

# DIRICHLET SERIES FOR SQUARES OF SUMS OF SQUARES: A SUMMARY

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This note is a summary version of the results that the authors presented at CNTA VII in Montreal and detailed proofs may be found in [2].

Let  $\sigma_k$  denote the sum of  $k$ th powers of the divisors of  $n$ . There is a beautiful formula for the generating functions of  $\sigma_k(n)$  (see Theorem 291 in Chapter XVII of [8])

$$(1) \quad \sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^s} = \zeta(s)\zeta(s-k), \quad \Re(s) > \max\{1, k+1\}$$

which is in terms of only the Riemann Zeta function  $\zeta(s)$ . Following Hardy and Wright, by standard techniques, one can prove the following remarkable identity due to Ramanujan (also see Theorem 305 in Chapter XVII of [8] or [13])

$$(2) \quad \sum_{n=1}^{\infty} \frac{\sigma_a(n)\sigma_b(n)}{n^s} = \frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)}$$

for  $\Re(s) > \max\{1, a+1, b+1, a+b+1\}$ . In this paper, we identify other arithmetical functions enjoying similarly explicit representations. We begin by generalizing the above result and proving that

**Theorem 1.** *If  $f_i$  and  $g_i$  are completely multiplicative arithmetical functions, then we have*

$$\sum_{n=1}^{\infty} \frac{(f_1 * g_1)(n) \cdot (f_2 * g_2)(n)}{n^s} = \frac{L_{f_1 f_2}(s) L_{g_1 g_2}(s) L_{f_1 g_2}(s) L_{g_1 f_2}(s)}{L_{f_1 f_2 g_1 g_2}(2s)}$$

where  $L_f(s) := \sum_{n=1}^{\infty} f(n)n^{-s}$  is the Dirichlet series corresponding to  $f$  and the convolution  $f * g$

$$f * g(n) := \sum_{d|n} f(d)g(n/d),$$

provided that  $\Re(s)$  is greater than all the abscissa of absolute convergence of  $L_{f_i}(s)$  and  $L_{g_i}(s)$ .

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This result recovers Hardy and Wright's formulae (1) and (2) immediately. In fact, Rankin and Selberg discovered the formula of Theorem 1 when they considered the convolution  $L$ -functions attached to automorphic forms (e.g. see Chapter 13 in [9] or [11]), as was pointed out by Ram Murty to the authors.

More generally, for certain classes of Dirichlet series,  $\sum_{n=1}^{\infty} A(n)n^{-s}$ , our Theorem 1 can be applied to obtain closed forms for the series  $\sum_{n=1}^{\infty} A^2(n)n^{-s}$ . In particular, if the generating function  $L_f(s)$  of an arithmetic function  $f$  is expressible as a sum of products of two  $L$ -functions:

$$L_f(s) = \sum_{\chi_1, \chi_2} a(\chi_1, \chi_2) L_{\chi_1}(s) L_{\chi_2}(s)$$

for appropriate coefficients  $a(\chi_1, \chi_2)$  and Dirichlet characters  $\chi_i$ , then we are able to find a simple closed form (in term of  $L$ -functions) for the generating function  $L_f^2(s) := \sum_{n=1}^{\infty} f^2(n)n^{-s}$ .

One of our central applications is to the study of the number of representations as a sum of squares. Let  $r_N(n)$  be the number of solutions to  $x_1^2 + x_2^2 + \cdots + x_N^2 = n$  (counting permutations and signs). Hardy and Wright record a classical closed form, due to Lorenz, of the generating function for  $r_2(n)$  in the terms of  $\zeta(s)$  and a Dirichlet  $L$ -function, namely,

$$\sum_{n=1}^{\infty} \frac{r_2(n)}{n^s} = 4\zeta(s)L_{-4}(s)$$

where  $L_{\mu}(s) = \sum_{n=1}^{\infty} \left(\frac{\mu}{n}\right) n^{-s}$  is the *primitive  $L$ -function* corresponding to the *Kronecker symbol*  $\left(\frac{\mu}{n}\right)$ . Define

$$\mathcal{L}_N(s) := \sum_{n=1}^{\infty} \frac{r_N(n)}{n^s} \quad \text{and} \quad \mathcal{R}_N(s) := \sum_{n=1}^{\infty} \frac{r_N^2(n)}{n^s}.$$

