Closed form solutions of linear odes having elliptic function coefficients

George Labahn

School of Computer Science,

University of Waterloo

Joint Work with R. Burger and M. van Hoeij

Introduction

- Solve

$$A_n(x)y^{(n)}(x) + \cdots + A_1(x)y'(x) + A_0(x)y(x) = 0$$

where the $A_i(x)$ are elliptic functions.

- Issues :

(a) Why should you care ? (General history)
(b) Why did I care ? (History of Maple's dsolve)
(c) Solve in terms of what?
(d) How do we solve?

Elliptic Functions

- $f(x)$ is *doubly-periodic* if there exist two periods T, T' such that

$$f(x+T) = f(x), \qquad f(x+T') = f(x)$$

- *elliptic* $=$ doubly-periodic $+$ analytic (except poles)

- Examples:

  $\wp(x)$, $\wp'(x)$, sn (x), cn (x), dn (x), etc.

Example : Lamé's equation

Lamé's equation: given by

$$y''(x) - [n(n+1)\wp(x) + B]y(x) = 0$$

where n is a positive integer, B any constant, and $\wp(x)$ the Weierstrass $\wp$ function.

Equation comes from studying Laplace's equation

$$\frac{\partial^2 u(x,y,z)}{\partial x^2} + \frac{\partial^2 u(x,y,z)}{\partial y^2} + \frac{\partial^2 u(x,y,z)}{\partial z^2} = 0$$

in confocal elliptic orthogonal curvilinear coordinates

and solving using seperation of variables.

Additional Examples (from Kamke)

2.26: $y'' = [A\wp(x) + B]y$

2.27: $y'' + \left(a\,\mathrm{sn}^2 x + b\right) y = 0$

2.28: $y'' = \left(\frac{1}{30}\wp^{(4)}(x) + \frac{7}{3}\wp''(x) + a\wp(x) + b\right) y$

2.72: $y'' + a\wp'(x)y' + [\alpha + \beta\wp(x) - 4na\wp^2(x)]y = 0$

2.73: $y'' + \frac{\wp^3 - \wp\wp' - \wp''}{\wp' + \wp^2} y' + \frac{(\wp')^2 - \wp\wp' - \wp\wp''}{\wp' + \wp^2} y = 0$

2.74: $y'' + k^2 \frac{\mathrm{sn}x\,\mathrm{cn}x}{\mathrm{dn}x} y' + n^2 y\,\mathrm{dn}^2 x = 0$

and also 2.439–2.441, 3.9–3.14, 3.28, 4.10

Klein on the younger generation

Als ich studierte, galten die elliptischen Funktionen—in Nachwirkung der Jacobischen Tradition—als der unbestrittene Gipfel der Mathematik, und jeder von uns hatte den selbstverständlichen Ehrgeiz, hier selbst weiterzukommen. Und jetzt? Die junge Generation kennt die elliptischen Funktionen kaum mehr.

—Felix Klein (1849-1925)

Klein on the younger generation

When I was a student, elliptic functions were considered, in the tradition of Jacobi, to be the height of mathematics, and each of us dreamed of making a contribution in this area. And now? The younger generation scarcely even knows what an elliptic function is.

—Felix Klein (1849-1925)

History of dsolve: 1983-1991

First version written by undergraduate students at Waterloo: Bruce Sutherland, Andre Trudel 1983.

- implements standard algorithms for both linear and nonlinear equations
- user options: solve via laplace transform; solve in terms of series; solve numerically

History of dsolve - linear: 1983-1991

- Standard algorithms for linear ODEs:

  - e.g. constant coefficients, Euler equations, etc.

- Kovacic's algorithm for 2nd order equations (Carolyn Smith 1983)

  - first decision procedure of any kind in Maple.

- Later specialized routines for some special ODEs

  - e.g. Bessel's equation.

Some Facts: 1983-1991

dsolve*(ode,y(x))*

- either returned a complete solution or nothing
- some algorithms (e.g. Kovacic), but mostly heuristics,
- output a bit clumsy to use

History: 1992-1993

Progress in 1992 and 1993.

