# Brouwer's Fixed Point Theorem: Methods of Proof and Generalizations

by

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# Abstract

The familiar Brouwer fixed point theorem says that any continuous self-map $f$ on a compact convex subset $X$ of finite dimensional Euclidean space $\mathbf{E}$ must leave at least one point fixed. This result is easy to state, but notoriously complicated to prove. We will give a sample of the various methods of proof available, ranging from the degree-theoretical methods used by Brouwer in the early 20th century, up to a recent proof based on an alternate change of variables formula for multiple integrals. We will also explore extensions of the theorem based on generalizations the space $\mathbf{E}$, the set $X$, and the function $f$.

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# Chapter 1

# Background and Preliminaries

## 1.1 The Topological Fixed Point Property

A fixed point of a function $f : X \to X$ is an element $x \in X$ that satisfies $f(x) = x$. Given a set $X$, it is possible to ask what types of functions on $X$ have a fixed point. Alternatively, we could consider a class of functions, and investigate the kinds of sets on which a function in our class will have a fixed point. It is the latter course of investigation that is the main subject here. A set $X$ is said to have the **topological fixed point property** (tfpp) provided every continuous self–map on $X$ has a fixed point. Clearly the space $\mathbf{R}^n$ does not have the topological fixed point property. It is not hard to see the closed unit interval does have the fixed point property.

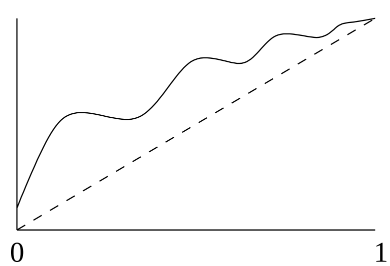


Figure 1.1: The fixed point property on $[0, 1]$.

As the above picture illustrates, if $f : [0, 1] \to [0, 1]$ is a continuous function with no fixed point in $[0, 1)$, then $f(0) \neq 0$. As we let $x$ vary continuously from 0 to 1, by our assumption, and continuity of $f$, $f(x)$ must always stay between $x$ and 1. This forces $f(1) = 1$.

In general, it is not known exactly what type of sets possess the tfpp. At the very least, we might expect that such a set would have to contain its limit points. Otherwise a continuous function might be able to shift each element of $X$ closer to a missing limit point. Thus it seems reasonable that compactness be a required property. However this is clearly not sufficient. For example, the set $[0,1] \cup [2,3]$ does not have the tfpp. In light of this, we might also ask that our set have no "holes" in it. For in this case we might consider a rotation around the missing set. Thus, it might be prudent to restrict ourselves to contractible sets. That is, sets that can be continuously deformed to a point. Of course, this property alone would not guarantee a fixed point for $f$. For example $(0,1)$ does not have the tfpp. The next logical step would be to consider whether sets with both properties, sets that are compact and contractible, have the tfpp. This question was posed by Borsuk in 1932, and remained open for more than 20 years. It was answered in the negative when Kinoshita [38] gave a beautiful example of a compact, contractible set without the topological fixed point property. What we need is something a little stronger than contractibility. Any convex set is contractible. It turns out that compact and convex is sufficient to ensure the existence of fixed points. Also, among convex sets, compactness is necessary [39].

However, there do exist sets that are nonconvex that do have the tfpp. One interesting example is the $\sin(\frac{1}{x})$ circle. This set consists of the closure of the set $\{(x,y) : y = \sin(\frac{1}{x}), 0 < x \leq \frac{1}{\pi}\}$, together with an arc joining the points $(0,1)$ and $(\frac{1}{\pi}, 0)$.

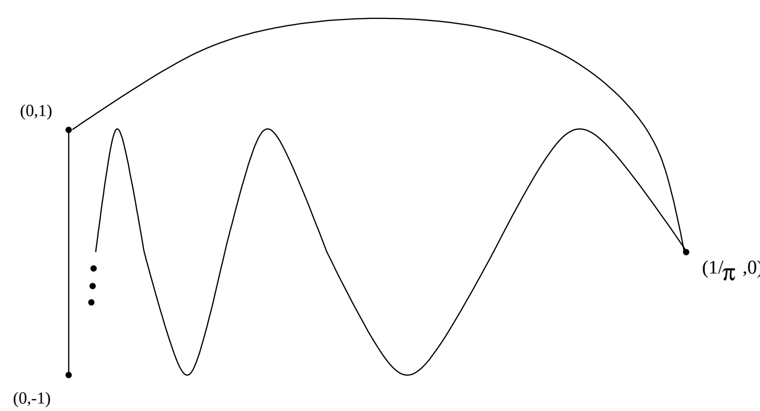


Figure 1.2: The $\sin\left(\frac{1}{x}\right)$ circle.

An argument similar to that above for the unit interval shows that this set has the tfpp. This "nowhere left to go" reasoning is sometimes referred to as the "dog chasing a rabbit" argument.

The following is an interesting example of a set in $\mathbf{R}^2$ that is neither convex, nor compact, but still has the topological fixed point property.

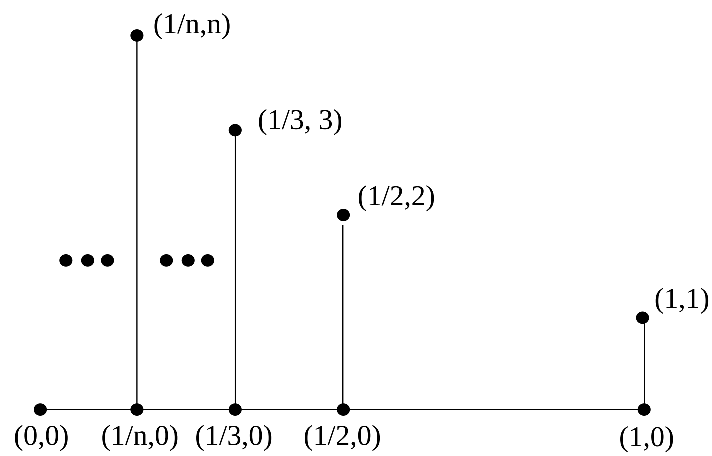


Figure 1.3: An unbounded, nonclosed set with the tfpp.

The figure consists of a base segment $X_0 = [0,1]$ on the $x$–axis, with vertical segments $X_n$ starting at the point $(\frac{1}{n}, 0)$ and extending to $(\frac{1}{n}, n)$. This set is neither closed nor bounded, thus certainly not compact. To see that $X$ has the fixed point property, let $f : X \to X$ be continuous. Denote by $x_n$ the point $(\frac{1}{n}, 0)$. There are three cases to consider. First, if there is an $n \in \mathbf{N}$ with $f(x_n) = x_n$, then we are done. Else, suppose there is an $n$ such that $f(x_n) \in X_n \setminus \{x_n\}$. Now let $r$ be a retraction of $X$ onto $X_n$. Then $r \circ f : X_n \to X_n$ is continuous, and thus has a fixed point, say $x_0$. By our assumption, $x_0$ cannot be $x_n$, and hence $x_0 = r(f(x_0)) = f(x_0)$. The final case to consider is that for every $n \in \mathbf{N}$, $f(x_n)$ is not in $X_n$. Then let $r$ be the retraction of $X$ onto $X_0$, and a similar argument as in the previous case gives us a fixed point for $f$.

Thus among nonconvex sets, compactness and contractibility do not have a direct relationship with the topological fixed point property. In general, it is not known what types on nonconvex sets have the property. For a discussion of various types of nonconvex sets, and the tfpp, see [6].

Some useful facts about the topological fixed point property are immediately obtained.

**Remark 1** *The topological fixed point property is a topological invariant.*

PROOF. Suppose $X$ has the tfpp, and let $h : X \to Y$ be a homeomorphism. Suppose $f : Y \to Y$ is continuous. Then $h^{-1} \circ f \circ h : X \to X$ is continuous, and by supposition has a fixed point, $x$. Then $f(h(x)) = h(x)$, and $Y$ has the tfpp.

■

A retraction of a set $X$ onto a subset $Y \subseteq X$ is a continuous function $r : X \to f(X) = Y$ such that $f|_Y$ is the identity map.

