873325879755287689011159112589937552456051735288 3\,\mu^{32}\\
&+2560035162389553795532308995393136195150124950311615977337746882373795 2\,\mu^{31}\\
&+1553229242054507255424391035895773857328914672133691392148351077359078 4\,\mu^{30}\\
&+1156832218663946531381756849278544704528754209988805839700393187884179 2\,\mu^{29}\\
&-2214708203652097074631347570742694571575686207373091356023844323529785 6\,\mu^{28}\\
&-1109099786927026844259690302360408472782922912695230541098800639964646 4\,\mu^{27}\\
&+3137333344301967778909076325021308408529658683556333307729425689923800 32\,\mu^{26}\\
&-4554036012147449433310719460785449057269936598991285122253124195816371 2\,\mu^{25}\\
&-5140734933563885164320304783310581577809651725155392929500866435416496 4\,\mu^{24}\\
&+2756788113191429066176875065621037647403771311089652475315859602880854 4\,\mu^{23}\\
&+9482002979662382932439292764111018633225767374524442118524830157395406 40\,\mu^{22}\\
&-1033914023540447669495498477867599455560965757615056524355540134477696 0\,\mu^{21}\\
&+2246604618848690937396518052315714499902912090899500497995459304711520 0\,\mu^{20}\\
&-3871001372403979336442431869350020041351355435054845650197576178377420 8\,\mu^{19}\\
&+1306770482228863324049535224944868100104332559596346258818462592030361 6\,\mu^{18}\\
&-4194929360472073669056884670821040263482206371497081111849316181939200 0\,\mu^{17}\\
&-1422466757531409259644230670259925269038101357368001551679388625109657 73\,\mu^{16}\\
&+2021661641168213314089737119042295418339737051685108347027499720935854 4\,\mu^{15}\\
&+2656420001725881479099533652404127070615557054567162814017093547918601 6\,\mu^{14}\\
&+1643445420795858527546559747386800799123951714167099951792247307436866 24\,\mu^{13}\\
&+6407293448679391355714879244657596557160387054725517846178800616914464 0\,\mu^{12}\\
&+6870723763707097667343084848617456567248108833668089377005191139229670 4\,\mu^{11}\\
&-7302341206273611007697121670291933693399323142781327256453078015131507 2\,\mu^{10}\\
&+1016119478608334668224608695557221990805841252027829499225067190021089 28\,\mu^{9}\\
&-2858517661934825384689463853755094497626902259883826022163713521908211 22\,\mu^{8}\\
&-1273212807710471736478485476336544121161530574056303299314346526643404 64\,\mu^{7}\\
&-2567140880600854761751231162071138676826883925029661123963880076273971 68\,\mu^{6}\\
&-2571938222225542937444938289737724038763312290564947462969483693478133 12\,\mu^{5}\\
&+9310186023920002722186056127585563254009770459979183083269262201397739 20\,\mu^{4}\\
&+1986066106425567145468762597517802839695718885598475553669534044740853249
\end{aligned}
$$