Simple closed forms for  $\mathcal{L}_N(s)$  are known for  $N = 2, 4, 6$  and  $8$ , due to the known explicit formula of  $r_N(n)$  for these  $N$  (e.g. §91 in [12]); indeed the corresponding  $q$ -series were known to Jacobi,

$$\begin{aligned} \mathcal{L}_2(s) &= 4\zeta(s)L_{-4}(s), \\ \mathcal{L}_4(s) &= 8(1 - 4^{1-s})\zeta(s)\zeta(s-1), \\ \mathcal{L}_6(s) &= 16\zeta(s-2)L_{-4}(s) - 4\zeta(s)L_{-4}(s-2), \\ \mathcal{L}_8(s) &= 16(1 - 2^{1-s} + 4^{2-s})\zeta(s)\zeta(s-3). \end{aligned}$$

Now applying an extension of Theorem 1 to these  $\mathcal{L}_N$ , we obtain the following closed forms for  $\mathcal{R}_N(s)$ :

**Theorem 2.** *We have*

$$\mathcal{R}_2(s) = \frac{(4\zeta(s)L_{-4}(s))^2}{(1 + 2^{-s})\zeta(2s)}, \quad \Re(s) > 1,$$

$$\mathcal{R}_4(s) = 64 \frac{(8 \cdot 2^{3-3s} - 10 \cdot 2^{2-2s} + 2^{1-s} + 1)\zeta(s-2)\zeta^2(s-1)\zeta(s)}{(1 + 2^{1-s})\zeta(2s-2)}, \quad \Re(s) > 3,$$

$$\mathcal{R}_6(s) = 16 \frac{(17 - 32 \cdot 2^{-s}) \zeta(s-4) L_{-4}^2(s-2) \zeta(s)}{(1 - 16 \cdot 2^{-2s}) \zeta(2s-4)} - \frac{128}{(1 + 4 \cdot 2^{-s})} \frac{L_{-4}(s-4) \zeta^2(s-2) L_{-4}(s)}{\zeta(2s-4)}, \quad \Re(s) > 5,$$

and

$$\mathcal{R}_8(s) = 256 \frac{(32 \cdot 2^{6-2s} - 3 \cdot 2^{3-s} + 1) \zeta(s-6) \zeta^2(s-3) \zeta(s)}{(1 + 2^{3-s}) \zeta(2s-6)} \quad \Re(s) > 7.$$

An immediate application of Theorem 2 is to the estimation of the average order of  $r_N^2(n)$ . Using the usual hyperbola method of Dirichlet and a convolution argument, we can prove that

**Theorem 3.** *We have*

$$(3) \quad \sum_{n \leq x} r_2^2(n) = 4x \log x + 4\alpha x + O(x^{2/3}),$$

$$\sum_{n \leq x} r_4^2(n) = W_4 x^3 + O(x^2 \log^5 x),$$

$$\sum_{n \leq x} r_6^2(n) = W_6 x^5 + O(x^4)$$

and

$$\sum_{n \leq x} r_8^2(n) = W_8 x^7 + O(x^6)$$

where  $\alpha := 2\gamma + \frac{8}{\pi} L'_{-4}(1) - \frac{12}{\pi^2} \zeta'(2) + \frac{1}{3} \log 2 - 1 = 2.0166216 \dots$  and

$$W_N := \frac{1}{(N-1)(1-2^{-N})} \frac{\pi^N}{\Gamma^2(\frac{1}{2}N)} \frac{\zeta(N-1)}{\zeta(N)}, \quad (N \geq 3).$$

Here  $\gamma$  is the Euler's constant.

