# Constructible Convex Sets

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*ABSTRACT.* We investigate when closed convex sets can be written as countable intersections of closed half-spaces in Banach spaces. It is reasonable to consider this class to comprise the *constructible convex sets* since such sets are precisely those that can be defined by a countable number of linear inequalities, hence are accessible to techniques of semi-infinite convex programming. We also explore some model theoretic implications. Applications to set convergence are given as limiting examples.

*Key words*: Convex Sets, Countable Intersections, Biorthogonal Systems, Mosco Convergence, Slice Convergence, Martin's Axiom, Kunen's Space.

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## 1 Introduction

We consider a real Banach space $X$, and shall call a closed convex set $C \subset X$ *constructible* if it is the countable intersection of closed half-spaces in $X$. This concept and consequent study was motivated by a recent paper by Azagra and Ferrera [1], who show that in a separable Banach space $X$, every closed convex set $C$ may be realized as the zero set of a finite non-negative $C^\infty$-smooth convex function $f$:

$$C := \{x : f(x) = \min_X f\}. \tag{1}$$

The key to their proof is to write each half-space in the intersection as the set where an appropriately constructed nonnegative $C^\infty$-smooth convex function vanishes, and then to take an appropriately weighted sum of those functions. It follows, [1], that every constructible convex

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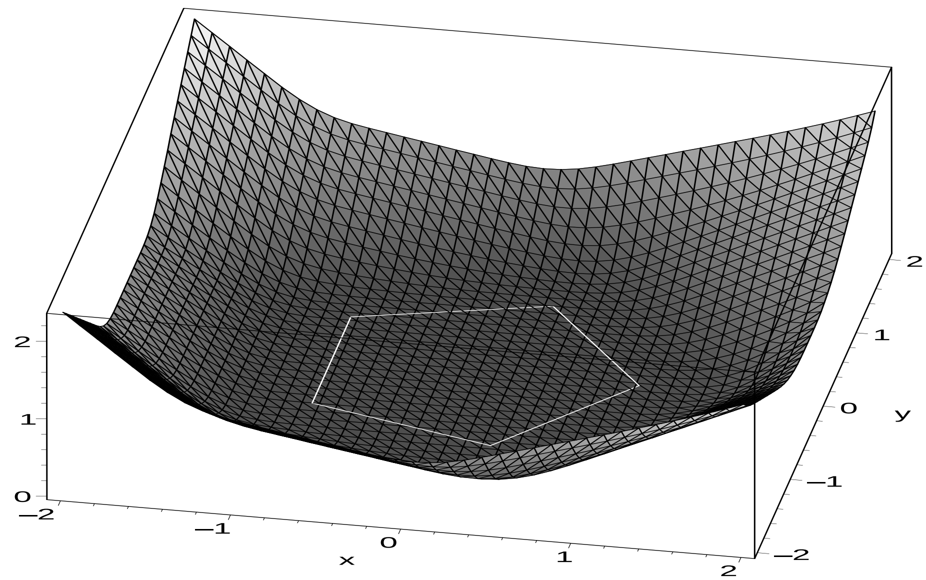


Figure 1: **The function associated with a pentagon**

set in a Banach space satisfies (1), for a corresponding function $f_C$, see also Proposition 4.1. This is illustrated for the regular pentagon in Figure 1. Moreover, the convex program

$$\inf_{x \in C} \langle x^*, x \rangle$$

will be approximated by a sequence of linear programs

$$\inf_{x \in C_N} \langle x^*, x \rangle$$

where $C_N$ is determined by the first $N$ half-spaces. Indeed, this is precisely the abstract form of the moment problem analyzed in [3].

Additionally, it follows, as in Corollary 5 of [1], that every closed constructible convex set is the *Mosco limit* of a quite explicit sequence of infinitely smooth convex sets: $C_n := f^{-1}([0, 2^{-n}])$ where $f$ is a $C^\infty$-smooth convex function. It is also interesting to consider convergence properties of $f_{C_N}$ where $\{C_N\}$ as before is the intersection of the first $N$ half-spaces.

Thus, we will explore when a convex set in a (non-separable) Banach space is constructible. In this direction, let us mention a recent interesting work of Granero *et. al.* [10] studies several types of uncountable "almost" biorthogonal systems that in turn are related to constructibility as will be discussed in the next section. Although we will recapture many of the results in [10] that characterize Banach spaces in which nonconstructible convex sets exist, there are several other results therein that are outside the focus of our paper, but are of interest in understanding the structure of Banach spaces in which nonconstructible convex sets exist.

The structure of our paper is as follows. In the second section our primary focus is on the 'construction' of nonconstructible sets, and characterizations of Banach spaces in which such sets exist. In particular,we shall establish that if any nonconstructible closed convex set exists in a Banach space, then the overwhelming majority of closed bounded convex sets in that space are not constructible: the collection of nonconstructible sets under the Hausdorff metric contains a dense open subset of the closed bounded convex sets. The third section's primary focus is to determine classes of constructible sets in certain nonseparable Banach spaces; for example, weakly compact convex sets are constructible in spaces whose duals are weak*-separable. The last section examines some set-convergence results related to constructibility.

Throughout this paper, basic Banach space concepts and definitions are taken from [7], and set-theoretic topological notions from [12] while notions concerning set-convergence can be found in [2]. The *distance* from a point $x$ to a set $A$ is defined by $d(x, A) := \inf\{\|x-a\| : a \in A\}$, and the *gap* between two sets $A$ and $B$ is defined by $d(A, B) := \inf\{\|a - b\| : a \in A,\ b \in B\}$. We define the *excess function* of a set $A$ over $B$ by $e(A, B) := \sup\{d(a, B) : a \in A\}$, and the *Hausdorff distance* between two sets $A$ and $B$ by $\rho_H(A, B) := \max\{e(A, B), e(B, A)\}$.

## 2 Nonconstructible Sets in Banach Spaces

Although the primary focus of this section will center on the characterization Banach spaces in which nonconstructible closed convex sets exist, we will begin with a couple of basic results relating to the constructibility of an individual convex set that will be useful throughout our paper. The first observation contains some basic facts concerning constructible sets, many of which we will use frequently without explicit mention.