**Remark 2** *The topological fixed point property is preserved under retractions.*

PROOF. Suppose $X$ has the tfpp, and let $r : X \to Y$ be a retraction. Suppose $f : Y \to Y$ is continuous. Then $f \circ r : X \to X$ is a continuous self–map of $X$, so for some $x$, $f \circ r(x) = x$. Since $x \in Y$, $r(x) = x$, and $x$ is a fixed point of $f$. Thus $Y$ has the tfpp.

■

Brouwer's theorem is the assertion that a compact convex set in $\mathbf{R}^n$ has the topological fixed point property. In this thesis we give a brief survey of some of the main results in topological fixed point theory, with a particular focus on Brouwer's fixed point theorem. It has been estimated that while 95% of mathematicians can state Brouwer's theorem, less than 10% know how to prove it [27]. Chapter two may help to remedy this. We present several different proofs using tools from various fields, their conception ranging in time periods from 1910, up to only a few years ago. Some of the proofs are analytic, while others rely more on combinatorial and algebraic methods. It is the author's hope that any mathematician will relate to one of the methods demonstrated.

Brouwer's theorem has been generalized in numerous ways. In chapter three, we highlight what we hope are some of the main points in this development for single valued mappings. The basic extensions are Schauder's and Tychonoff's fixed point theorems. We also mention fixed point properties of closed bounded sets based on boundary conditions, and assumptions involving compactness. In chapter four, we give the multifunction analog of Brouwer's theorem, and also of some of the results in chapter three. The final chapter is a brief note meant to illustrate the wide range of applications that Brouwer's theorem and its descendants have had in mathematics.

### 1.1.1 Background on Simplexes and Triangulations

The closed convex hull of a subset $A = \{a_i \ldots, a_k\} \subseteq \mathbf{R}^n$ is the set $\overline{\text{conv}}(A) = \{\sum_{i=1}^k \lambda_i a_i : \lambda_i \geq 0, \sum \lambda_i = 1\}$. A subset $S$ of $\mathbf{R}^n$ is called a **$k$–simplex**, or **$k$-dimensional simplex**, provided there exists a set $V = \{v_0, \cdots, v_k\}$, such that the vectors $(v_1 - v_0), (v_2 - v_0), \cdots, (v_k - v_0)$ are linearly independent, and $S = \overline{\text{conv}}\{v_0, \cdots, v_k\}$. The $v_i$ are called the *vertices* of $S$. If context is clear, we may simply write $S = \{v_0, \ldots, v_k\}$ to refer to the simplex $S$. A **$p$–face** of $S$ is the closed convex hull of any subset of $p$ points in $V$. The ordering of the set $V$ determines an orientation of the simplex. If two orderings differ by an even permutation, then they induce the same orientation. If they differ by an odd permutation, they induce opposite orientations of the simplex $S$. A finite collection $K$ of simplexes that contains the faces of each of its members, and is such that any two members of $K$ who intersect do so in a face, is called a **simplicial complex**. A **simplicial subdivision** of a $k$–simplex $S$ is obtained by adding vertices to the vertex set of $S$, and then adding new faces to $S$ in such a way that a simplicial complex is obtained. A $k$–simplex in this new collection is called a **$k$–subsimplex** of the complex. A triangulation of a topological space $X$ consists of a simplicial complex $K$, and a homeomorphism $h : |K| \to X$, where $|K|$ is the complex $K$ thought of as a subset of Euclidean space, endowed with the subspace topology.

A **labeling** of a $k$–simplex, or simplicial complex, $S$ is a function $\mu$ that maps the set of vertices $V = \{v_0, v_1, \cdots v_k\}$ of $S$ to the set of integers $\{0, \cdots, k\}$. A labeling of a $k$–simplex is said to be **proper** provided this mapping is a bijection. A simplicial subdivision $S'$ of a $k$–simplex $S$ is properly labeled provided $\mu|_S$ is a proper labeling, and for any vertex $v \in S'$ contained in a face of $S$ carrying the labels $i_0, i_1, \cdots, i_m$, $\mu(v)$ is one of $i_0, i_1, \cdots, i_m$. A $k$–subsimplex $S$ of a properly labeled simplicial subdivision is called **distinguished** if $\mu$ maps $S$ onto the set $\{0, 1, \cdots, k\}$. It is true that any properly labeled simplicial subdivision of a $k$–simplex $S$ contains an odd number of distinguished $k$–subsimplexes.

### 1.1.2 Background in Analysis and Topology

Let $\mathbf{E}$ be a topological space, and $X \subseteq \mathbf{E}$. A function $f$ is **continuous** if the inverse image under $f$ of an open set is open. An **open cover** of $X$ is a collection of open sets whose union contains $X$. We say $X$ is **compact** provided every open cover has a finite subcover. This is equivalent to stating that every collection of closed sets in $X$ with the finite intersection property has nonempty intersection. A metric space is compact provided every sequence has a convergent subsequence. In $\mathbf{R}^n$ compactness is equivalent to closed and bounded. If $f$ is continuous, and $X$ is compact, then $f(X)$ is compact. Any closed subset of a compact space is compact. If points in $X$ can be separated by disjoint open sets, then we say $X$ is **Hausdorff**. In a Hausdorff space compact subsets are closed. A compact Hausdorff space is **normal**, by which we mean closed sets can be separated by disjoint open sets.

A family $\{U_i\}_{i\in I}$ of subsets of $X$ is called **neighbourhood finite** (nbd–finite) if each $x$ in $X$ has a neighbourhood $V$ such that $V \cap U_i \neq \emptyset$ for at most finitely many $i \in I$. For $X$ Hausdorff, a family $\{\beta_i\}_{i\in I}$ of continuous real valued maps is a **partition of unity** on $X$, provided the supports of the $\beta_i$ form a nbd–finite closed covering of $X$, $0 \leq \beta_i(x) \leq 1$, and $\sum \beta_i(x) = 1$, for each $x$ in $X$. Given an open cover $\{U_i\}_{i\in I}$ of $X$, we say a partition of unity $\{\beta_i\}_{i\in I}$ is **subordinate** to this cover if for each $i$, the support of $\beta_i$ lies in $U_i$. If $X$ is (para)compact then any open cover of $X$ admits a partition of unity subordinate to it. A **topological vector space** is a vector space equipped with a topology such that scalar multiplication and vector addition are continuous. In what follows, we must be able to topologically distinguish points. Thus we assume that all spaces are Hausdorff.

### 1.1.3 Upper Semicontinuous Multifunctions

Let $X$ and $Y$ be topological spaces. A **multifunction** $F : X \to 2^Y$ is a mapping that sends points $x$ in $X$ to subsets $F(x)$ of $Y$. We write $\cup_{x\in X} F(x) = F(X)$. For $y \in Y$, we define the inverse of $F$ at $y$ to be $F^{-1}(y) = \{x \in X : y \in F(x)\}$. For $B \subseteq Y$, $F^{-1}(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$. The **graph of** $F$ is the set $Gr(F) = \{(x, y) \in X \times Y : y \in F(x)\}$. We say that $F$ is **upper semicontinuous (*usc*)** at $x \in X$ if for every neighbourhood $V$ of $F(x)$, there exists a neighbourhood $U$ of $x$ with $F(U) \subseteq V$. We say that $F : X \to 2^Y$ is *usc* if it is *usc* at every $x \in X$. If $Y$ is compact, and the images $F(x)$ are closed, then $F$ is *usc* if and only if $Gr(F)$ is closed in $X \times Y$. In this case, if $Y$ is compact, we also have that $F$ is *usc* if $x_n \to x$, $y_n \to y$, and $y_n \in F(x_n)$, together imply that $y \in F(x)$. In a topological vector space, a *usc* multifunction with nonempty, compact, convex images is called a *cusco* for short. Some facts about *cuscos* are immediate.

**Proposition 1** *Let $X$ be a compact subset of a topological vector space. If $F : X \to 2^X$ is a cusco, and $f : X \to X$ is linear, then $f \circ F : f(X) \to 2^{f(X)}$ is a cusco.*

**Proposition 2** *Let $X \subseteq Y$ be subsets of a Banach space, with $X$ closed. If $F : X \to 2^X$ is a cusco, and $f : Y \to X$ is continuous, then $F \circ f : Y \to 2^Y$ is a cusco.*

# Chapter 2

# Brouwer's Fixed Point Theorem

Brouwer's fixed point theorem is the assertion that the class of compact convex sets in $\mathbf{R}^n$ has the fixed point property. As is often the case, Brouwer was not the first to prove "his" theorem. The result has its roots at least as far back as 1817, when Bolzano's intermediate value theorem appeared. In 1883 Poincaré generalized this result in what is known as the Bolzano-Poincaré-Miranda theorem.