- more decision procedures
  - *rational solver* : from Manuel Bronstein (1992)
  - *exponential solver* : breaking through the order 2 barrier
    implementation by S. Schwendimann (1993)
- ability to return *partial* solutions using the new *DESol* function.
- try to reduce to second order linear ODEs

Simple Examples

$$x^2y'''(x)-(3x^2-x)y''(x)+(4x^2-2x-n^2)y'(x)-(2x^2-x-n^2)y(x) = 0;$$

$$y(x) = \_C1\ e^x + \_C2\ e^x \int J_n(x)dx + \_C3\ e^x \int Y_n(x)dx$$

$$y'''(x) - 3y''(x) + (x^2+3)y'(x) + (x^3+7)y(x) = 0;$$

$$y(x) = \_C1\, e^{-x} + e^{-x} \int DESol(w''(x) - 6w'(x) + (x^3-12)w(x), w(x))d$$

Special Functions : 1994-1996

- Need for more methods to solve ODEs having special functions as solutions.
- Attempt to find a fast front-end for finding *special function* solutions of second order linear ODEs (via heuristics)
- Sometimes trivial

$$dsolve(x^2 y''(x) + x y'(x) + (x^2 - a^2) y(x) = 0, y(x));$$

$$y(x) = \_C1\ J_a(x) + \_C2\ Y_a(x)$$

Special Functions

But what about (from Abramowitz and Stegun: 9.1.49-9.1.56)?

$$y''(x) + (\lambda^2 - \frac{a^2 - \frac{1}{4}}{x^2})y(x) = 0 \quad \Rightarrow y(x) = \_C1\sqrt{x}J_a(\lambda x) + \cdot$$

$$y''(x) + (\frac{\lambda^2}{4x} - \frac{a^2 - 1}{4x^2})y(x) = 0 \quad \Rightarrow y(x) = \_C1\sqrt{x}J_a(\lambda\sqrt{x}) +$$

$$y''(x) + \lambda^2 x^{p-2} y(x) = 0 \quad \Rightarrow y(x) = \_C1\sqrt{x}J_{\frac{1}{p}}(\frac{2\lambda}{p}x^{\frac{p}{2}}) -$$

$$x^2 y''(x) + (1-2p)xy'(x)$$

$$+(\lambda^2 q^2 x^{2q} + p^2 - a^2 q^2)y(x) = 0 \quad \Rightarrow y(x) = \_C1x^p J_a(\lambda x^q) + \cdot$$

$$y''(x) - \frac{2a-1}{x}y'(x) + \lambda^2 y(x) = 0 \quad \Rightarrow y(x) = \_C1x^a J_a(\lambda x) + \cdot\cdot$$

$$y^{(n)}(x) - (-1)^n x^{-n} y(x) = 0$$

etc. etc.

Heuristic Special Function Solver

For common second order linear ODEs (e.g. Bessels, Legendre, Whittaker, Hypergeometric, ...) do:

- transform to new ODE via $x \to ax^b$
- convert new ODE to normal form: $y''(x) + I(x)y(x) = 0$
- obtain a set of "common" invariants with variables a, b

For a given ODE we compute its invariant $\hat{I}(x)$, match to set of existing common invariants and build solution.

Worked very well.

Additional Improvements (1997-)

- a new *exponential* solver (M. van Hoeij 1997)
- handling functions in coefficients (G. Labahn 1998)
  - known (e.g. $sin(x)$) or unknown (e.g. $f(x)$)
- using differential factorization (G. Labahn 1999)
- LCLM differential factorizations for orders 3 and 4 (MVH 2000)
- finding symmetric products for orders 3 and 4 (MVH 2000)

Additional Improvements (1997-)

- recognizing MeijerG ODE for higher order (G. Labahn 2001)
  - basis in terms of Meijer G functions (with ax^b arguments)
  - Meijer G functions converted to special functions
- improved recognition of Hypergeometric ODEs (E. Cheb-Terrab 2001)
  - includes $\frac{ax^b+c}{dx^b+e}$ arguments
- new Kovacic algorithm (MVH 2001)
- blended in with newer version of dsolve (E. Cheb-Terrab)

How Good?

- Handles 90% of Kamke's linear ODES. Kamke has approximately
  - 500 second order examples,
  - 82 third order examples,
  - 44 fourth order examples,
  - 11 fifth order examples.
- Handles 95% of Kamke that one can expect Maple to do.
- Implies dsolve/linear can do nearly all *easy* problems. Rest?