**Fact 2.1.** *(a) Any translate of a constructible set is constructible.*

*(b) If $C$ is constructible, then $\lambda C$ is constructible for all $\lambda \neq 0$. However, there are examples where $C$ is constructible but $0\,C = \{0\}$ is not constructible.*

*(c) A nonempty intersection of countably many constructible sets is constructible.*

*(d) If $0 \in C$ and $C$ is constructible, then we can represent $C = \cap_{n=1}^{\infty} f_n^{-1}(-\infty, 1]$.*

*Proof.* Most of the proofs are straightforward exercises involving the definition. Let us point out for the 'however' portion of (b), we can use the fact that $\{0\}$ is not constructible in any Banach space whose dual is not weak*-separable (see Proposition 3.1 below). Hence in such a space, let $C$ be a closed half-space (which is clearly constructible) while $\{0\} = 0\,C$ is not constructible. $\square$

Suppose $C \subset X^*$. Let us say $C$ is *weak*-constructible* if it is the countable intersection of weak*-closed half-spaces, that is half-spaces determined by elements of $X$. The following observation, gives dual characterizations of constructible and of weak*-constructible sets.

**Proposition 2.2.** *(a) Let $C$ be a closed convex subset of $X$ containing the origin. Then $C$ is constructible if and only if its polar $C^o := \{\phi \in X^* : \phi(x) \leq 1 \text{ for all } x \in C\}$ is weak*-separable.*

*(b) Let $W$ be a weak*-closed convex set in $X^*$ containing the origin. Then $W$ is weak*-constructible if and only if its pre-polar $W^o = \{x \in X : \phi(x) \leq 1 \text{ for all } \phi \in W\}$ is norm separable in $X$.*

*Proof.* $\Rightarrow$: The proofs of (a) and (b) are similar, so we prove only (a). Since $0 \in C$ we can write the countable intersection as $C = \cap_{n=1}^{\infty} \phi_n^{-1}(\infty, 1]$. Now let $W = \overline{\text{conv}}^{w^*}(\{\phi_n\} \cup \{0\})$. Because $\phi_n(x) \leq 1$ for all $x \in C$, it follows that $\phi(x) \leq 1$ for all $x \in C$, and all $\phi \in W$; thus $W \subset C^o$. If $W \neq C^o$, then there exist $\phi \in C^o \setminus W$ and $x_0 \in X$ such that $\phi(x_0) > 1 > \sup_W x_0$ (we know this since $0 \in W$). Thus $\phi_n(x_0) < 1$ for all $n$ and so $x_0 \in C$. This with $\phi(x_0) > 1$ contradicts that $\phi \in C^o$. Consequently, $W = C^o$, and so $C^o$ is weak*-separable.

$\Leftarrow$: Let $C$ be a closed convex set containing the origin, and suppose that $C^o$ is weak*-separable. Choose a countable weak*-dense collection $\{\phi_n\}_{n=1}^{\infty} \subset C^o$. Clearly, $C \subset \cap_{n=1}^{\infty} \phi_n^{-1}(-\infty, 1]$. Moreover, if $x_0 \notin C$, then there is a $\phi \in X^*$ such that $\phi(x_0) > 1 > \sup_C \phi$. Then $\phi \in C^o$. The weak*-density of $\{\phi_n\}_{n=1}^{\infty}$ in $C^o$ implies there is a $\phi_n$ such that $\phi_n(x_0) > 1$. Thus $C = \cap_{n=1}^{\infty} \phi_n^{-1}(-\infty, 1]$ as desired. □

Now we give a simple criterion which we will use to build closed convex sets that are not constructible.

**Lemma 2.3.** *Let $C$ be a closed convex subset of a Banach space $X$. Suppose there is an uncountable sequence $\{x_i\}_{i \in I}$ such that $x_i \notin C$ for all $i \in I$, but $\frac{x_i + x_j}{2} \in C$ for all $i \neq j$. Then $C$ is not constructible.*

*Proof.* By translation we may assume $0 \in C$ and thus we may suppose $C = \cap_{\alpha} f_{\alpha}^{-1}(-\infty, 1]$. We will show that there are uncountably many $f_{\alpha}$'s in this representation. Indeed, for each $i$, we find $\alpha_i$ such that $f_{\alpha_i}(x_i) > 1$. Now $\frac{x_i + x_j}{2} \in C$ and so $f_{\alpha_i}(x_j) < 1$ for $i \neq j$. Therefore, if $i \neq j$, one has $f_{\alpha_i}(x_i) > 1$ and $f_{\alpha_j}(x_i) < 1$. This shows $f_{\alpha_i} \neq f_{\alpha_j}$ for $i \neq j$ and so there are necessarily uncountably many $f_{\alpha}$'s in this representation. □

The following proposition can be deduced by piecing together various results in [10], our goal here is to provide a self-contained—and we hope more instructive in the sense it pertains to constructibility—proof that does not rely on the results from [10].

**Theorem 2.4.** *Let $X$ be a Banach space, then the following are equivalent.*

*(a) There is an uncountable sequence $\{x_{\alpha}\} \subset X$ such that $x_{\alpha} \notin \overline{\text{conv}}(\{x_{\beta} : \beta \neq \alpha\})$.*

*(b) There is a bounded closed convex subset of $X$ that is not constructible.*

*(c) There is a closed convex subset in $X$ that is not constructible.*

*(d) There is a weak*-closed convex subset of $X^*$ that is not weak*-separable.*

*(e) There is a ball of an equivalent dual norm in $X^*$ that is not weak*-separable.*

*(f) There is an equivalent norm on $X$ whose unit ball is not constructible.*

*(g) There is a bounded uncountable system $\{x_\alpha, \phi_\alpha\} \subset X \times X^*$ such that $\phi_\alpha(x_\alpha) = 1$ and $|\phi_\alpha(x_\beta)| \leq a$ for some $a < 1$ and all $\alpha \neq \beta$.*

*Proof.* (a) $\Rightarrow$ (b): Suppose (a) holds, then for some $N > 0$ there are uncountably many $\{x_\alpha\}$ such that $\|x_\alpha\| < N$, so we may and do assume $\|x_\alpha\| < N$ for all $\alpha$. By the separation theorem, for each $\alpha$, we find $f_\alpha \in X^*$ and $\delta_\alpha > 0$ such that $f_\alpha(x_\alpha) > f_\alpha(x_\beta) + \delta_\alpha$ for all $\alpha \neq \beta$. Now let $a_\alpha = f_\alpha(x_\alpha)$ and let

$$C = \{x : f_\alpha(x) \leq a_\alpha - \delta_\alpha/2 \text{ for all } \alpha\} \cap NB_{X^*}.$$

Then $x_\alpha \notin C$ for all $\alpha$, however, for $\alpha \neq \beta$ we have $f_\mu(\frac{x_\alpha + x_\beta}{2}) \leq a_\mu - \delta_\mu/2$ and so $\frac{x_\alpha + x_\beta}{2} \in C$ for all $\alpha \neq \beta$. Therefore, $C$ is not constructible by Lemma 2.3.