This technique can be adjusted to handle all integers  $N \geq 2$  except  $N = 3$  as below and so we establish all but the most difficult case of the following general conjecture due to Wagon (see [4]):

**Wagon's Conjecture.** *For integer  $N \geq 3$ ,  $\sum_{n \leq x} r_N^2(n) \sim W_N x^{N-1}$  as  $x \rightarrow \infty$ .*

Despite lacking the closed form for  $\mathcal{R}_N(s)$  in general for  $N \geq 5$ , we can use the singular series formula for  $r_N(n)$  given by Hardy (see p.342 of [7] or p.155 of [6]), which may be written as

$$r_N(n) \frac{\Gamma(N/2)}{\pi^{N/2}} n^{1-N/2} = \sum_{k=1}^{\infty} \sum_{\substack{1 \leq h \leq k \\ (h,k)=1}} \left( \frac{G(h,k)}{k} \right)^N e^{-2\pi i h n / k} + O(n^{1-N/4}),$$

where  $G(h,k) = \sum_{j=1}^k e^{2\pi i h j^2 / k}$  is the standard quadratic Gauss sum, to obtain

$$(4) \quad \sum_{n \leq x} r_N^2(n) = W_N x^{N-1} + O(x^{N-2} + x^{3N/4}) \quad (N \geq 5).$$

The error term in (3) is not as good as the best known result. Indeed M. Kühleitner proved in [10] that

$$\sum_{n \leq x} r_2^2(n) = 4x \log x + 4\alpha x + O(x^{1/2}(\log x)^{11/3}(\log \log x)^{1/3}).$$

However, for  $N \geq 5$ , the estimate  $O(x^{N-2})$  in (4) in fact is the best possible.

We similarly study the number of representations by a binary quadratic forms. Let  $r_{2,P}(n)$  be the number of solutions of the binary quadratic form  $x^2 + Py^2 = n$ . Define

$$\mathcal{L}_{2,P}(s) := \sum_{n=1}^{\infty} \frac{r_{2,P}(n)}{n^s} \quad \text{and} \quad \mathcal{R}_{2,P}(s) := \sum_{n=1}^{\infty} \frac{r_{2,P}(n)^2}{n^s}.$$

The closed forms of  $\mathcal{L}_{2,P}(s)$  have been studied by a number of people, particular by Glasser, Zucker and Robertson (see [5] and [14]). In finding the exact evaluation of lattice sums, they are interested in expressing a multiple sum, such as the generating functions of  $r_{2,P}(n)$ , as a product of simple sums. As a result, plenty of closed forms of Dirichlet series  $\sum_{(n,m) \neq (0,0)} (am^2 + bmn + cn^2)^{-s}$  in terms of  $L$ -functions have been found. One of the most interesting cases is when the binary quadratic forms have *disjoint discriminants*, i.e, have only one form per genus. Then there are simple closed forms for the corresponding  $\mathcal{L}_{2,P}(s)$  (see (9.2.8) in [1])

$$(5) \quad \mathcal{L}_{2,P}(s) = 2^{1-t} \sum_{\mu|P} L_{\epsilon_\mu \mu}(s) L_{-4P\epsilon_\mu/\mu}(s)$$

where  $P$  is an odd square-free number,  $t$  is the number of distinct factors of  $P$  and  $\epsilon_\mu := \left(\frac{-1}{\mu}\right)$ . Explicitly, (5) holds for all *type one* numbers. These include and may comprise:

$$P = 5, 13, 21, 33, 37, 57, 85, 93, 105, 133, 165, 177, 253, 273, 345, 357, 385, 1365.$$

It is known that there are only finitely many such disjoint discriminants. We call such  $P$  **solvable**. We have similar closed forms of  $L$ -functions for the quadratic form  $x^2 + 2Py^2$  with discriminant  $-8P$  (see (9.2.9) in [1]):

$$\mathcal{L}_{2,2P}(s) = 2^{1-t} \sum_{\mu|P} L_{\epsilon_\mu \mu}(s) L_{-8P\epsilon_\mu/\mu}(s)$$

for the *type two* integers

$$P = 1, 3, 5, 11, 15, 21, 29, 35, 39, 51, 65, 95, 105, 165, 231.$$

Again applying extensions of Theorem 1, we obtain closed forms for  $\mathcal{R}_{2,P}(s)$  and  $\mathcal{R}_{2,2P}(s)$ .