**Theorem 2.0.1** *Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be continuous, and suppose that $|x_i| \leq a_i$, for some prescribed set of reals $a_i > 0$, and $1 \leq i \leq n$. Further suppose that on each face $x_i = a_i$, we have $f_i(x) > 0$, and for $x_i = -a_i$, we have $f_i(x) < 0$. Then there exists $x$ such that $f(x) = 0$.*

Miranda's name has been attached to this theorem because in 1941 he proved that it was in fact equivalent to Brouwer's theorem. This wasn't the only equivalent result to preclude Brouwer's publication of his theorem. In 1904, Bohl used Green's Theorem to prove that there could be no retraction of the $n$–cube onto its boundary [8]. There does not seem to be evidence that Bohl made the short leap from this to deduce Brouwer's theorem. Following this, in 1909 Brouwer proved the theorem in $\mathbf{R}^3$. Then in 1910, using Kronecker indices, Hadamard published a proof of the theorem for general $n$ in the appendix of a book by Tannery [54]. It wasn't until 1912 that Brouwer himself published his proof in $\mathbf{R}^n$ [13]. He used simplicial approximations and the degree of a map to prove his theorem in the setting of an $n$–simplex.

| | | |
|---|---|---|
| **1883** | : | Bolzano–Poincaré–Miranda theorem. |
| **1904** | : | Bohl proves no retraction of $n$–cube onto its boundary. |
| **1909** | : | Brouwer proves Brouwer's theorem in $\mathbf{R}^3$. |
| **1910** | : | Hadamard proves Brouwer's theorem in $\mathbf{R}^n$. |
| **1912** | : | Brouwer proves Brouwer's theorem in $\mathbf{R}^n$. |

It seems to have been generally accepted that Brouwer knew the theorem to be true in 1910. In fact, Hadamard knew of the theorem through a letter from Brouwer himself, which he received that same year [26]. Still, it is clear that Brouwer was not the first to prove the Brouwer fixed point theorem. It is somewhat ironic that decades his name was the one attached to the result, when decades later his intuitionist philosophy dictated he reject his nonconstructive proof [14]. We state now, Brouwer's fixed point theorem.

**Theorem 2.0.2 (Brouwer's Fixed Point Theorem)** *Any compact convex subset of* $\mathbf{R}^n$ *has the fixed point property.*

Brouwer's original proof used complicated ideas such as the degree of a map. In the decades that followed, mathematicians searched for proofs that were simpler, or somehow better, or that used the language of a certain field. In this chapter we explore various proofs of the theorem, ranging from the early 20th century, up to the beginning of the 21st. We have divided the chapter into two sections; nonanalytic proofs, and analytic proofs. By analytic, we mean that a student with knowledge of calculus and some real analysis should be able to understand the proofs.

## 2.1 Nonanalytic Methods of Proof

In general, the proofs in this section are the earlier methods used to prove Brouwer's theorem. Brouwer himself used the notion of the degree of a map on the sphere, and it is a proof based on this literature that we present first. The second method we illustrate uses the famous KKM theorem. It is relatively easy to derive Brouwer's theorem from KKM, but the section has been shortened in that we leave out the proof of Sperner's lemma. Still, the proof of the lemma is not hard, and thus this may be the most elementary proof we show. The final nonanalytic proof we give uses the $n$th–homology groups of a topological space $X$. There is some difficulty in setting up the language in this section, but once it is in place, it provides an immediate proof of Brouwer's theorem.

### 2.1.1 The Degree of a Self–map on $\mathbf{S}^{n-1}$

The proof given by Brouwer in 1912 was based on the notion of the degree of a continuous function $f : \mathbf{S}^{n-1} \to \mathbf{S}^{n-1}$. The following version of the proof can be found in Dugundji [21]. When $n = 2$, the degree of $f$, $\deg(f)$, can be thought of as the net number of times $f(x)$ travels around the unit circle as we let $x$ make one counterclockwise trip around. Formally, choose set of points $\{x_0, x_1, \ldots, x_p\}$ taken in counterclockwise order around $\mathbf{S}^1$ such that $|x_{i+1} - x_i| < 1$. Then each segment $[x_i, x_{i+1}]$ is a 1–simplex in $\mathbf{R}^2$. Think of this segment as an arc on the unit circle, rather than a straight line segment. That is, let $[x_i, x_{i+1}]$ be the projection from the origin through the line segment onto to circle. Then the union $\bigcup [x_i, x_{i+1}] = T$ is a triangulation of $\mathbf{S}^1$.

The set $\{f(x_0), \ldots, f(x_p)\}$ will be another ordered set of points around $\mathbf{S}^1$, but the points may not follow each other counterclockwise around $\mathbf{S}^1$. The function $f$ may reverse the orientation of some pairs of these points. We say that the image simplex $[f(x_i), f(x_{i+1})]$ has positive orientation if as $x$ travels from $x_i$ to $x_{i+1}$, $f(x)$ traverses the segment $[f(x_i), f(x_{i+1})]$ in a counterclockwise manner. If $f(x)$ travels in the opposite direction, we say this 1–simplex has negative orientation.

Next, fix $x \in \mathbf{S}^1$ such that $x$ is not on the boundary of any of the image segments $[f(x_i), f(x_{i+1})]$. Then the number of positively oriented image segments containing $x$, minus the number of negatively oriented segments, is called the degree of $f$ at $x$ with respect to the given triangulation $T$. With a little effort it can be seen that this degree is actually independent of $x$ and $T$.

In general, an $n$–simplex $S$ in $\mathbf{R}^n$ is the convex hull of a set of $n + 1$ points, called vertices. By fixing the order of the vertices, we consider $S$ to be an ordered simplex. $S$ is said to be nondegenerate provided the volume of this hull in $\mathbf{R}^n$ is nonzero. That is, $S$ is nondegenerate provided its $n + 1$ vertices do not all lie on an $(n - 1)$–hyperplane. If we write $S = \overline{\text{conv}}\{x_0, \ldots, x_n\}$, and $x_i = (x_i^1, \ldots, x_i^n) \in \mathbf{R}^n$, then this nondegeneracy condition is equivalent to

$$\det(S) = \begin{vmatrix} x_0^1 & \cdots & x_0^n & 1 \\ x_1^1 & \cdots & x_1^n & 1 \\ \vdots & \cdots & \vdots & \vdots \\ x_n^1 & \cdots & x_n^n & 1 \end{vmatrix} \neq 0.$$

Further, if $\det(S) > 0$ we say the $n$–simplex $S$ is **positively oriented**. If $\det(S) < 0$, then $S$ is **negatively oriented**. From the rules for matrix determinants, we see that even permutations of the order of the vertices of $S$ will not change its orientation, odd ones will reverse the sign of $\det(S)$.

**Lemma 2.1.1** *Let $S$ and $S'$ be two oriented $n$–simplexes with*

$$\begin{aligned} S &= \{x_0, x_1, \ldots, x_n\}, \\ S' &= \{x_0', x_1, \ldots, x_n\}. \end{aligned}$$

*Then $S$ and $S'$ have the same orientation if and only if $x_0$ and $x_0'$ lie on the same side of the $(n-1)$–hyperplane $H$ containing $\{x_1, \ldots, x_n\}$.*

Proof. Let $S = \{x_0, x_1, \ldots, x_n\}$, and $S' = \{x_0', x_1, \ldots, x_n\}$. If $x_0$ and $x_0'$ lie on opposite sides of $H$ then $\bar{x} = tx_0 + (1-t)x_0' \in H$ for some $t \in (0,1)$. Then the n–simplex $\bar{S}$ with vertices $\{\bar{x}, x_1, \ldots, x_n\}$ is degenerate, and

$$\det(\bar{S}) = t\det(S) + (1-t)\det(S') = 0.$$

This is possible only when $S$ and $S'$ have opposite orientation.