(b) $\Rightarrow$ (c) is trivial, and (c) $\Rightarrow$ (d) follows from Proposition 2.2(a).

(d) $\Rightarrow$ (e): This follows from [10, Proposition 4.4]; we will present an argument based in part on some techniques therein. Let $W$ be a weak*-closed convex subset of $X^*$ that is not weak* separable, then $W \cap nB_{X^*}$ is not weak*-separable for some $n > 0$ (otherwise $W$ would be a countable union of weak*-separable sets). Thus, we assume without loss of generality that $W$ is bounded. Now let

$$m := \min\{e(W, C) : C = \overline{\text{conv}}^{w^*}(S),\ S \subset W \text{ is countable}\} \tag{2}$$

Then $m > 0$ or else $W$ would be weak*-separable. From this it follows that $W + \epsilon B_{X^*}$ is not weak*-separable where $0 < \epsilon < m$. Thus we may assume without loss of generality that $W$ is weak* compact convex and has nonempty norm interior.

Now we follow ideas of [10, Lemmas 4.2 and 4.3] to construct a dual ball in $X^*$ that is not weak*-separable. Fix $x_0 \in S_X$, then for some (rational) number $a \neq 0$ with $\inf_W x_0 < a < \sup_W x_0$, we have $x_0^{-1}(a) \cap W$ is not weak*-separable, otherwise $W$ would be weak* separable. Now let $K := x_0^{-1}(a) \cap W$, then $K$ is a weak* compact convex set, and so the symmetric convex set $B := \text{conv}(K \cup (-K))$ is also weak*-compact. Moreover B has nonempty norm interior because $x_0^{-1}(a) \cap W$ has nonempty norm interior relative to $x_0^{-1}(a)$. Finally, if $B$ were weak*-separable, we could find a countable collection $\{\lambda_n x_n - (1 - \lambda_n) y_n\}_{n=1}^\infty$ where $x_n, y_n \in K$ and $0 \leq \lambda_n \leq 1$ that is weak*-dense in $B$. Any net from this collection converging to $k \in K$ has $\lambda_{n_\alpha} \to 1$, and so it follows that $\{x_n\}_{n=1}^\infty$ is weak*-dense in $K$. This contradiction shows that $B$ is not weak*-separable, as desired.

(e) $\Rightarrow$ (f): This follows from Proposition 2.2(a).

(f) ⇒ (g): Suppose $B_X$ is a nonconstructible ball, then its polar, $B_{X^*}$ is not weak*-separable by Proposition 2.2(a). It follows from [10, Proposition 2.7] that $X$ has an uncountable sequence as in (g). However, here we will provide a shorter proof of this fact using different methods.

For $Y \subset X^*$, let $|x|_Y := \sup\{\phi(x) : \phi \in Y \cap B_{X^*}\}$, and define

$$(3) \qquad \lambda := \sup\{\alpha : \alpha\|x\| \leq |x|_Y \text{ where } Y \subset X^* \text{ is a separable subspace}\}.$$

Then $\lambda < 1$, or else for some separable subspace $Y$ we would have $|x|_Y \geq \|x\|$ and then $Y \cap B_{X^*}$ would be weak* dense in $B_{X^*}$—contradicting that $B_{X^*}$ is not weak* separable. Now, choose $l > 0$ so that $1 - l > \lambda$. For $Y \subset X^*$ separable, let $F_Y := \{x \in S_X : |\phi(x)| \leq 1 - l$ for all $\phi \in Y \cap B_{X^*}\}$. Now let

$$(4) \qquad \delta := \inf\{e(F_Y, Z) : Y \subset X^* \text{ and } Z \subset X \text{ are separable}\}.$$

We prove that $\delta > 0$: otherwise, choose $Y_n$ and $Z_n$ such that $e(F_{Y_n}, Z_n) \to 0$. Letting $Y = \overline{\text{span}}(\cup_n Y_n)$ and $Z = \overline{\text{span}}(\cup_n Z_n)$, we find that $e(F_Y, Z) = 0$. Then there is a countable set in $S \subset B_{X^*}$ such that $\sup\{\phi(z) : \phi \in S\} = \|z\|$ for all $z \in Z$. Thus for $\widetilde{Y} := \overline{\text{span}}(Y \cup S)$ we now have $(1 - l)\|x\| \leq |x|_{\tilde{Y}}$ for all $x \in X$; this is a contradiction because $1 - l > \lambda$. Hence $\delta > 0$.

Let $\eta > 0$ be such that $\eta < \min\{l, \delta\}$. We will find an uncountable system in $\{x_\alpha, \phi_\alpha\} \subset B_X \times B_{X^*}$ such that $|\phi_\alpha(x_\alpha)| \geq 1 - \eta + \eta^2$ for all $1 \leq \alpha < \omega_1$ while $|\phi_\alpha(x_\beta)| \leq 1 - \eta$ for all $\alpha \neq \beta$. Indeed, fix $x_1 \in B_X$ and $\phi_1 \in B_{X^*}$ such that $\phi_1(x_1) = 1$. Suppose for an ordinal $1 < \mu < \omega_1$ that $x_\alpha, \phi_\alpha$ have been chosen as prescribed for all $\alpha < \mu$. We denote

$$(5) \quad F_\mu := \{x \in S_X : |\phi_\alpha(x)| \leq 1 - l \text{ for all } \alpha < \mu\} \text{ and } X_\mu := \overline{\text{span}}(\{x_\alpha : \alpha < \beta\}).$$

Because $\eta < \delta$ as defined in (4), we can choose $x_\mu \in F_\mu$ such that $d(x_\mu, X_\mu) > \eta$. Now select $x^*_{\mu,1} \in S_{X^*}$ such that $x^*_{\mu,1}(x_\mu) = 1$ and choose $x^*_{\mu,2} \in S_{X^*}$ such that $x^*_{\mu,2}(x_\mu) > \eta$ while $x^*_{\mu,2}(X_\mu) = 0$. Let $\phi_\mu = (1-\eta)x^*_{\mu,1} + \eta x^*_{\mu,2}$. Then: $\phi_\mu(x_\mu) \geq 1 - \eta + \eta^2$; $|\phi_\mu(x_\alpha)| \leq 1 - \delta \leq 1 - \eta$ for all $\alpha < \mu$ because $x_\alpha \in X_\mu$; and, $\phi_\alpha(x_\mu) \leq 1 - l < 1 - \eta$ for $\alpha < \mu$ because $x_\mu \in F_\mu$. By transfinite induction, we construct a sequence as we claimed. Scaling the $\phi_\alpha$'s so that $\phi_\alpha(x_\alpha) = 1$ produces a system as in (g) where $a = (1 - \eta)/(1 - \eta + \eta^2)$.