**Theorem 4.** *Let  $P$  be a solvable square-free integer and let  $t$  be the number of distinct factors of  $P$ . We have for  $P$  respectively of type one and type two:*

$$\begin{aligned} \mathcal{R}_{2,P}(s) &= 2^{2(1-t)} \sum_{\mu_1, \mu_2 | P} L_{\epsilon_{\mu_1^* \mu_2^*} \mu_1^* \mu_2^*}(s) L_{-4P\epsilon_{\mu_1^* \mu_2^*} / \mu_1^* \mu_2^*}(s) \zeta(2s)^{-1} \\ &\quad \times \prod_{p|2P} \left\{ 1 + \left[ \left( \frac{\epsilon_{\mu_1^* \mu_2^*} \mu_1^* \mu_2^*}{p} \right) + \left( \frac{-4P\epsilon_{\mu_1^* \mu_2^*} / \mu_1^* \mu_2^*}{p} \right) \right] p^{-s} \right\}^{-1}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{R}_{2,2P}(s) &= 2^{2(1-t)} \sum_{\mu_1, \mu_2 | P} L_{\epsilon_{\mu_1^* \mu_2^*} \mu_1^* \mu_2^*}^2(s) L_{-8P \epsilon_{\mu_1^* \mu_2^*} / \mu_1^* \mu_2^*}^2(s) \zeta(2s)^{-1} \\ &\quad \times \prod_{p|2P} \left\{ 1 + \left[ \left( \frac{\epsilon_{\mu_1^* \mu_2^*} \mu_1^* \mu_2^*}{p} \right) + \left( \frac{-8P \epsilon_{\mu_1^* \mu_2^*} / \mu_1^* \mu_2^*}{p} \right) \right] p^{-s} \right\}^{-1} \end{aligned}$$

where  $\epsilon_\mu = \left( \frac{-1}{\mu} \right)$  and  $\mu_i^* = \mu_i / (\mu_1, \mu_2)$ .

In particular, the prime cases provide:

**Corollary 5.** *We have*

$$\mathcal{R}_{2,p}(s) = \frac{2\zeta^2(s)L_{-4p}^2(s)}{(1+2^{-s})(1+p^{-s})\zeta(2s)} + \frac{2L_p^2(s)L_{-4}^2(s)}{(1-2^{-s})(1+p^{-s})\zeta(2s)}$$

for  $p = 5, 13, 37$ , while

$$\mathcal{R}_{2,2}(s) = \frac{4\zeta^2(s)L_{-8}^2(s)}{(1+2^{-s})\zeta(2s)}.$$

Similarly,

$$\mathcal{R}_{2,2p}(s) = \frac{2\zeta^2(s)L_{-8p}^2(s)}{(1+2^{-s})(1+p^{-s})\zeta(2s)} + \frac{2L_{-p}^2(s)L_8^2(s)}{(1-2^{-s})(1-p^{-s})\zeta(2s)},$$

for  $p = 3, 11$  while

$$\mathcal{R}_{2,2p}(s) = \frac{2\zeta^2(s)L_{-8p}^2(s)}{(1+2^{-s})(1+p^{-s})\zeta(2s)} + \frac{2L_p^2(s)L_{-8}^2(s)}{(1-2^{-s})(1-p^{-s})\zeta(2s)}$$

for  $p = 5, 29$ .