■

As in the discussion for $n = 1$, we shall need to look at triangulations living on the unit sphere $\mathbf{S}^{n-1}$. Any set of $n$ points on $\mathbf{S}^{n-1}$ that do not lie on the same $n-2$–hyperplane determine an $(n-1)$–simplex $S$. We say a simplex $S$ is proper if $\mathrm{diam}(S) < 1$. In this case the projection from the origin through $S$, onto $\mathbf{S}^{n-1}$ determines a set $\tilde{S}$ that is proper in the same sense, i.e., $\mathrm{diam}(\tilde{S}) < 1$. Such a projection will be called the spherical $(n-1)$–simplex corresponding to $S$. The vertex set of $\tilde{S}$ is the same as that of $S$. The ordering and orientation of the projection is inherited from the original simplex. A spherical $(n-1)$–simplex $\tilde{S}$ will be said to be degenerate in the case that the simplex determined by its vertices, along with the point $\mathbf{0}$, is a degenerate $n$–simplex. A triangulation of $\mathbf{S}^{n-1}$ is a finite collection $T = \bigcup \tilde{S}$ of nondegenerate ordered spherical $(n-1)$–simplexes covering $\mathbf{S}^{n-1}$ and satisfying two properties. First, members of $T$ intersect only in a common face, and second, for any $\tilde{S} \in T$, each $(n-2)$–face of $\tilde{S}$ is shared with exactly one other member of $T$. The vertices of $T$ are the union of the vertices of the $\tilde{S} \in T$. A function that maps vertices of $T$ into $\mathbf{S}^{n-1}$ is called a proper vertex map provided that for each $\tilde{S} \in T$, the simplex determined by the image of the vertices of $\tilde{S}$ is proper, and hence has an associated proper spherical simplex.

With the above definitions in place, let $T$ be a triangulation of $\mathbf{S}^{n-1}$. Using the orientability of $\mathbf{S}^{n-1}$, assume all simplexes are oriented positively. Let $f : T \to \mathbf{S}^{n-1}$ be a proper vertex map. Fix $x \in \mathbf{S}^{n-1}$ such that $x$ is not on the boundary of $f(\tilde{S})$ for any $\tilde{S} \in T$. Let $p(f, T, x)$ be the number of positively oriented spherical simplexes $f(\tilde{S})$ containing $x$, and $n(f, T, x)$ the number of negatively oriented spherical simplexes containing $x$. Then we define the degree of $f$ with respect to $T$ and $x$ to be

$$\deg(f, T, x) = p(f, T, x) - n(f, T, x).$$

**Lemma 2.1.2** *For a given triangulation $T$ of $\mathbf{S}^{n-1}$, with each $\tilde{S} \in T$ positively oriented, and a proper vertex map $f$, $\deg(f, T, x)$ is independent of the choice of $x$.*

PROOF. We prove the case where for each $\tilde{S} \in T$, $f(\tilde{S})$ determines a nondegenerate spherical $(n-1)$–simplex. Let $f : T \to \mathbf{S}^{n-1}$ be a proper vertex map. Pick any two points $y$,$z \in \mathbf{S}^{n-1}$ that do not lie on the boundary of $f(\tilde{S})$ for any $\tilde{S} \in T$. We will show that $\deg(f, T, y) = \deg(f, T, z)$. To this end, let $C$ be an arc in $\mathbf{S}^{n-1}$ joining $y$ and $z$ that doesn't pass through any face of dimension less than $(n-2)$ of any spherical $(n-1)$–simplex of $f(\tilde{S})$. We consider what happens to $\deg(f, T, x)$ as we let $x$ move from $y$ to $z$.

Clearly, for the degree to change, $x$ has to travel across some $(n-2)$–face of some simplex in $f(T)$. Call one such simplex $f(\tilde{S_1})$, where $\tilde{S_1} = \{x_0, x_1, \ldots, x_{n-1}\}$, and the $(n-2)$–face $x$ crosses is $A = \{f(x_1), f(x_2), \ldots, f(x_{n-1})\}$. Now, to $\tilde{S_1}$ there corresponds exactly one other simplex in $T$ that shares the face $\{x_1, \ldots, x_{n-1}\}$. Call this simplex $\tilde{S_2}$, and write $\tilde{S_2} = \{x_0', x_2, x_1, \ldots, x_{n-1}\}$, where the ordering of the vertices is chosen to give $\tilde{S_2}$ positive orientation. Note that $f(\tilde{S_2})$ is a simplex in $f(T)$ that shares the $(n-2)$–face that our point $x$ crosses. We write

$$f(\tilde{S_1}) = \{f(x_0), f(x_1), f(x_2), \ldots, f(x_{n-1})\}$$
$$f(\tilde{S_2}) = \{f(x_0'), f(x_2), f(x_1), \ldots, f(x_{n-1})\}.$$

There are two cases to consider. First, suppose $f(x_0)$, and $f(x_0')$ are on the same side of the hyperplane determined by the vertices of $A$. As $x$ travels through the face $A$ it either enters both $f(\tilde{S_1})$ and $f(\tilde{S_2})$, or it leaves both. Then by lemma 2.1.1, the simplexes $f(\tilde{S_1})$ and $f(\tilde{S_2})$ have opposite orientation. In either case, the net change in $p(f, T, x) - n(f, T, x)$ is zero.

Next suppose $f(x_0)$ and $f(x_0')$ lie on opposite sides of $A$. As $x$ passes through $A$ it must leave one of $f(\tilde{S}_1)$ or $f(\tilde{S}_2)$, and enter the other. Again, by the lemma, $f(\tilde{S}_1)$ and $f(\tilde{S}_2)$ have the same orientation. Again, there is no net change in $\deg(f, T, x)$.

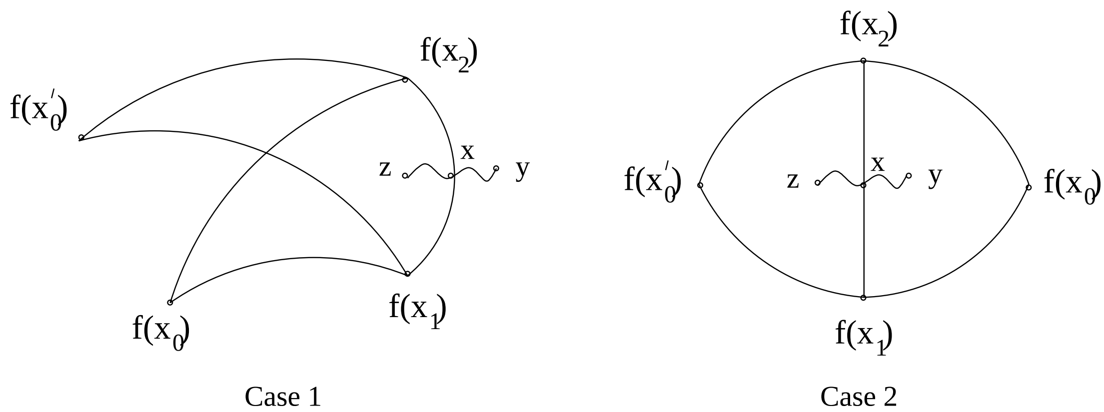


Figure 2.1: The two cases in the proof of lemma 2.1.2.

We address the case when some of the $f(\tilde{S})$, $\tilde{S} \in T$, may be degenerate. To do this we will reduce it to the previous one by approximating $f$ with another proper vertex map $g$. Suppose $f$ maps $\tilde{S}$ to a degenerate simplex $f(\tilde{S})$. This means that some vertex in $\tilde{S}$ gets mapped to the $(n-3)$–hyperplane containing the remaining $(n-2)$ vertices. Now, $y$ and $z$ lie in the interior of $f(\tilde{S})$ for each $\tilde{S}$. Thus we can slightly perturb $f$ at the offending vertex so that its image is no longer in the $(n-1)$–hyperplane, and we haven't altered the orientation or number of simplexes $f(\tilde{S})$ containing $y$ or $z$. That is, we can find an $\epsilon > 0$ and a proper vertex map $g : T \to \mathbf{S}^{n-1}$ that has no degenerate $g(\tilde{S})$, and $|f(x) - g(x)| < \epsilon$ for each vertex $x \in T$. We have that $\deg(f, T, y) = \deg(g, T, y)$, and $\deg(f, T, z) = \deg(g, T, z)$. So via $g$, and the previous nondegenerate case, we see $\deg(f, T, y) = \deg(f, T, z)$.