(g) ⇒ (a): this is an immediate consequence of the separation theorem. □

There are several other conditions equivalent to those listed in Theorem 2.4 that can be found in [10], we now highlight a few such conditions. A bounded family $\{x_\alpha : 1 \leq \alpha < \omega_1\}$ such that $x_\alpha \notin \overline{\text{conv}}(\{x_\beta : \alpha < \beta < \omega_1\})$ is called a *convex right-separated $\omega_1$-family* in $X$. A Banach space has *property $HL(1)$* if for every family of open half-spaces $\{U_i\}_{i \in I}$ of $X$ there exists a countable subfamily $\{U_{i_n}\}_{n \geq 1}$ such that $\cup_{n \geq 1} U_{i_n} = \cup_{i \in I} U_i$.

Clearly, if a Banach space $X$ has property $HL(1)$, then every closed convex subset of $X$ is constructible. The converse, however, while true as noted in the following remark is far more subtle (see also Example 3.6 below).

**Remark 2.5.** *([10]) For a Banach space $X$, the following are equivalent:*

*(a) Any of the equivalent conditions in Theorem 2.4*

*(b) $X$ has a convex right-separated $\omega_1$-family.*

*(c) $X$ does not have property $HL(1)$.*

*(d) There is a convex subset of $X^*$ that is not weak*-separable.*

*Proof.* The equivalence of (b), (c) and (d) is readily established in [10, Proposition 6.2]. Moreover, is clear that Theorem 2.4(a) implies the existence of a convex right-separated $\omega_1$-family. However, the converse appears to be much more subtle and decidedly nontrivial— see [10, Proposition 7.3] and also the introduction to Sections 4 and 6 in [10] where the terminology used in their Proposition 7.3 is introduced and discussed. □

Again, we refer the reader to [10] for an interesting comprehensive study on subtle issues related to "almost" biorthogonal systems in Banach spaces. Our present goal, however, is to provide some refinements to add to the list of equivalent conditions listed in Theorem 2.4 and Remark 2.5; the proof below also provides an explicit description of a nonconstructible ball.

**Theorem 2.6.** *For a Banach space $X$, the following are equivalent.*

*(a) There is an uncountable sequence $\{x_i\}$ such that $x_i \notin \overline{\text{conv}}(\{x_j : j \neq i\})$.*

*(b) Each unit ball of an equivalent norm on $X$ is the Hausdorff metric limit of a sequence of unit balls that are not constructible.*

*(c) Each bounded closed convex set is a the limit in the Hausdorff metric of closed convex sets that are not constructible.*

*Proof.* (a) $\Rightarrow$ (b): Suppose $X$ has an uncountable sequence as in (a). By Theorem 2.4 (or more directly by [10, Proposition 2.2]) there is a bounded family $\{x_i, f_i\}_{1\leq i<\omega_1} \subset X \times X^*$ and $0 \leq \eta < 1$ such that

$$f_i(x_i) = 1 \quad \text{and} \quad |f_i(x_j)| \leq \eta \ \text{ for } i \neq j.$$

Let $B_X$ be the unit ball of an arbitrary equivalent norm $\|\cdot\|$ on $X$, and let $\epsilon > 0$. Choose $\delta > 0$ such that $1 - 4\delta > \eta$. By normalizing we may assume $\|f_i\| = 1$ for all $i$ and we choose $N > 0$ such that $\|x_i\|/N < \epsilon$ for all $i$ (in particular, $1/N < \epsilon$). Now let $B$ be a ball of the equivalent norm be defined by

$$B := (1+\epsilon)B_X \cap \{x : |f_i(x)| \leq 1 + \frac{1-2\delta}{N}\}.$$

Then $B_X \subset B \subset (1+\epsilon)B_X$ so it suffices to show that the ball $B$ is not constructible.

For this, we choose $y_i \in B_X$ such that $f_i(y_i) > 1 - \frac{\delta}{N}$, and we let $z_i = y_i + \frac{x_I}{N}$. Then $\|z_i\| < 1 + \epsilon$, however $f_i(z_i) > 1 - \frac{\delta}{N} + \frac{1}{N}$ and so $z_i \notin B$ for all $i$. On the other hand $\|\frac{z_i+z_j}{2}\| < 1+\epsilon$ for all $i, j$ and for $i \neq j$, and all $k$, $|f_k(z_i + z_j)| \leq 1 + \frac{\delta}{N} + 1 + \frac{1-4\delta}{N}$ since at least one of $i \neq k$ or $j \neq k$. Thus $\frac{z_i+z_j}{2} \in B$ for all $i \neq j$. According to Lemma 2.3, the ball $B$ is not constructible.

Now (b) $\Rightarrow$ (a) follows from Theorem 2.4. The proof of (a) $\Rightarrow$ (c) this is similar to that of (a) $\Rightarrow$ (b), while (c) $\Rightarrow$ (a) follows from Theorem 2.4. □

Before refining (b) and (c) above, let us observe that (c) is not necessarily true for unbounded convex sets. Indeed, consider a half-space, say $H := \{x : \phi(x) \geq 0\}$ where $\|\phi\| = 1$. Then any closed convex set $C$ such that $\rho_H(C, H) < 1$ can be written as $C := \{x : \phi(x) \geq a\}$ where $|a| < 1$. So $H$ is not a limit in the Hausdorff metric of nonconstructible sets. In our refinements of (b) and (c) we will use the following lemma.

**Lemma 2.7.** *If $C$ is a closed convex set with nonempty interior and $\rho_H(C_n, C) \to 0$, then $C$ is constructible if each $C_n$ is constructible.*

*Proof.* Suppose $C_n$ is constructible for each $n$ and $\rho_H(C_n, C) \to 0$. By translation, we may assume $B_{2\delta} \subset C$ for some $\delta > 0$. It follows from the Hausdorff convergence that $B_\delta \subset C_n$ for all $n$ sufficiently large. By passing to this tail, we may assume $B_\delta \subset C_n$ for all $n$. Now write $C_n = \cap_k f_{n,k}^{-1}(-\infty, a_{n,k}]$ where $\|f_{n,k}\| = 1$. Then $a_{n,k} \geq \delta$ for all $n, k$. Let $\epsilon_n := \rho_H(C_n, C)$, because $\|f_{n,k}\| = 1$ for all $n, k$, it follows that $C \subset \cap_{n,k} f_{n,k}^{-1}(-\infty, a_{n,k} + \epsilon_n]$. Now suppose $x_0 \notin C$. Then for some $n_1 > 0$ there is a $0 < \lambda < 1$ such that $\lambda x_0 \notin C_n$ for $n > n_1$. Choose $n_0 > n_1$ so that $\epsilon_{n_0} < (1-\lambda)\delta$. Now for some $k_0$, $f_{n_0,k_0}(\lambda x_0) > a_{n_0,k_0} \geq \delta$. Therefore $f_{n_0,k_0}(x_0) > a_{n_0,k_0} + (1-\lambda)\delta > a_{n_0,k_0} + \epsilon_{n_0}$. Thus $C = \cap_{n,k} f_{n,k}^{-1}(-\infty, a_{n,k} + \epsilon_n]$ as desired. □