# Appendix II

*The irreducible minimal polynomial of degree 120 for $B_4(B_4-2)$*

$$
\begin{aligned}
& x^{120} - 240\,x^{119} + 28320\,x^{118} - 2192000\,x^{117} + 125279998\,x^{116} - 5643368000\,x^{115} + 208854888448\,x^{114} - 6536547204864\,x^{113} + 176734354324979\,x^{112} \\
& -4196783750187360\,x^{111} + 88682696222696320\,x^{110} - 1685586976417383296\,x^{109} + 29076538840135752476\,x^{108} - 458679555102463845696\,x^{107} \\
& +6660099378384010584704\,x^{106} - 89518363094168939739136\,x^{105} + 1119319071298983395756109\,x^{104} - 13076868950817574984732240\,x^{103} \\
& +143300638913687024583031328\,x^{102} - 1478069037167442383841049088\,x^{101} + 14394582540337886032550052682\,x^{100} - 132735194531395911305471295872\,x^{99} \\
& +1161897699784925337527097977344\,x^{98} - 9677400297988188435979783880192\,x^{97} + 76857694433619164817379768106223\,x^{96} \\
& -583186936030240782254276681610176\,x^{95} + 4235532630931065075350379908751872\,x^{94} - 29492920294372518598860578015066368\,x^{93} \\
& +197203607116952506532616974664947416\,x^{92} - 1268033628603596659429063886443735424\,x^{91} + 7851550781813449394299977955008736512\,x^{90} \\
& -46874997440508848888432649452532745216\,x^{89} + 270150983433166145183446383060568451709\,x^{88} - 1504660922028293811090939826648576959088\,x^{87} \\
& +8107676533925727162494493064571099723040\,x^{86} - 42307037463758784409505431313943980013952\,x^{85} \\
& +213991172111445517948808864702874179043118\,x^{84} - 1050104212860197222522190733885015588587200\,x^{83} \\
& +5003659606005464816826850045391059295896064\,x^{82} - 23169045996546249018953127314043361444671744\,x^{81} \\
& +104333039907651695095949044999492179942993431\,x^{80} - 457236047395522550638159992276751141531760800\,x^{79} \\
& +1951461508123338242216621674859300356339393920\,x^{78} - 8116373688646739982015937446540737643388827264\,x^{77} \\
& +32916584737711769696171594333597519051658466980\,x^{76} - 130248854855179751897609481271395862492441820608\,x^{75} \\
& +503133978061586866292547074316114562806824201600\,x^{74} - 1898355089247802351298264074204380268732565888000\,x^{73} \\
& +6999691368389185026843994241534919908136581610473\,x^{72} - 25234943723224276866875981157563454497455229865040\,x^{71} \\
& +88992544210922676185813389057909291784925829309088\,x^{70} - 307135909013851370509982634495899793862091508964096\,x^{69} \\
& +1037823860394761095672449539340447469375011838389386\,x^{68} - 3434916275859916265274530996483204700433492248877312\,x^{67} \\
& +11139990166614190634243136019389671926734433904628736\,x^{66} - 35416034662403670166679746579385212043296882302706688\,x^{65} \\
& +110414402901859637549487735829569571721838562235437139\,x^{64} - 337691735279915915766318811409406637996082074518363264\,x^{63} \\
& +1013529699478994714914288376693362986467347596455088128\,x^{62} - 2986211858934802916427243680878687806857833243459921408\,x^{61} \\
& +8640005146936487346785442781687401700013882605388659408\,x^{60} - 24555699960329825824627974340337660753342323417731612928\,x^{59} \\
& +68575012600178857996284186889662631575556791935390307840\,x^{58} - 188226107925343194103263273522214055593388270952343046144\,x^{57} \\
& +507942239618005427805715369583872124725422921037671624275\,x^{56} - 1347983649709247624140692271950050653476744137989837301584\,x^{55} \\
& +3518856465490246200303203113180599317398494409561031972832\,x^{54} - 9037986653696461348031982875567469886222691599727865341312\,x^{53} \\
& +22845345773475308016340656154694303336062601123480600683914\,x^{52} - 56843234374977873735919615200783481418507756275773371617216\,x^{51} \\
& +139254966659808221215335963045232508726372612014524731180032\,x^{50} - 335959084659801854785828212407323997100377244438203817513216\,x^{49} \\
& +798355109159405801345927228788789991932705946280576582104553\,x^{48} - 1869078915450794673707390589396655341034131192688172517958816\,x^{47} \\
& +4311878060262565584511453032496296652296216620062797604275840\,x^{46} - 9803832804670317369528551341240686624528854561348493567580288\,x^{45} \\
& +21973387277450918872403998857858519302949645340697466786097316\,x^{44} - 48556639151962971177903079404113687551076710627061144017337536\,x^{43} \\
& +105810078867005707240674387648786627249352898624231666757838208\,x^{42} - 227408012170604334090552684775585389806323453129997125714366464\,x^{41} \\
& +482119864814578864737156955099274395610459652048216847513801751\,x^{40} - 1008419587101452099997676431659108982211823021990216983274986864\,x^{39} \\
& +2081261161619243659604575177214407715522135947654420913272216160\,x^{38} - 4239046046670475622617577864270299983675157673444675240355521536\,x^{37} \\
& +8521535169219597995143668176037045491467263674850132732684786734\,x^{36} - 16909165763986713112466186382140648721068695026385981636687162752\,x^{35} \\
& +33122446355246487384546094740122121420781996226444572484716069376\,x^{34} - 64055303982833468064886897792297955649835867296103976588895731200\,x^{33} \\
& +122307507387088918303056244327600508890090759603876872637926773885\,x^{32} - 230593399619283157478765056345061799175520003628114993692666971072\,x^{31} \\
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& +1433199693898536167175644348574600092186989495251117229063317212888\,x^{28} \\
& -2570418000838212144016450582433188481789143703397998702927395380608\,x^{27} \\
& +4553884870485946770630220406073989196032627308730614827510280326400\,x^{26} \\
& -7970793680282940792263826670423300603074330369712914677219541562368\,x^{25} \\
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& +109791977874550931408987112257522750141842076144182880187122536832074\,x^{20} \\
& -179220123235454966517323829724597906433553408722401788305736079731520\,x^{19} \\
& +289229143753847315463646474423359651480458725331320643805421831522816\,x^{18} \\
& -461528363311560683702446173348065755266647462471310477238225618050816\,x^{17} \\
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& -1136885069022665792814880141319638980558553659762725500746311915275616\,x^{15} \\
& +1754575561723453001181198763624934386518264158564979594600627199283328\,x^{14} \\
& -2675077774503866351961187172277154905584939886468384596764865202453888\,x^{13} \\
& +4025310265347883281142197175554098375821428143293980885824587912643868\,x^{12} \\
& -5978786046089548290522450806992678710115303806311938060553694075508800\,x^{11} \\
& +8773301942289539705998465428158115865482266531510980896141248738327168\,x^{10} \\
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& +18459534806770826925707982025203817657537491945881969892326265603729395\,x^{8} \\
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& +58188662649500017013662850797409770337561065374869894270432888758735870\,x^{4} \\
& +19860661064255671454687625975178028396957188855984755536695340447408532 49
\end{aligned}
$$