Closed forms for  $\mathcal{L}_{2,P}(s)$  are also accessible for some  $P$  other than those of *type one* or *type two*. For example, (see Table VI of [5]) one has

$$\mathcal{L}_{2,3}(s) = (2 + 4^{1-s}) \zeta(s) L_{-3}(s)$$

and hence we obtain

$$(6) \quad \mathcal{R}_{2,3}(s) = 4 \frac{1 + 2^{3-2s}}{1 + 3^{-s}} \frac{(\zeta(s) L_{-3}(s))^2}{\zeta(2s)}.$$

We may also derive many formulae for non-square free integers via modular transformations [1]. We contain ourselves with the simplest example which is

$$\mathcal{R}_{2,4}(s) = \frac{4 - 2^{2-s} + 2^{4-2s}}{1 + 2^{-s}} \frac{(\zeta(s) L_{-4}(s))^2}{\zeta(2s)}$$

as a consequence of a quadratic transformation leading to

$$\mathcal{L}_{2,4}(s) = (2^{-1} - 2^{-1-s} + 4^{-s}) \mathcal{L}_2(s).$$

There are some simple closed forms of the generating functions for more general binary quadratic forms found in [5]. Let

$$\mathcal{L}_{(a,b,c)}(s) := \sum_{(n,m) \neq (0,0)} \frac{1}{(am^2 + bmn + cn^2)^s} = \sum_{n=1}^{\infty} \frac{r_{(a,b,c)}(n)}{n^s}$$

and  $\mathcal{R}_{(a,b,c)}(s) := \sum_{n=1}^{\infty} \frac{r_{(a,b,c)}(n)^2}{n^s}$  where  $r_{(a,b,c)}(n)$  is the number of representations of  $n$  by the quadratic form  $ax^2 + bxy + cy^2$ . Then, we have (e.g. (26) of [15])

$$\sum_{h(D)} \mathcal{L}_{(a,b,c)}(s) = \omega(D)\zeta(s)L_D(s)$$

where the sum is taken over the  $h(D)$  inequivalent reduced quadratic forms of discriminant  $D := b^2 - 4ac$  and  $\omega(-3) = 6, \omega(-4) = 4$  and  $\omega(D) = 2$  for  $D < -4$ . In particular, for  $c = 2, 3, 5, 11, 17, 41$ ,  $h(D) = 1$  and the result is especially simple:

$$\mathcal{L}_{(1,1,c)}(s) = 2\zeta(s)L_D(s).$$

Hence from Theorem 1, we have

$$\mathcal{R}_{(1,1,c)}(s) = \frac{4(\zeta(s)L_D(s))^2}{(1 + |D|^{-s})\zeta(2s)},$$

with similar formulae for  $(a, b, c) = (1, 1, 1)$  and  $(1, 0, 1)$ . Thanks to the *On-Line Encyclopedia of Integer Sequences*

<http://www.research.att.com/~njas/sequences/>

we discover that the sequence 2, 3, 5, 11, 17, 41 is exactly the so-called Euler “lucky” numbers which are the numbers  $n$  such that  $m \rightarrow m^2 - m + n$  has prime values for  $m = 0, \dots, n - 1$ .

Applying standard complex integration methods to  $\mathcal{R}_{2,P}(s)$  and  $\mathcal{R}_{2,2P}(s)$ , we derive the asymptotic formula

**Theorem 6.** *Let  $P$  be a solvable square-free integer. Let  $x > 1$  and  $\epsilon > 0$ . We have for either  $N = P$  of type one or  $N = 2P$  of type two:*

$$\sum_{n \leq x} r_{2,N}(n)^2 = \frac{3}{N} \left( \prod_{p|2N} \frac{2p}{p+1} \right) (x \log x + \alpha(N)x) + O(N^{\frac{1}{4}+\epsilon} x^{\frac{3}{4}+\epsilon})$$

where the implicit constants are independent of both  $x$  and  $P$  and

$$\alpha(N) := 2\gamma + \sum_{p|2N} \frac{\log p}{p+1} + 2 \frac{L'_{-4N}(1)}{L_{-4N}(1)} - \frac{12}{\pi^2} \zeta'(2) - 1$$

and  $\sum_{p|n}$  is the summation over all prime factors of  $n$ .