**Corollary 2.8.** *(a) Let $(\mathcal{B}, \rho_H)$ be the collection of unit balls of equivalent norms on $X$ endowed with the Hausdorff metric. Let $\mathcal{N}$ be the collection of nonconstructible unit balls. Then either $\mathcal{N}$ is empty, or it is a dense open set in $\mathcal{B}$.*

*(b) Let $(\mathcal{C}, \rho_H)$ be the collection of closed and bounded convex subsets of a Banach space $X$ endowed with the Hausdorff metric. Let $\mathcal{NC}$ be the subcollection of nonconstructible sets in $\mathcal{C}$. Then either $\mathcal{NC}$ is empty, or it contains a dense open subset of $\mathcal{C}$.*

*Proof.* (a) If $\mathcal{N}$ is not empty, then it is dense in $\mathcal{B}$ by Theorem 2.6. Now Lemma 2.7 shows that $\mathcal{N}^c$ is closed.

(b) Suppose $\mathcal{NC}$ is not empty. Let $C \in \mathcal{NC}$ and let $\epsilon > 0$. We shall show there is a set $C_2$ in the interior of $\mathcal{NC}$ such that $\rho_H(C, C_2) < \epsilon$. Indeed, let $C_1 = C + \frac{\epsilon}{2} B_X$. According to Theorem 2.6 there is a nonconstructible set, say $C_2$, such that if $\rho_H(C_2, C_1) < \frac{\epsilon}{4}$ then $C_2$ has nonempty interior, and according to Lemma 2.7, $C_2$ is in the interior of $\mathcal{NC}$. □

**Remark 2.9.** *Under additional set-theoretic axioms, there are nonseparable Banach spaces in which all closed convex sets are constructible. These are known to include: (i) the $C(K)$ space* *of Kunen constructed under the Continuum Hypothesis (CH) [12], and (ii) the space of Shelah constructed under the diamond principle [13], neither of which satisfy Theorem 2.6 (a).*

The condition (a) in Theorem 2.6 is formally weaker than $X$ having an uncountable biorthogonal system. In contrast to the previous remark there are general conditions under which nonseparable spaces are known to have uncountable biorthogonal systems.

**Remark 2.10.** *Suppose $X$ is a nonseparable Banach space such that*

*(i) $X$ is a dual space, or*

*(ii) $X = C(K)$, for $K$ compact Hausdorff, and one assumes Martin's axiom along with the negation of the Continuum Hypothesis.*

*Then $X$ admits an uncountable biorthogonal system.*

*Proof.* (i) If $X$ is nonseparable dual space, then $X$ admits an uncountable biorthogonal system ([6], [14]).

(ii) A nonseparable $C(K)$ space always admits an uncountable biorthogonal system under Martin's axiom and the negation of the continuum hypothesis; this is a deep recent result of S. Todorcevic (see for example, [9, p. 5]). □

Thus, the answer to *'does a continuous function space always admit an uncountable biorthogonal system?'* really does depend on the model theoretic extension of Zermelo-Fraenkel set theory with the Axiom of Choice.

In a related light, consider the question

> Does every non-separable $C(K)$ contain a closed convex set entirely composed of support points (the tangent cone is never linear)?

In [4] it is shown that this is equivalent to $C(K)$ admitting an uncountable *semi-biorthogonal system*, and Kunen's space is an example where this happens without there being an uncountable biorthogonal system assuming the Continuum Hypotheses. Thus, the answer is 'yes' except perhaps when Martin's Axiom fails (along with CH).

Let us also note that the case where every norm closed convex set in a dual space is constructible is completely determined with the help of Remark 2.10.

**Corollary 2.11.** *A dual space $X^*$ is separable if and only if every norm-closed convex set in $X^*$ is constructible.*

*Proof.* If $X^*$ is separable, then every closed convex subset of $X^*$ can be written as a countable intersection of half-spaces by [1].

Conversely, if $X^*$ is not separable, there is an uncountable biorthogonal system in $X^*$ (as cited in Remark 2.10). Thus Theorem 2.4 ensures that there is a closed convex set in $X^*$ that cannot be represented as a countable intersection of half-spaces. □

Recall that a space $(X, \|\cdot\|)$ has the *Mazur intersection property* if every closed convex set is the countable intersection of norm balls, [11]. One may wonder how nonconstructible sets relate to the Mazur intersection property. We close this section with an observation along those lines.

**Remark 2.12.** *(a) If $X$ is non-separable, and either $X$ has the Mazur intersection property or $X^*$ has the weak-star Mazur intersection property, then $X$ has a nonconstructible closed convex set.*

*(b) If $B_X$ is the unit ball of a nonseparable space $X$ with the Mazur intersection property, then $B_X$ is not constructible*

*Proof.* (a) Under the assumption in (a), [11, Proposition 4.5] shows that is an uncountable system as in Theorem 2.4(a).

(b). This follows because if $B_X$ were constructible, the $B_{X^*}$ would be weak*-separable by Proposition 2.2(a), and then $X^*$ would contain a separable 1-norming (hence proper) subspace. Consequently, $X$ would not have the Mazur intersection property (cf. [11, p. 501]) □

# 3 Weak*-separability and Constructible Sets

The previous section showed that there is an abundance of nonconstructible closed convex sets in Banach spaces provided there is one such set. In this section, we will examine the flip side of that result. That is, we will look more closely at when certain convex sets, such as weakly compact or weak*-compact, are constructible. Although the proof of the first result is rather simple, a somewhat unexpected consequence is that every weakly compact convex set in a Banach space $X$ is constructible if and only if there exists at least one closed bounded constructible convex set in $X$.