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Back to : Elliptic Functions

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Weierstrass $\wp$ Function

- Doubly-periodic with poles (of second order) at $mT + m'T'$, for all $m, m' \in \mathbb{Z}$
- $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$, ($g_2$, g_3 constants)
  $\Rightarrow \wp'', \wp''', \ldots$ expressible in terms of $\wp$ and $\wp'$:
  $\wp'' = 6\wp^2 - \frac{1}{2}g_2$, $\wp''' = 12\wp\wp'$, etc.
- Any elliptic function can always be expressed as

$$R_1(\wp) + R_2(\wp)\wp' \quad \in \quad \mathbb{K}(\wp, \wp'),$$

  where $R_1(\wp)$, $R_2(\wp)$ are rational functions of $\wp$.
  Note: Not true in case of periodic functions.
- Similar properties for $sn(x)$, etc.

Doubly-Periodic of the 2nd Kind

- $F(x)$ is *doubly-periodic of the second kind* if there exist two periods T, T', and two constants s, s' such that

$$F(x+T) = sF(x), \qquad F(x+T') = s'F(x)$$

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- $F'(x)$ also doubly-periodic of the second kind, (same T, T', s, s'). Hence $\frac{F'(x)}{F(x)}$ is doubly-periodic

$$\frac{F'(x+T)}{F(x+T)} = \frac{sF'(x)}{sF(x)} = \frac{F'(x)}{F(x)} \qquad \Longrightarrow \qquad \frac{F'(x)}{F(x)} \in \mathbb{K}(\wp, \wp')$$

- $\Rightarrow F(x) = e^{\int^x f(u)du}$, for some $f \in \mathbb{K}(\wp, \wp')$.

Picard's Theorem (c. 1879)

- Consider the nth order homogeneous linear ODE

$$A_n(x)y^{(n)}(x) + \cdots + A_1(x)y'(x) + A_0(x)y(x) = 0$$

where the $A_i(x)$ are elliptic functions.

If the general solution of ode is uniform (i.e., path-independent), then ode possesses at least one solution that is doubly-periodic of the second kind.

Classical Soln of Lamé's Eqn

$$y(x) = \frac{\sigma(x-a_1)\sigma(x-a_2)\cdots\sigma(x-a_n)}{\sigma^n(x)} e^{x\sum_{i=1}^n \zeta(a_i)},$$

a_1, a_2,. . . ,a_n are constants satisfying

$$\frac{\wp'(a_1)+\wp'(a_2)}{\wp(a_1)-\wp(a_2)} + \frac{\wp'(a_1)+\wp'(a_3)}{\wp(a_1)-\wp(a_3)} + \cdots + \frac{\wp'(a_1)+\wp'(a_n)}{\wp(a_1)-\wp(a_n)} = 0$$

$$\frac{\wp'(a_2)+\wp'(a_1)}{\wp(a_2)-\wp(a_1)} + \frac{\wp'(a_2)+\wp'(a_3)}{\wp(a_2)-\wp(a_3)} + \cdots + \frac{\wp'(a_2)+\wp'(a_3)}{\wp(a_2)-\wp(a_n)} = 0$$

$$\vdots$$

$$\frac{\wp'(a_{n-1})+\wp'(a_1)}{\wp(a_{n-1})-\wp(a_1)} + \quad \cdots \quad \cdots \quad + \frac{\wp'(a_{n-1})+\wp'(a_n)}{\wp(a_{n-1})-\wp(a_n)} = 0$$

$$(2n-1)\sum_{i=1}^n \wp(a_i) = B.$$

Our "closed form" solutions

- First-order right hand factors in $\overline{\mathbb{K}}(\wp, \wp')[D_x]$

  (Picard's Thm implies these exist for many odes)

- e.g., Lamé's Equation, for $n = 1$:

$$D_x - \frac{\wp'(u) - \sqrt{4B^3 - g_2B - g_3}}{2(\wp(u) - B)}, \quad D_x - \frac{\wp'(u) + \sqrt{4B^3 - g_2B - g_3}}{2(\wp(u) - B)}$$

i.e., solutions

$$y_1(x), \;\; y_2(x) = \exp\left(\frac{1}{2}\int^x \frac{\wp'(u) \pm \sqrt{4B^3 - g_2B - g_3}}{\wp(u) - B}du\right)$$