■

Thus we may write $\deg(f, T)$ without any ambiguity. From the above argument we obtain the following useful lemma.

**Lemma 2.1.3** *Let* $f, g : T \to \mathbf{S}^{n-1}$ *be proper vertex maps, and* $x \in \mathbf{S}^{n-1}$ *not on the boundary of any of any* $f(\tilde{S})$, $\tilde{S} \in T$. *There exists an* $\epsilon > 0$ *such that if* $|f(y) - g(y)| < \epsilon$ *for all vertices in* $T$, *then* $\deg(f, T, x) = \deg(g, T, x)$.

Given a triangulation $T$ we can refine $T$ by adding a finite set of points to its vertex set. We then need only add faces to $T$ to preserve the triangulation properties. Barycentric subdivision is one example of a refinement procedure. See Armstrong [1] or Dugundji [21] for references.

Until now, the function $f$ has been defined only on the vertices of $T$. We can extend the above concepts to a continuous $f : \mathbf{S}^{n-1} \to \mathbf{S}^{n-1}$, since any such function induces a vertex map on a $T$ by restricting its domain to the set of vertices of $T$. For an arbitrary triangulation $T$, $f$ may not be a proper vertex map. It follows from the continuity of $f$, and the compactness of $\mathbf{S}^{n-1}$, that we can find a refinement $T'$ of $T$ such that $\operatorname{diam}(f(\tilde{S})) < 1$ for each $\tilde{S} \in T'$. Thus we can speak of the degree of a continuous function $f$ with respect to a triangulation $T$.

**Lemma 2.1.4** $\deg(f, T)$ *is independent of the choice of* $T$.

Proof. Let $T_1$ and $T_2$ be two triangulations of $\mathbf{S}^{n-1}$, both inducing proper vertex maps of $f$. Then let $T_3$ be a common refinement of both. We will show that if $T'$ is a refinement of $T$, then $\deg(f, T) = \deg(f, T')$.

Suppose $T'$ is a refinement of $T$ formed by adding a single vertex to $T$. Note that if the image of just the new simplexes created by adding this new point covers all of $\mathbf{S}^{n-1}$, then there must have been a simplex in the original triangulation that violated the property $\operatorname{diam}(f(S)) < 1$. Thus we can pick $x \in \mathbf{S}^{n-1}$ that is not in the image of any of the newly created simplexes. Thus $\deg(f, T, x) = \deg(f, T', x)$. By induction, we see that any refinement of $T$ will not alter the degree. Thus via the common refinement, we see $\deg(f, T_1) = \deg(f, T_2)$.

■

From this point we may write $\deg(f)$ to refer to the degree of $f$.

**Theorem 2.1.5** *If* $f, g : \mathbf{S}^{n-1} \to \mathbf{S}^{n-1}$ *are homotopic, then* $\deg(f) = \deg(g)$.

Proof. Let $f, g : \mathbf{S}^{n-1} \to \mathbf{S}^{n-1}$, and $F : \mathbf{S}^{n-1} \times I \to \mathbf{S}^{n-1}$ be a homotopy of $f$ and $g$. So $F$ is continuous, and $F(x, 0) = f(x)$, $F(x, 1) = g(x)$. First note that by compactness of $\mathbf{S}^{n-1} \times I$, $F$ is uniformly continuous. Thus $\exists \delta > 0$ such that for any $t \in I$, and $x, y \in \mathbf{S}^{n-1}$ satisfying $|x - y| < \delta$, we have $|F(x, t) - F(y, t)| < 1$. Choose a triangulation $T$ of $S^{n-1}$

such that $\text{diam}(\tilde{S}) < \delta$ for all $\tilde{S} \in T$. Then for each $t$, the function $F(\cdot, t)$ induces a proper vertex map of $T$.

Now, by the previous lemma, there is an $\epsilon > 0$ such that if $|F(x,t) - h(x)| < \epsilon$ for every vertex $x \in T$, then $\deg(F(\cdot,t)) = \deg(h)$. Fix such an $\epsilon$. Again, by uniform continuity of $F$, choose $\delta > 0$ such that for any $x \in \mathbf{S}^{n-1}$, $|F(x,t) - F(x,t')| < \epsilon$ whenever $|t - t'| < \delta$. Thus we have shown that the function mapping $t$ to $\deg(F(\cdot,t))$ is a continuous integer valued function, and hence, must be a constant function. Then $\deg(f) = \deg(g)$.

■

**Example 1** *The degree of* $id : \mathbf{S}^n \to \mathbf{S}^n$ *is* 1.

PROOF. Follows from the definition.

■

With the above machinery in place, we can prove that the unit sphere is not a retract of the unit ball. This result is a well known equivalence of Brouwer's theorem.

**Lemma 2.1.6** *Suppose* $f : \mathbf{S}^{n-1} \to \mathbf{S}^{n-1}$ *has a continuous extension to* $\mathbf{B}^n$*. Then* $f$ *is homotopic to a constant function.*

PROOF. Let $\bar{f} : \mathbf{B}^n \to \mathbf{S}^{n-1}$ be a continuous extension of $f$. Define $F : \mathbf{S}^{n-1} \times I \to \mathbf{S}^{n-1}$ by $F(x,t) = \bar{f}((1-t)x)$. Then $F$ is a homotopy of $f$ to the constant map that sends $x$ to $\bar{f}(0)$.

■

**Theorem 2.1.7** $\mathbf{S}^{n-1}$ *is not a retract of* $\mathbf{B}^n$*.*

PROOF. Suppose $r : \mathbf{B}^n \to \mathbf{S}^{n-1}$ is a retraction. Then $r|_{\mathbf{S}^{n-1}} = id : \mathbf{S}^{n-1} \to \mathbf{S}^{n-1}$ would have a continuous extension to $\mathbf{B}^n$. Thus, by the previous lemma, $id : \mathbf{S}^{n-1} \to \mathbf{S}^{n-1}$ is homotopic to a constant function. On the other hand, the degree of the constant map is 0, while the degree of $id : \mathbf{S}^{n-1} \to \mathbf{S}^{n-1}$ is 1. By theorem 2.1.5, these functions cannot be homotopic.

■

From here, we easily deduce Brouwer's theorem. Assume the continuous map $f : \mathbf{B}^n \to \mathbf{B}^n$ is fixed point free. Then define $g$ to be the function that maps a point $x$ to the projection from $\mathbf{0}$ through $x$ onto $\mathbf{S}^{n-1}$. Then $g$ is a continuous retract of the unit ball onto the unit sphere, violating the above theorem.

### 2.1.2 The KKM Theorem

The following section looks at a proof of Brouwer's theorem that follows from an elegant combinatorial result on simplicial subdivisions proven in 1928 by Sperner. A year after this lemma was published, Knaster, Kuratowski and Mazurkiewicz used it to prove the so called KKM theorem, from which they deduced the Brouwer fixed point theorem. The ease at which they obtained Brouwer's theorem caused speculation as to the possible equivalence of these three theorems. The question remained open for almost 50 years, until in 1974 Yoseloff [56] showed that Brouwer implies Sperner's lemma. Below we show how to obtain the KKM theorem from Sperner's lemma, and then apply it to obtain Brouwer's theorem.

**Theorem 2.1.8 (Sperner's Lemma)** *Any properly labeled simplicial subdivision of a $k$–simplex has an odd number of distinguished $k$–subsimplexes.*

We leave out the proof of this lemma, but note that it is not difficult. See for example [27], where a proof using only a counting argument and mathematical induction is given. As such, the proof of Brouwer's theorem given in this section may be the most elementary.