**Proposition 3.1.** *For a Banach space $X$, the following are equivalent.*

*(a) $X^*$ is weak*-separable.*

*(b) Every weakly compact convex set in $X$ is constructible.*

*(c) There exists a closed, bounded, convex constructible subset of $X$.*

*Proof.* (a) ⇒ (b): Let $Y$ be a separable total subspace in $X^*$ and let $\tau$ be the Hausdorff topology on $X$ induced by pointwise convergence on $Y$, and fix a countable dense collection $\{\phi_n\}_{n=1}^{\infty} \subset Y$. Now $\tau$ is a topology weaker than the weak topology so if $C$ is weakly compact, then it is $\tau$-compact since $\tau$ is a weaker Hausdorff topology. Thus, $C$ is $\tau$-closed, and by translation we may assume $0 \in C$. By the separation theorem, given any $x_0 \notin C$, choose $\phi_{x_0} \in Y$ such that $\sup_C \phi_{x_0} < 1 < \langle \phi_{x_0}, x_0 \rangle$. Since $C$ is bounded, there exists a $\phi_{n_{x_0}}$ such that $\sup_C \phi_{n_{x_0}} < 1 < \langle \phi_{n_{x_0}}, x_0 \rangle$. Hence, $C = \cap_{x_0 \notin C} \phi_{n_{x_0}}^{-1}(-\infty, 1]$, which is a countable intersection since there are only countably many $\phi_n$'s.

(b) ⇒ (c) is trivial, so we prove (c) ⇒ (a): Let $C$ be a closed, bounded, convex constructible set. We may assume $0 \in C$. By Fact 2.1, we know that $\frac{1}{n}C$ is constructible for each $n$. Consequently $\{0\} = \cap_{n=1}^{\infty} \frac{1}{n} C$ is constructible. Then we write $\{0\} = \cap \phi_n^{-1}(-\infty, a_n]$, and so $\{0\} = \cap_{n=1}^{\infty} \phi_n^{-1}(0)$. Therefore, $\{\phi_n\}_{n=1}^{\infty}$ separates points in $X$ which implies $X^*$ is weak*-separable. □

The following example demonstrates that (b) of the preceding proposition cannot be strengthened to include all weakly (i.e. norm) closed convex sets, nor can (b) be strengthened to include all weak*-compact convex sets in the event $X$ is a dual space. That said, a partial redress in the dual situation is given in Proposition 3.5 below.

**Example 3.2.** *(a) Let $X$ be a nonseparable dual space whose predual is separable (e.g. $X = \ell_\infty$). Then $X^*$ is weak*-separable, but $X$ admits nonconstructible closed bounded convex sets.*

*(b) Consider $\ell_1(\Gamma)$ as the dual to $c_0(\Gamma)$ where $\Gamma = 2^{\aleph_0}$. Then $\ell_1(\Gamma)^*$ is weak*-separable, however there is a weak*-compact ball in $\ell_1(\Gamma)$ that is not constructible.*

*Proof.* (a) Now $X^*$ is weak* separable as the second dual of a separable space. Also, Corollary 2.11 shows $X$ admits nonconstructible closed convex sets (indeed many by Theorem 2.6).

(b) First, $\ell_1(\Gamma) \subset \ell_\infty$, see e.g. [5, p. 211]. Therefore, there is a countable total set of functionals over $\ell_1(\Gamma)$ and so $\ell_1(\Gamma)^*$ is weak*-separable. However, consider $\{e_\gamma\}_{\gamma\in\Gamma}$ and $\{e_\gamma^*\}_{\gamma\in\Gamma}$ the usual bases of $c_0(\Gamma)$ and $\ell_1(\Gamma)$. Let $B_1$ be the ball of the usual norm on $\ell_1(\Gamma)$ and consider the weak*-compact ball $B = \{x : x \in B_1 \text{ and } e_\gamma(x) \le 1/2 \text{ for all } \gamma \in \Gamma\}$. Then $e_\gamma^* \notin B$ for all $\gamma$, while $(e_{\gamma_1} + e_{\gamma_2})/2 \in B$ for all $\gamma_1 \neq \gamma_2$. According to Lemma 2.3, $B$ is not constructible. With a little thought, using the technique as in Theorem 2.6, one can see that every dual ball on $\ell_1(\Gamma)$ is a Hausdorff metric limit of dual balls that are not constructible. □

Further comparison with Proposition 3.1 will show that in Banach spaces with a lot of structure, the existence of one bounded constructible convex set implies that all closed convex sets are constructible. By a 'lot of structure' we mean that the dual unit ball is Corson compact which is defined as follows. Given an uncountable set $\Gamma$, we set $\Sigma(\Gamma) := \{x = (x_\gamma) \in \mathbb{R}^\Gamma : \operatorname{supp} x \ \text{ is at most countable}\}$, where $\operatorname{supp} x = \{\gamma \in \Gamma : x_\gamma \neq 0\}$ and we equip $\Sigma(\Gamma)$ with the pointwise topology. Then a compact set is *Corson compact* if it can be identified, up to a homeomorphism, as a subset of $\Sigma(\Gamma)$, [7].

**Proposition 3.3.** *Suppose $(B_{X^*}, w^*)$ is Corson compact. Then the following are equivalent.*

*(a) $X$ is separable.*

*(b) Every closed convex subset in $X$ is constructible.*

*(c) There is at least one linearly bounded constructible convex set in $X$.*

*Proof.* (a) $\Rightarrow$ (b) follows from [1] (or Proposition 3.1) while (b) $\Rightarrow$ (c) is trivial.

(c) $\Rightarrow$ (a): We proceed by contraposition. Suppose (a) is not true. The Corson hypothesis implies that there is an M-basis ([7]) $\{x_i, f_i\}_{i \in I}$ on $X$, in which $I$ is uncountable and such that $\{i : f(x_i) \neq 0\}$ is countable for each $f \in X^*$. Suppose $C$ is a closed convex set that is a countable intersection of half-spaces. By translation, we may and do assume that $0 \in C$.

Now write $C := \cap_{n=1}^{\infty} \phi_n^{-1}(-\infty, a_n]$. Because $0 \in C$, we know that $a_n \geq 0$ for each $n$, and so $\phi_n^{-1}(0) \subset C$ for each $n$. Now let $I_n := \{i : \phi_n(x_i) \neq 0 \text{ for each } n\}$. Then $I_n$ is countable for each $n$ (again using the Corson property) and so $A = I \setminus (\cup_{n=1}^{\infty} I_n)$ is an uncountable set. Now the closed linear span of $\{x_i : i \in A\}$ is a nonseparable subspace that is contained in $C$. As $C$ was a translate of our original set, we know that each closed convex set that is a countable intersection of half-spaces contains a nonseparable affine space, and hence is certainly not linearly bounded. Thus (c) is not true. $\square$

The dual ball of each weakly compactly generated (WCG) space is Corson compact in the weak* topology, [7], so the above result shows, in particular, that nonseparable reflexive Banach spaces do not have linearly bounded constructible convex sets. It is tempting to ask whether the preceding result can be strengthened to include all spaces with M-bases. The answer is, in fact, 'no' in a strong sense as shown in the following example.