Akin to Wagon’s conjecture, we make the following conjecture.

**Quadratic Conjecture.** *For any square-free  $P$ ,*

$$\sum_{n \leq x} r_{2,P}(n)^2 \sim \frac{3}{P} \left( \prod_{p|2P} \frac{2p}{p+1} \right) x \log x$$

as  $x \rightarrow \infty$ .

In view of Theorem 6, (3) and (6), our conjecture is true for solvable  $P$  and for  $P = 1, 3$ . We have also confirmed it for  $P = 7$  and 15 from the representations of

$$\mathcal{L}_{2,7}(s) = 2(1 - 2^{1-s} + 2^{1-2s})\zeta(s)L_{-7}(s)$$

and

$$\mathcal{L}_{2,15}(s) = (1 - 2^{1-s} + 2^{1-2s})\zeta(s)L_{15}(s) + (1 + 2^{1-s} + 2^{1-2s})L_{-3}(s)L_5(s)$$

again given in [5], which leads to

$$\mathcal{R}_{2,7}(s) = 4 \frac{(1 - 3 \cdot 2^{-s} + 2^{2-2s}) (\zeta(s) L_{-7}(s))^2}{(1 + 2^{-s})(1 + 7^{-s}) \zeta(2s)}$$

and

$$\begin{aligned} \mathcal{R}_{2,15}(s) &= \frac{2(1 - 3 \cdot 2^{-s} + 2^{2-2s}) (\zeta(s) L_{-15}(s))^2}{(1 + 2^{-s})(1 + 3^{-s})(1 + 5^{-s}) \zeta(2s)} \\ &\quad + \frac{2(1 + 3 \cdot 2^{-s} + 2^{2-2s}) (L_{-3}(s) L_5(s))^2}{(1 - 2^{-s})(1 - 3^{-s})(1 - 5^{-s}) \zeta(2s)}, \end{aligned}$$

and may be analyzed by the methods above.

For the negative  $P$ , we have studied only the case  $P = -1$ . By the elementary formula due to Sierpinski [16]

$$r_{2,-1}(n) = 2d(n) - 4d\left(\frac{n}{2}\right) + 4d\left(\frac{n}{4}\right)$$

we obtain

$$\mathcal{L}_{2,-1}(s) = 2(1 - 2^{1-s} + 2^{1-2s}) \zeta(s)^2$$

and

$$\mathcal{R}_{2,-1}(s) = 4 \frac{(1 - 3 \cdot 2^{-s} + 4 \cdot 2^{-2s}) \zeta(s)^4}{(1 + 2^{-s}) \zeta(2s)}$$

where  $d(n)$  is the divisor function and  $d(x) = 0$  if  $x$  is not an integer.

We also studied  $\mathcal{L}_N(s)$ , for all  $N \geq 2$ , and obtained a Bessel-series in  $r_{N-1}(m)$ :

$$\begin{aligned} \mathcal{L}_N(s) &= \sum_{n>0} \frac{r_N(n)}{n^s} = \frac{2N \Gamma(s - \frac{N-3}{2})}{\Gamma(s+1)} \pi^{\frac{N-1}{2}} \zeta(2s - N + 1) \\ (7) \quad &\quad + \frac{4N \pi^{s+1}}{\Gamma(s+1)} \sum_{m>0} \frac{m^{\frac{1}{2}s} r_{N-1}(m)}{m^{\frac{N-3}{4}}} \sum_{n>0} \frac{n^{\frac{N+1}{2}}}{n^s} K_{s - \frac{N-3}{2}}(2n\pi\sqrt{m}). \end{aligned}$$

Here  $K_s$  is the *modified Bessel function of the second kind*. This is especially attractive for  $N = 3, 5, 7, 9$  since in these cases  $r_{N-1}(n)$  is an explicit divisor function.