**Theorem 2.1.9 (KKM)** *Let $S = \overline{conv}\{v_0, v_1, \cdots, v_k\}$ be a $k$–simplex. Suppose $A_0, A_1, \cdots, A_k$ are closed subsets of $S$ such that*

$$\overline{conv}\{v_{i_o}, v_{i_1}, \cdots, v_{i_m}\} \subseteq \bigcup_{j=0}^{m} A_{i_j}$$

*holds for any subset $\{v_{i_j}\}$ of $\{v_i\}_{i=0}^k$. Then $\bigcap_{i=0}^k A_i \neq \emptyset$.*

Proof. Let $S = \overline{\text{conv}}\{v_0, v_1, \cdots, v_k\}$ be a $k$–simplex. For each $n \in \mathbf{N}$, there is a simplicial subdivision $S^n$ of $S$ such that the diameter of each $k$–subsimplex of $S^n$ is less than $\frac{1}{n}$. Dress $S$ with a proper labeling of its vertices, so $v_i$ is labeled with $i$. We extend this to a proper labeling of $S^n$ as follows. For each vertex $q$ of $S^n$, there is a smallest face $\overline{\text{conv}}\{v_{i_o}, v_{i_1}, \cdots, v_{i_m}\}$ of $S$ containing $q$. Then by our assumption, $q \in A_{i_j}$ for some $j$, $0 \leq j \leq m$. Assign to $q$ the label $i_j$. Thus we obtain the desired labeling on $S^n$.

Now, by Sperner's lemma, $S^n$ has a distinguished $k$–subsimplex, say $\overline{\text{conv}}\{q_o^n, q_1^n, \cdots, q_k^n\}$. Upon relabeling if necessary, we can assume that the vertex $q_i^n$ carries the label $i$, and thus is a member of $A_i$ by our construction. Now compactness of $S$ guarantees a convergent subsequence of $\{q_i^n\}_{n=1}^{\infty}$ for each $i$. Since the diameter of the $k$–subsimplexes goes to zero, these sequences must converge to a common point. Since each $A_i$ is closed in $S$, this limit must be in $\bigcap_{i=0}^k A_i$.

■

From the above lemma, we can easily deduce Brouwer's theorem. Let $S$ be a $k$–simplex with vertices $\{v_0, v_1, \cdots, v_k\}$, and $f$ be a continuous self–map on $S$. For $x \in S$ we have $x = \sum_{i=0}^k x_i v_i$, with $\sum_{i=0}^k x_i = 1$, and $x_i \geq 0$. Define

$$A_i = \{x \in S : f_i(x) \leq x_i\}.$$

From the continuity of the components of $f$, we see that each $A_i$ is indeed a closed subset of $S$. By applying the KKM theorem to these $A_i$ we get some point $y \in S$ such that $f_i(y) \leq y_i$ for each $i$. But since $\sum y_i = 1 = \sum (f(y))_i$, we must have $y_i = (f(y))_i$ for each $i$. That is, $f(y) = y$.

Of course, this theorem extends to any closed convex subset in $\mathbf{R}^n$ since any such a set is a retract of an $n$–simplex.

### 2.1.3 Via Homology Groups

To give a good sampling of the methods with which Brouwer's theorem may be proven, it is necessary to discuss homology groups. Of all the methods discussed, it is from the language of the material in this section that Brouwer's theorem flows most naturally. The difficulty is that this language takes quite a bit of work to establish. For this reason, we will show how homology groups are defined on simplicial complexes, and compute some of these groups for triangulations of the ball and sphere. We will discuss, though not prove, how these ideas are extended to arbitrary topological spaces, and use the homology groups of $\mathbf{B}^n$ and $S^{n-1}$ to prove Brouwer's theorem. For references, see [1].

The first thing we must do is define $H_q(K)$, the $q$th–homology group of $K$, where $K$ is a simplicial complex. To do so, we consider the set of all $q$–simplexes in $K$. Each $q$–simplex can be oriented in one of two ways. To each $q$–simplex $z$ in $K$ we assign one of those orientations to be positive, and call $-z$ the $q$–simplex $z$ oriented in the opposite way.

The $q$th–chain group of $K$, denoted $C_q(K)$, is the free abelian group generated by these oriented $q$–simplexes. The elements of $C_q(K)$ are called $q$–chains. The boundary function $\partial : C_q(K) \to C_{q-1}(K)$ maps a $q$–chain to its $(q-1)$–dimensional boundary, a $(q-1)$–chain in $C_{q-1}(K)$. It is defined for each $q$–simplex $z = (v_0, \ldots, v_q)$ by

$$\partial(v_0, \ldots, v_q) = \sum_{i=0}^{q} (-1)^i (v_0, \ldots \hat{v}_i, \ldots, v_q),$$

where $(v_0, \ldots, \hat{v}_i, \ldots, v_q)$ is the simplex formed by deleting $v_i$ from the vertex set of $z$. Then $\partial$ is extended linearly for longer $q$–chains.

Let $Z_q(K)$ be the kernel of $\partial : C_q(K) \to C_{q-1}(K)$. Thus $Z_q(K)$ consists of those $q$–chains that have zero boundary. We call these chains $q$–cycles. Next, we define $B_q(K)$ to be the image of the map $\partial : C_{q+1}(K) \to C_q(K)$. Elements of $B_q(K)$ are called bounding $q$–cycles, as they bound the $(q+1)$–simplex that was their preimage under $\partial$.

**Proposition 3** *$B_q(K) \subseteq Z_q(K)$.*

Proof. We need only show $\partial(\partial(z)) = 0$ for each $(q+1)$–simplex $z \in C_{q+1}(K)$. Indeed, for such a $z = (v_0, \ldots, v_{q+1})$ we have

$$\begin{aligned}
\partial(\partial(z)) &= \partial\left(\sum_{i=0}^{q+1} (-1)^i (v_0, \ldots, \hat{v}_i, \ldots v_{q+1})\right) \\
&= \sum_{i=0}^{q+1} (-1)^i \partial(v_0, \ldots, \hat{v}_i, \ldots v_{q+1}) \\
&= \sum_{i=0}^{q+1} (-1)^i \left(\sum_{j=0}^{i-1} (-1)^j (v_0, \ldots, \hat{v}_j, \ldots, \hat{v}_i, \ldots, v_{q+1})\right) \\
&+ \sum_{i=0}^{q+1} (-1)^i \left(\sum_{j=i+1}^{q+1} (-1)^{j+1} (v_0, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_{q+1})\right) \\
&= 0
\end{aligned}$$

■

We can now define the $q$th–homology group of a simplicial complex $K$.

$$H_q(K) = \frac{Z_q(K)}{B_q(K)}$$

For $q = 0$, $Z_0(K) = C_0(K)$, which is generated by the vertex set of $K$. If $K$ is connected we can connect any two vectors $v$, $w$ by an edge path, whose boundary is $v - w$. Then every vertex represents the same elements in $H_0(K)$. That is, $H_0(K) \cong \mathbf{Z}$.

For $q > 0$, we intuitively think of $H_q(K)$ as measuring in a sense, the $q$–dimensional holes in $K$. For example, if $K$ is the simplicial complex formed by taking an $n$–simplex with all of its faces, then we would expect every $q$–cycle to be a bounding cycle, and thus $H_q(K) = 0$, the trivial group, for $0 < q \leq n$. This is, in fact, the case.

**Proposition 4** *Let $K$ be the simplicial complex formed by an $n$–simplex with all of its faces. Then $H_q(K) = 0$ for $0 < q \leq n$.*

PROOF. Label one of the vertices of $K$ as $v$. Call the simplicial complex determined by removing $v$ and its associated faces, $L$. Define $d_q : C_q(K) \to C_{q+1}(K)$ by

$$d_q(v_0, \ldots, v_q) = \begin{cases} (v, v_0, \ldots, v_q) & \text{if } v \neq v_i,\ 0 \leq i \leq q, \\ 0 & \text{otherwise,} \end{cases} \tag{2.1}$$

for $q$–simplexes , and extend linearly to all of $C_q(K)$. Given a $q$–simplex $z = (v_0, \ldots, v_q)$, if $v$ is not a vertex of $z$, then

$$\begin{aligned} \partial(d_q(z)) &= \partial(v, v_0, \ldots, v_q) \\ &= (v_0, \ldots, v_q) = \sum_{i=0}^{q} (-1)^{i+1} (v, v_0, \ldots, \hat{v}_i, \ldots, v_q) \\ &= z - d_{q-1}(\partial(z)). \end{aligned}$$

If $v$ is a vertex of $z$, then without loss of generality, write $z = (v, v_0, \ldots, v_{q-1})$. Then $\partial(d_q(z)) = 0$, and

$$\begin{aligned} d_{q-1}(\partial(z)) &= d_{q-1}\left( \sum_{i=0}^{q} (-1)^i (v, v_0, \ldots, \hat{v}_i, \ldots, v_{q-1}) \right) \\ &= \sum_{i=0}^{q} (-1)^i d_{q-1}(v, v_0, \ldots, \hat{v}_i, \ldots, v_{q-1}). \end{aligned}$$

Each term in the sum having $v$ as a vertex will get mapped to zero. Thus the only remaining term is $d_{q-1}(v_0, \ldots v_{q-1}) = z$. Thus we have proven in any case that $\partial(d_q(z)) = z - d_{q-1}(\partial(z))$.