**Example 3.4.** *Let $X = \ell_1(\Gamma)$ with $\Gamma = 2^{\aleph_0}$. Then $X$ has a norming M-basis, nevertheless, every weakly compact convex subset of $X$ is constructible.*

*Proof.* Now $X = \ell_1(\Gamma)$ has a norming M-basis (namely its usual basis with dual coordinate functionals in $c_0(\Gamma)$ ) and hence a projectional resolution of the identity; see [7]. Nevertheless, as in Example 3.2, $X^*$ is weak* separable. Accordingly, every weakly compact subset of $X$ is constructible by Proposition 3.1. $\square$

The following result determines when every weak*-closed convex subset of a dual space is weak*-constructible.

**Proposition 3.5.** *For a Banach space $X$, the following are equivalent.*

*(a) $X$ is separable*

*(b) Every weak*-closed convex subset of $X^*$ is weak*-constructible.*

*(c) There is an equivalent dual norm on $X^*$ whose unit ball is weak*-constructible.*

*Proof.* (a) $\Rightarrow$ (b): According to Proposition 2.2(b), every weak*-closed convex set containing the origin is weak*-constructible, hence by translation, every weak*-closed convex set is weak*-constructible.

(b) $\Rightarrow$ (c) is trivial, while (c) $\Rightarrow$ (a) follows because Proposition 2.2(b) ensures that the unit ball of the pre-dual norm on $X$ is separable. $\square$

In particular, this proposition shows that in a dual to a separable space, every weak*-closed convex set is constructible. This may be about as strong a general result as we can hope for. Indeed, Examples 3.2(b) and 3.4 show that $\ell_1(2^{\aleph_0})$ is a nonseparable dual space with weak*-separable dual in which some bounded constructible sets exist and yet not every weak*-compact ball is constructible.

Finally, let us observe that $\ell_1(\Gamma)$ provides yet another example that sheds light on properties of constructibility.

**Example 3.6.** *There is a Banach space $X$ with an uncountable decreasing sequence of closed convex sets $\{C_\alpha\}_{1\leq\alpha<\omega_1}$ such that $C_\alpha \subset C_\beta$ for $\alpha > \beta$, $C := \cap_{1\leq\alpha<\omega_1} C_\alpha$ is constructible, but $C \neq \cap_{n=1}^{\infty} C_{\alpha_n}$ for any countable subcollection of $\alpha_n$'s.*

*Proof.* Let $X = \ell_1(\Gamma)$ where $\Gamma = [0, \omega_1)$. Then $C := \{0\}$ is a constructible subset of $\ell_1(\Gamma)$ as in Example 3.4. Now let $C_\alpha = \{x : e_\gamma(x) = 0, \gamma \leq \alpha\}$. It is easy to check that these sets have the claimed properties. $\square$

In particular, the previous example shows that isolating the property $HL(1)$ to the complement of a single closed convex set $C$ is strictly stronger than the constructibility $C$. That is we can have an uncountable cover of $\ell_1([0,\omega_1)) \setminus \{0\}$ by open half-spaces with no countable subcover, and yet $\{0\}$ is constructible.

# 4 Set Convergence and Constructibility

We begin by recalling two fundamental notions of set convergence. Let $\{C_n\}_{n\geq1}$ be a sequence of closed convex subset of a Banach space $X$, and let $C$ be a closed convex subset of $X$. The sequence $\{C_n\}_{n\geq1}$ is said to converge *slice* to the closed convex set $C$ if $d(W, C) = \lim_{n\to\infty} d(W, C_n)$ for each closed bounded convex set $W \subset X$. The sequence $\{C_n\}_{n\geq1}$ converges *Mosco* to $C$ if the following two conditions are met:

(i) for each $x \in C$, there exist $x_n \in C_n$ such that $\|x_n - x\| \to 0$.

(ii) $x_{n_k} \in C_{n_k}$ and $x_{n_k} \to_{\text{weakly}} x$, imply $x \in C$.

In general, slice convergence implies Mosco convergence, and the two forms of convergence coincide in reflexive spaces; see [2, Chapter 5].

The following results from [1] illustrate the relevance of constructibility to set convergence.

**Proposition 4.1.** *(a) ([1, Theorem 1]) Suppose $C$ is a constructible set, then there is a $C^\infty$-smooth convex function $f_C : X \to [0,\infty)$ such that $C = f_C^{-1}(0)$.*

*(b) ([1, Corollary 5]) Suppose $f : X \to [0,\infty)$ is a $C^\infty$-smooth convex function. Then $C_n := \{x : f(x) \leq \frac{1}{n}\}$ are $C^\infty$-smooth convex bodies, and $\{C_n\}_{n\geq 1}$ converges Mosco to $C := f^{-1}(0)$. In particular, every constructible set is a Mosco limit of $C^\infty$-smooth convex bodies.*

For the reader's convenience we outline the basic construction of $f_C$ in (a). First represent $C = \cap_{n=1}^\infty \phi_n^{-1} x_n(-\infty, \alpha_n]$ where $\|\phi_n\| = 1$ for each $n$, then choose $\theta : \mathbb{R} \to [0,\infty)$ to be an appropriate $C^\infty$-smooth convex function such that $\theta(t) = 0$ for all $t \leq 0$ and $\theta(t) = t + b$ for all $t > 1$ and $-1 < b < 0$. The $C^\infty$-smooth convex function $f_C$ is defined by

$$f_C(x) := \sum_{n=1}^{\infty} \frac{\theta(\phi_n(x) - \alpha_n)}{(1 + |\alpha_n|)2^n}. \tag{6}$$

Also, let us note that the sets $C_n$ in (b) are infinitely smooth convex bodies as a consequence of the implicit function theorem (see [1]).

The next example illustrates that the Mosco convergence in (b) of the previous proposition doesn't necessarily translate into slice convergence in the nonreflexive setting.