Algebraic Form

- Change independent variable from x to $z = \wp$

$$\wp' = \sqrt{4z^3 - g_2 z - g_3} = \sqrt{\omega(z)}, \quad \frac{d}{dx} \to \sqrt{\omega(z)}\frac{d}{dz}$$

so ode is in $\mathbb{K}\left(z, \sqrt{\omega(z)}\right)[D_z]$

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- Looking for factors in $\overline{\mathbb{K}}\left(z, \sqrt{\omega(z)}\right)[D_z]$

  (solved by Michael Singer, for arbitrary order)

  We give an efficient algorithm for 2nd order case

Elliptic function solutions

- Find solutions of the form $R_1(\wp) + R_2(\wp)\wp'$ for

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- Change independent variable from x to $z = \wp$ so now looking for solutions

$$R_1(z) + R_2(z)\sqrt{\omega(z)} \in \mathbb{K}(z, \sqrt{\omega(z)})$$

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$$R_1(z) + R_2(z)\sqrt{\omega(z)} \in \mathbb{K}(z, \sqrt{\omega(z)})$$

- Similar to finding rational solutions ($\in \mathbb{K}(x)$) of a linear ode
(e.g. use singularities, indicial equation, etc)

(but solutions will not often be of this form)

Product of Solutions

- Suppose $A_{n-1}(x) = 0$ in

$$A_n(x)y^{(n)}(x) + \cdots + A_0(x)y(x) = 0$$

and that $\{y_1(x), \ldots, y_n(x)\}$ form a basis of solutions, all doubly-periodic of second kind.

- Then $Y(x) := y_1(x) \cdots y_n(x)$ is doubly-periodic

$$\Longrightarrow \; Y(x) \in \mathbb{K}(\wp, \wp')$$

Symmetric Power ODE

- Let y_1, y_2 be solns of $y''(x) + A(x)y(x) = 0$. Then y_1^2, $y_1 y_2$, y_2^2 are solutions of

$$Y'''(x) + 4A(x)Y'(x) + 2A'(x)Y(x) = 0.$$

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- From a solution Y of Symmetric Power compute

$$C = \sqrt{(Y')^2 - 2YY'' - 4AY^2}.$$

Then solutions y_1, y_2 of ode are:

$$y_1(x) = \exp \int^x \frac{Y' - C}{2Y}\, du, \qquad y_2(x) = \exp \int^x \frac{Y' + C}{2Y}\, du$$

Example : Lamé's Equation

- Lamé's Equation, for $n = 1$:

$$y'' - (2\wp(x) + B)\,y = 0 \tag{-9}$$

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$$y'' - (2\wp(x) + B)\, y = 0 \qquad \text{(-8)}$$

- Symmetric Power ODE:

$$Y''' - 4\,(2\wp(x) + B)\, Y' - 4\wp'(x)Y = 0 \qquad \text{(-8)}$$

with solution $Y = \wp(x) - B \in \mathbb{K}(\wp, \wp')$

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- Solutions to Lamé's equation therefore

$$y_1(x),\;\; y_2(x) = \exp\left(\frac{1}{2}\int^x \frac{\wp'(u) \pm \sqrt{4B^3 - g_2 B - g_3}}{\wp(u) - B}\,du\right)$$

Case 1: Two Hyperexp. Solutions

- Let y_1, y_2 be two independent hyperexponential solutions of

$$y''(x) + A(x)y(x) = 0$$

i.e., $\frac{y_1'}{y_1}$, $\frac{y_2'}{y_2} \in \mathbb{K}(\wp, \wp')$ By Abel's Theorem, y_1, y_2 have nonzero constant Wronskian C:

$$y_2' y_1 - y_1' y_2 = C$$

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- Divide by $y_1 y_2 = Y$ and rearrange : $Y = \frac{C}{\frac{y_2'}{y_2} - \frac{y_1'}{y_1}}$

(r.h.s. terms are all $\in \mathbb{K}(\wp, \wp')$, so Y also in $\mathbb{K}(\wp, \wp')$)

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(r.h.s. terms are all $\in \mathbb{K}(\wp, \wp')$, so Y also in $\mathbb{K}(\wp, \wp')$)