Now, given a $q$–cycle $z$ in $Z_q(K)$, $d_q(z)$ is a $(q+1)$–chain, and $\partial(d_q(z)) = z - d_{q-1}(\partial(z)) = z$. Thus every $q$–cycle is a bounding cycle. That is, $Z_q(K) = B_q(K)$, and so $H_q(K) = 0$, the trivial group.

■

Next, let $K$ be as above; an $n$–simplex with all of its faces. Let $K^{n-1}$ be the boundary of $K$, with all of its faces, again thought of as a simplicial complex. Thus $K^{n-1}$ is composed of all the $q$–simplexes of $K$ with dimension less than $n$. We will compute the $q$th–homology groups of $K^{n-1}$.

**Proposition 5** *For* $0 \leq q \leq n-1$, $H_q(K^{n-1}) = 0$.

PROOF. It follows from the above example, since for $0 \leq q \leq n-1$, $K$ and $K^{n-1}$ have the same $q$–simplexes.

■

**Proposition 6** $H_{n-1}(K^{n-1}) = \mathbf{Z}$.

PROOF. $K^{n-1}$ has no $n$–simplexes. Thus none of its $(n-1)$–cycles can be bounding cycles. Then $B_{n-1}(K^{n-1}) = 0$, and $H_{n-1}(K^{n-1}) = Z_{n-1}(K^{n-1})$. Further, since $K$ and $K^{n-1}$ have the same $(n-1)$–cycles, $Z_{n-1}(K^{n-1}) = Z_{n-1}(K)$. Now, from Proposition 4, $H_{n-1}(K) = 0$, and $Z_{n-1}(K) = B_{n-1}(K)$. But since there is only one $n$–simplex in $K$, $C_n(K)$ is generated by a single element. Thus $B_{n-1}(K)$ has a single generator. It follows that $H_{n-1}(K^{n-1})$ is infinite cyclic.

■

From here, our proof of Brouwer's theorem follows by arguing that $K$ and $K^{n-1}$ are triangulations of $\mathbf{B}^n$ and $\mathbf{S}^{n-1}$ respectively, and that homology groups are a homotopy invariant. Thus we can define $H_q(\mathbf{B}^n) = H_q(K)$, and $H_q(\mathbf{S}^{n-1}) = H_q(K^{n-1})$. We also need the following theorems.

**Theorem 2.1.10** *A continuous function* $f : |K| \to |L|$ *induces a homomorphism* $f_{q^\star} : H_q(K) \to H_q(L)$ *for each* $q$.

**Theorem 2.1.11** *If $f : |K| \to |K|$ is the identity, then $f_{q^\star} : H_q(K) \to H_q(K)$ is the identity homomorphism. Also, if $f : |K| \to |L|$, $g : |L| \to |M|$ are continuous, then $(g \circ f)_{q^\star} = g_{q^\star} \circ f_{q^\star}$.*

We give a brief discussion of the proof of Theorem 2.1.10. The details and the proof of Theorem 2.1.11 can be found in [1]. Theorem 2.1.10 is proven first for a class of mappings, called simplicial maps, that take simplexes of $K$ linearly into simplexes of $L$. Given $s : |K| \to |L|$, we define $s_q : C_q(K) \to C_q(L)$ naturally by $s_q(v_0, \dots, v_q) = (s(v_0), \dots, s(v_q))$, when each $s(v_i)$ is distinct, and zero otherwise. It is then shown that $s_q$ sends $q$–cycles and bounding $q$–cycles in $C_q(K)$ into respectively $q$–cycles and bounding $q$–cycles in $C_q(L)$. Thus, $s_q$ induces a homomorphism $s_{q^\star} : H_q(K) \to H_q(L)$. To extend this idea to any continuous function $f : |K| \to |L|$ it is shown that we can subdivide $K$ and $L$ into finer simplicial complexes for which we can find a simplicial map $s$ that is as close as we want to $f$.

With the above machinery, we can prove Brouwer's theorem. Suppose $f : \mathbf{B}^n \to \mathbf{B}^n$ is continuous, and fixed point free. Then, as is familiar by now, define $g : \mathbf{B}^n \to \mathbf{S}^{n-1}$ by mapping $x$ to the intersection of the ray extending from $f(x)$ through $x$, with $\mathbf{S}^{n-1}$. Then $g$ is a continuous function. Let $\iota : \mathbf{S}^{n-1} \to \mathbf{B}^n$ be the inclusion mapping, $\iota(x) = x$. Then by Theorem 2.1.10, both $g$ and $\iota$ induce homomorphisms, $g_\star$ and $\iota_\star$, of the $(n-1)$th homology groups.

$$H_{n-1}(\mathbf{S}^{n-1}) \xrightarrow{\iota_\star} H_{n-1}(\mathbf{B}^n) \xrightarrow{g_\star} H_{n-1}(\mathbf{S}^{n-1})$$

Now, since $g \circ \iota : \mathbf{S}^{n-1} \to \mathbf{S}^{n-1}$ is the identity, by Theorem 2.1.11, $G_\star \circ \iota_\star$ is the identity homomorphism. Then $g_\star$ must be onto. But from Proposition 4, $H_{n-1}(\mathbf{B}^n) = 0$, while Proposition 6 tells us $H_{n-1}(\mathbf{S}^{n-1}) = \mathbf{Z}$. Thus we have a contradiction, so $f$ must have had a fixed point.

## 2.2 Analytic Methods of Proof

From section 2.1.1, we could have proceeded to define the characteristic function of a nonvanishing vector field $f : \mathbf{S}^{n-1} \to \mathbf{S}^{n-1}$. Then using facts about the degree of this function, we would have obtained another classical result in topology, known as the Hairy ball theorem. Until the 1970's, both this result and Brouwer's theorem were proven using combinatorial arguments, homology theory, differential forms, or geometric topology. In 1978, John Milnor published self described "strange" proofs of these results that are nicely analytic in nature. This description prompted subsequent authors to attempt "less strange" versions of his proof, as can be seen in Rogers [48] and Gröger [32].

Milnor's proof of the standard change of variables formula, the Weierstrauss approximation theorem, and the observation that $(1+t^2)^{\frac{n}{2}}$ is not a polynomial in $t$ for odd $n$, to obtain a contradiction in a volume computation, and then prove the hairy ball theorem. Recently, Lax used a more sophisticated approximation technique, along with some standard results in single variable calculus, to prove an alternate change of variables formula in multiple integrals [40]. This new change of variables formula can be used to obtain the traditional one [41], and also has the advantage of yielding Brouwer's theorem as an almost immediate corollary. We end this section with a proof credited to Garcia that again uses the Weierstrauss approximation theorem, but invokes Green's theorem as its main machinery.

There are many other proofs of Brouwer that are analytic in nature. See, for example, Samelson [49], Kannai [37], Baez-Duarte [5], and Su [53].

### 2.2.1 The Hairy Ball Theorem

Milnor's proof is interesting not only because of its analytic nature, but because it follows from a calculation of volume in $\mathbf{R}^n$, and the fact that $(1+t^2)^{\frac{n}{2}}$ is not a polynomial for odd $n$. This version of the original proof can be found in [29].

**Lemma 2.2.1** *Let $f : A \to \mathbf{R}^n$ be continuously differentiable over a neighbourhood of the compact set $A$. Then there exists a Lipschitz constant $L$ such that for all $x$,$y$, $\in A$,*

$$\|f(x) - f(y)\| \leq L\|x - y\|.$$

Proof. Cover $A$ with a finite number of balls $U_1, U_2, \ldots, U_p$, such that $f$ is continuously differentiable on $\overline{U_k}$, $1 \leq k \leq p$. First we obtain a Lipschitz constant for $f$ on $U_k$. By continuity of the partials on $\overline{U_k}$, we can choose a constant $c_{ij}^k = \max_{i,j,k} \left\{ \frac{\partial f_i}{\partial x_j}(x) : x \in \overline{U_k} \right\}$. Using the triangle inequality, and the Mean Value property we obtain for $x$, $y \in U_k$,

$$\|f(x) - f(y)\| \leq \sum_{i=1}^{n} |f_i(x) - f_i(y)| \leq \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}^k \|x - y\| = L_k \|x - y\|,$$

where $L_k = \sum_{i,j=1}^{n} c_{ij}^k$.