Most pleasantly, for ‘jellium’, which is the Wigner sum analogue of Madelung’s constant, we have

$$\mathcal{L}_3(1/2) = -\pi + 3\pi \sum_{m>0} \frac{r_2(m)}{\sinh^2(\pi\sqrt{m})},$$

in which the exponential convergence is entirely apparent.

Some similar identities are

$$\mathcal{L}_5(3/2) = -\frac{10}{9} \pi^2 \left( 1 + 3 \sum_{m>0} \frac{r_4(m)}{\sinh^2(\pi\sqrt{m})} \right),$$

$$\mathcal{L}_7(5/2) = -\frac{28}{45} \pi^3 \left( 1 + 3 \sum_{m>0} \frac{r_6(m)}{\sinh^2(\pi\sqrt{m})} \right),$$

$$\mathcal{L}_9(7/2) = -\frac{8}{35} \pi^4 \left( 1 + 3 \sum_{m>0} \frac{r_8(m)}{\sinh^2(\pi\sqrt{m})} \right),$$

and

$$R_{2,4}(2) = 8G^2$$

where  $G := \sum_{n \geq 0} (-1)^n (2n+1)^{-2}$  is *Catalan's constant*.

For a survey of other rapidly convergent lattice sums of this type see [1] and [3]. Unfortunately we have not been able to extend this analysis to  $R_N(s)$ .

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#### REFERENCES

- [1] J.M. Borwein and P.B. Borwein, *Pi and the AGM. A study in analytic number theory and computational complexity*, CMS, Monographs and Advanced Texts, 4. John Wiley & Sons, New York, 1987. Paperback, 1998.
- [2] J.M. Borwein and K.K.S. Choi, "On Dirichlet Series for Sums of Squares", *The Ramanujan Journal*, special issue for Robert Rankin, to appear (2002).
- [3] R. E. Crandall, "New representations for the Madelung constant," *Experimental Mathematics*, **8:4** (1999), 367-379.
- [4] R. Crandall and S. Wagon, "Sums of squares: Computational aspects," (2001) preprint.
- [5] M. Glasser and I. Zucker, "Lattice Sums," in *Theoretical Chemistry : Advances and Perspectives*, **5** (1980), 67-139.
- [6] E. Grosswald, *Representations of Integers as Sums of Squares*, Springer-Verlag, 1985.
- [7] G.H. Hardy, *Collected Papers*, Vol I, Oxford University Press, 1969.
- [8] G.H. Hardy and E.M. Wright, *An Introduction to the Theory of Numbers*, 5th Ed., Oxford, 1979.
- [9] H. Iwaniec, *Topics in Classical Automorphic Forms*, Graduate Studies in Mathematics, Vol 17, AMS, 1997.
- [10] M. Kühleitner, "On a question of A. Schinzel concerning the sum  $\sum_{n \leq x} (r(n))^2$ ", *Österreichisch-Ungarisch-Slowakisches Kolloquium Über Zahlentheorie (Maria Trost, 1992)*, 63-67, *Grazer Math. Ber.*, **318**, Karl-Franzens-Univ. Graz, Graz, 1993.
- [11] M.R. Murty, *Problems in Analytic Number Theory*, GTM 206, Springer, New York, 2000.
- [12] H. Rademacher, *Topics in Analytic Number Theory*, Springer-Verlag, 1973.
- [13] S. Ramanujan, "Some formulae in the analytic theory of numbers" , *Messenger of Math.*, **45** (1916), 81-84.
- [14] M.M. Robertson and I.J. Zucker, "Exact Values for Some Two-dimensional Lattice Sums," *J. Phys. A: Math. Gen.* **8** (1975), 874-881.
- [15] D. Shanks, "Calculation and Applications of Epstein Zeta Functions", *Math. Comp.*, **29** (1975), 271-287.
- [16] W. Sierpinski, "Über die Darstellungen ganzer Zahlen als Differenz von zwei Quadraten", *Wiadom. Mat.*, **11** (1907), 89-110.

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