**Example 4.2.** *Let $X$ be a separable nonreflexive Banach space, then there is a $C^\infty$-smooth convex function $f_C : X \to [0,\infty)$ as defined in (6) such that $C_n := \{x : f_C(x) \leq 1/n\}$ does not converge slice to $C := \{x : f_C(x) = 0\}$.*

*Proof.* The proof, which we outline, follows the idea of [3, Theorem 1]. Let $\{\phi_n\}_{n=1}^\infty$ be a collection in $B_{X^*}$ that is total but not norm dense in $B_{X^*}$, and such that $\{0\} = \cap_{n=1}^\infty \phi_n^{-1}(-\infty, 0]$. Now let $f(x)$ be defined as in (6). Then $C = \{0\}$ and $\{C_n\}_{n\geq 1}$ converges Mosco to $C$. However, if $x \in B_X$ is such that $\phi_k(x) = 0$ for $k = 1, \ldots, n$, then $f(x) \leq \sum_{k=n+1}^\infty 2^{-k} < \frac{1}{n}$. Thus $C_n$ contains $F_n := \{x \in B_X : \phi_k(x) = 0,\ k = 1, 2, \ldots n\}$ and so the proof of [3, Theorem 1] shows that $\{C_n\}_{n\geq 1}$ does not converge slice to $\{0\}$. Indeed, choose $\phi \in S_{X^*} \setminus Y$ where $Y = \overline{\text{span}}(\{\phi_n\})$. Choose $F \in S_{X^{**}}$ such that $F(Y) = \{0\}$ and $F(\phi) > 0$. Let $\delta$ be such that $0 < \delta < F(\phi)$. Now let $W = \{x \in B_X : \phi(x) > \delta\}$, and apply Goldstine's theorem [7] to show that $d(W, F_n) = 0$ for all $n$, while clearly $d(W, C) > \delta$. Thus $\{C_n\}_{n\geq 1}$ does not converge slice to $C$. ☐

In [3] it is shown that in non-reflexive spaces monotone decreasing sequences of closed convex sets are prone to converge Mosco but not slice (or indeed in the Wisjman sense).

We now examine some stability properties of constructibility under set convergence. First we need a lemma that may have some interest in its own right.

**Lemma 4.3.** *Suppose $Z$ is a closed subspace of $X$, and suppose there is a subset of $Z$ with nonempty interior in $Z$, that is constructible as a subset of $X$. Then $Z$ is itself constructible.*

*Proof.* Let $C$ be a constructible subset of $X$ such that $C$ has nonempty interior relative to $Z$. By translation, we may assume $\delta B_Z \subset C$ for some $\delta > 0$. Now suppose $Z$ is not constructible, then $Z^o = Z^\perp$ is not weak*-separable by Proposition 2.2(a). Also, $Z^\perp \subset C^o \subset Z^\perp + \frac{1}{\delta} B_{X^*}$. It follows from Proposition 2.2(a) that there is a countable collection $\{f_n + g_n\}_{n=1}^\infty$ where $f_n \in Z^\perp$ and $\|g_n\| \leq \frac{1}{\delta}$ whose weak* closure contains $Z^\perp$. However, $\overline{\text{span}}^{w^*}(\{f_n\}_{n=1}^\infty) \neq Z^\perp$ and because $Z^\perp$ is a linear space, there is a $z^\perp \in Z^\perp$ such that $d(z^\perp, \overline{\text{span}}^{w^*}(\{f_n\}_{n=1}^\infty)) > \frac{1}{\delta}$ which contradicts that $Z^\perp$ is in the weak*-closure of $\{f_n + g_n\}_{n=1}^\infty$. □

As is standard, in what follows we use $c_0$ to denote $c_0(\mathbb{N})$ and $\ell_\infty$ to denote $\ell_\infty(\mathbb{N})$.

**Example 4.4.** *The sequence space $c_0$ considered as a subspace of $\ell_\infty$ is not constructible. Consequently, no bounded set with nonempty interior relative to $c_0$ is constructible as a subset of $\ell_\infty$. In particular the unit ball of $c_0$ is not constructible when viewed as a subset of $\ell_\infty$.*

*Proof.* Now $c_0^o = c_0^\perp$ which is isomorphic to $(\ell_\infty/c_0)^*$ is not weak*-separable. According to Proposition 2.2(a), $c_0$ is not constructible in $\ell_\infty$; consequently neither are any of its closed convex subsets with nonempty interior by Lemma 4.3. □

Lemma 4.3 gives a convenient way to improve the stability result in Lemma 2.7 for Hausdorff convergence.

**Proposition 4.5.** *Suppose $Z$ is a subspace of $X$ and $C$ has nonempty interior relative to $Z$. Suppose there is a sequence of constructible sets $\{C_n\}_{n \geq 1}$ such that $C_n \subset Z$ and $\rho_H(C_n, C) \to 0$. Then $C$ is constructible.*

*Proof.* The Hausdorff convergence implies that some $C_n$ has interior relative to $Z$, and therefore $Z$ is constructible by Lemma 4.3. Now Lemma 2.7 implies that $C$ is constructible relative to $Z$, that is there are $f_n \in Z^*$ such that $C = \cap f_n^{-1}(-\infty, a_n])$. Now extend these $f_n$ to functionals on $X$, say $\tilde{f}_n$. Then $C = \cap \tilde{f}_n^{-1}(-\infty, a_n] \cap Z$, so as the intersection of constructible sets, $C$ is constructible. □

With a little more machinery, namely that *Attouch-Wets* convergence is preserved under polarity [2, Corollary 7.2.12] one can show that the preceding proposition holds with Hausdorff convergence replaced by Attouch-Wets convergence. Indeed, after translation, $C_n$ (eventually) and $C$ contain some open ball, it follows that their polars are (eventually) uniformly bounded, and hence converge Hausdorff. Then, using Proposition 2.2(a) one need only verify that the Hausdorff limit of weak*-separable sets is weak*-separable. However, the previous stability result does not extend to slice convergence as is observed in the following example.

**Example 4.6.** *Consider* $C = B_{c_0}$ *and* $C_n = \{x \in C : x(i) = 0$ *for all* $i > n\}$ *as subsets of* $\ell_\infty$. *Then each* $C_n$ *is constructible,* $C_n$ *converges slice to* $C$*, but* $C$ *is not constructible.*

*Proof.* As compact subsets of $\ell_\infty$, the sets $C_n$ are constructible by Proposition 3.1, but $C$ is not constructible according to Example 4.4. Now let $W$ be any closed bounded convex subset of $\ell_\infty$. Then $d(W, C_n) \geq d(W, C)$. Let $\epsilon > 0$, and let $x_0 \in C$ be such that $d(W, C) > d(W, x_0) - \frac{\epsilon}{2}$. For some $n_0 > 0$, $d(x_0, C_n) < \frac{\epsilon}{2}$ for all $n > n_0$. Thus $d(W, C_n) < d(W, C) + \epsilon$ for all $n > n_0$. Thus $C_n$ converges slice to $C$. □

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