- $\Rightarrow$ **symmetric power method still works for this**

Case 2: One Hyperexp. Solution

$$y'' - \left(6\wp + 1 - \frac{g_2}{2\wp} + \frac{2}{\wp}\wp'\right) y = 0$$

- Has a solution $y_1 = e^x \wp$, doubly periodic of 2nd kind

  i.e., operator has a right hand factor

$$D_x - \frac{y_1'}{y_1} = D_x - \left(1 + \frac{\wp'}{\wp}\right)$$

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$$D_x - \frac{y_1'}{y_1} = D_x - \left(1 + \frac{\wp'}{\wp}\right)$$

- However $y_1^2 \notin \mathbb{K}(\wp, \wp')$, no 2nd solution y_2 such that $y_1 y_2 \in \mathbb{K}(\wp, \wp')$

  $\Rightarrow$ **symmetric power method fails**

One Approach

$$L = D_x^2 - r(x), \quad \text{where} \quad r(x) = a(\wp) + b(\wp)\wp' \qquad \text{(-10)}$$

- Transform independent variable from x to $z = \wp$:

$$\wp' = \sqrt{4z^3 - g_2 z - g_3} = \sqrt{\omega(z)}, \quad D_x = \sqrt{\omega(z)}D_z$$

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- L becomes a differential operator in $\mathbb{K}(z, \sqrt{\omega(z)})[D_z]$:

$$L = \omega(z)D_z^2 + \frac{\omega'(z)}{2}D_z - a(z) - b(z)\sqrt{\omega(z)}$$

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- $\hat{L} =$ symmetric product $(L, \overline{L}) \in \mathbb{K}(z)[D_z]$ 4th order
- Then $(D_z - r_1 - \overline{r}_1) \in \mathbb{K}(z)[D_z]$ is right hand factor of $\hat{L}$

Recovering Factors of L

- If $D_z - c(z)$ is a right hand factor of $\hat{L}$, set

$$v(z) = \frac{c(z)}{2}$$

$$t(z) = v(z)^2\omega(z) + v'(z)\omega(z) + \frac{1}{2}v(z)\omega'(z)$$

$$u(z) = \frac{1}{2b(z)}(a'(z) + 4a(z)v(z) - 4t(z)v(z) - t'(z))$$

- If $a(z) = u(z)^2 + t(z)$, then we have

$$L = (D_x + s(x))\,(D_x - s(x))$$

where $s(x) = u\,(\wp(x)) + v\,(\wp(x))\,\wp'(x)$

Example (completed)

$$y'' - \left(6\wp + 1 - \frac{g_2}{2\wp} + \frac{2}{\wp}\wp'\right) y = 0$$

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- Check symmetric power for solns $Y \in \mathbb{K}(\wp, \wp')$
  $\Rightarrow$ none found, so at most one hyperexp. soln
- Transform variable $x \to z = \wp$, construct $\overline{L}$, $\hat{L}$ and find right hand factor of $\hat{L}$: $\left(D_z - \frac{2}{z}\right)$

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- Transform variable $x \to z = \wp$, construct $\overline{L}$, $\hat{L}$ and find right hand factor of $\hat{L}$: $(D_z - \frac{2}{z})$
- $\Rightarrow v(z) = \frac{1}{z}$, $u(z) = 1$ so right hand factor of ode is $D - \left(1 + \frac{\wp'}{\wp}\right)$, i.e., a solution is

$$y_1(x) = e^{\int^x (1 + \frac{\wp'(u)}{\wp(u)}) du} = e^x \wp$$

Summary

- Have given algorithm for factoring 2nd order odes in $\overline{\mathbb{K}}(\wp, \wp')[D_x]$

- Algorithm to find elliptic solutions currently implemented in Maple 9.0 (DEtools[dperiodic_sols]).

  (includes case of two hyperexp. solutions).

- Remaining case (of one hyperexponential solution) will be in next version of Maple.

Future Work

- Higher order odes - e.g., Beke-like algorithm or van Hoeij factorization algorithm
- For second order, complete Kovacic-like algorithm
- Alternate approach : in case of input with parameters, when do we have solutions that are elliptic or doubly-periodic of second kind ?

  (joint with A. Fredet)