Next we consider the set of $x,y \in A$ such that $x$ and $y$ are not both in one of the $U_k$. This set can be expressed as $W = (A \times A) \setminus \bigcup_{k=1}^{p} U_k \times U_k$. Consider the function $g : W \to \mathbf{R}^n$ defined by $g(x, y) = \|x - y\|$. This function is continuous over the compact set $W$, and thus achieves its minimum. Since $x \neq y$, we have $\min_{y\in W} \|x - y\| = \epsilon > 0$. So we obtain the following bound:

$$\|f(x) - f(y)\| \leq \epsilon^{-1}\text{diam}f(A)\|x - y\|.$$

Choose $L = \max\{L_1, L_2, \ldots, L_p, \epsilon^{-1}\text{diam}f(A)\}$, and we have obtained the desired Lipschitz constant.

■

**Lemma 2.2.2** *Suppose $A \subseteq \mathbf{R}^n$ is compact, and $v : A \to \mathbf{R}^n$ is continuously differentiable in a neighbourhood of $A$. Then there exists an interval $(-\epsilon, \epsilon)$ on which the function $t \mapsto |f_t(A)|$ is a polynomial.*

Proof. We will apply the Change of Variables formula to obtain the desired expression for the volume of $f_t(A)$. Thus we need to show that $f_t$ is one–to–one and continuously differentiable, and that $D_x f_x$ (the derivative of $f_t$ at $x$) is invertible for $x \in A$. To this end, by the previous lemma, let $L$ be the Lipschitz constant for $v$, and suppose $|t| < L^{-1}$. Then $f_t(x) - f_t(y)$ implies that $\|v(x) - v(y)\| = t^{-1}\|x - y\| \leq L\|x - y\|$. From out choice of $t$, this is possible only if $x = y$. Thus $f_t$ is one–to–one. Continuous differentiability of $f_t$ follows from that of its component parts.

Next, $D_x f_t = I + t\left[\dfrac{\partial v_i}{\partial x_j}(x)\right]$ has a strictly positive determinant for $t$ sufficiently small, say less than $K$. Set $\epsilon = \min\{K, L^{-1}\}$. The for $|t| < \epsilon$, and $x \in A$, we have $D_x f_t$ is invertible. Thus we may express the volume of $f_t(A)$ by

$$\text{vol}f_t(A) = \int_A |\det(D_x f_t)|dx.$$

We may write $\det(D_x f_t) = 1 + ta_1(x) + t^2 a_2(x) + \cdots + t^n a_n(x)$, where each $a_i$ is a continuous function. Upon integrating this expression over $A$, we obtain

$$\text{vol}f_t(A) = \text{vol}A + t\alpha_1 + t^2\alpha_2 + \cdots + t^n\alpha_n,$$

where $\alpha_i = \int_A a_i(x)dx$.

■

**Lemma 2.2.3** *Suppose $v : \mathbf{S}^{n-1} \to \mathbf{R}^n$ is a normed vector field tangent to $\mathbf{S}^{n-1}$, continuously differentiable on a neighbourhood of $\mathbf{S}^{n-1}$. Then for $t > 0$ small enough, the function $f_t : \mathbf{S}^{n-1} \to (1+t^2)^{\frac{1}{2}}\mathbf{S}^{n-1}$ is onto.*

PROOF. First we show the function is well defined. Indeed, for $x \in \mathbf{S}^{n-1}$, the norm of $f_t(x)$ can be computed by considering $\|f_t(x)\|^2 = \|x + tv(x)\|^2 = \langle x + tv(x), x + tv(x)\rangle$. From this we obtain $\|f_t(x)\| = \sqrt{1+t^2}$.

To show $f_t$ is onto, first define $A$ to be the set $A = \{x \in \mathbf{R}^n : \dfrac{1}{2} \leq \|x\|\dfrac{3}{2}\}$, and extend $f_t$ to all of $A$ by setting $v(x) = \|x\|v\left(\dfrac{x}{\|x\|}\right)$ for $x$ no on the unit sphere. Note that $v$ is still continuously differentiable, and so by Lemma 2.2.1 has a Lipschitz constant $L$ on $A$.

Fix $w \in \sqrt{1+t^2}\mathbf{S}^{n-1}$, and let $z \in \mathbf{S}^{n-1}$ be such that $w = \sqrt{1+t^2}z$. Then for $t < \min\{\frac{1}{3}, L^{-1}\}$, the function $g : x \mapsto z - tv(x)$ maps $A$ into $A$, and is a contraction mapping. Thus by the Banach Contraction Principle, $g$ has a fixed point, say $x$, in $A$. So $x = z - tv(x)$. Since $z \in \mathbf{S}^{n-1}$, we have $\|x + tv(x)\|^2 = 1$. Expanding the inner product $\langle x+tv(x), x+tv(x)\rangle$ gives $\|x\| = (1+t^2)^{\frac{1}{2}}$. Then $y = (1+t^2)^{\frac{1}{2}}x \in \mathbf{S}^{n-1}$, and $y + tv(y) = w$. Thus $f_t$ is onto.

■

**Theorem 2.2.4 (Hairy Ball Theorem - Weak Version)** *The sphere $\mathbf{S}^{2k}$ does not possess a continuously differentiable field of unit tangent vectors.*

PROOF. Suppose the contrary, and let $v : \mathbf{S}^{2k} \to \mathbf{R}^n$ be such a vector field. Select $0 < a < 1 < b$, and extend $v$ to the set $A = \{x \in \mathbf{R}^n : a \leq \|x\| \leq b\}$ by defining $v(x) = \|x\|v\left(\dfrac{x}{\|x\|}\right)$, as in Lemma 2.2.3. Note that for $x \in \mathbf{S}^{n-1}$, and $r > 0$, $F_t(rx) = rx + tv(rx) = rf_t(x)$. Thus by Lemma 2.2.3, $f_t$ maps the sphere $S_r$ of radius $r$ onto $(1+t^2)^{\frac{1}{2}}S_r$. Thus $f_t(A) = (1+t^2)^{\frac{1}{2}}A$ for small enough $t$, and so

$$|f_t(A)| = (1+t^2)^{2k+1}2|A|.$$

This contradicts Lemma 2.2.2 since the right hand side, $\sqrt{1+t^2}(1+t^2)^k|A|$, cannot be a polynomial.

■

**Theorem 2.2.5 (Hairy Ball Theorem - Strong Version)** *There does not exist a continuous nonzero vector field $v$ tangent to $\mathbf{S}^{2k}$.*

Proof. Suppose $v : \mathbf{S}^{2k} \to \mathbf{R}^n$ is such a vector field. Let $m = \min\{\|v(x)\| : x \in \mathbf{S}^{2k}\} > 0$. By the Weierstrass approximation theorem, each component $v_i$ of $v$ can be approximated by a polynomial $p_i : \mathbf{S}^{2k} \to \mathbf{R}$ such that $\|p(x) - v(x)\| < \frac{m}{2}$ for all $x \in \mathbf{S}^{2k}$. Then $p$ is $C^1$ and nonzero, since $\|p(x)\| \geq \|v(x)\| - \|p(x) - v(x)\| \geq m - \dfrac{m}{2} = \dfrac{m}{2}$. Using $p$, we obtain a $C^1$ nonzero vector field $q$ that is tangent to $\mathbf{S}^{2k}$. Define $q(x) = p(x) - \langle p(x), x\rangle x$. Then $q$ inherits its continuous differentiability from $p$, and

$$\begin{aligned}
\|q(x)\| &\geq \|p(x)\| - \|q(x) - p(x)\| \\
&> \frac{m}{2} - |\langle p(x), x\rangle| \\
&= \frac{m}{2} - |\langle p(x) - v(x), x\rangle| \\
&\geq \frac{m}{2} - \|p(x) - v(x)\| \\
&> 0.
\end{aligned}$$

The proof is complete upon observing that $\dfrac{q(\cdot)}{\|q(\cdot)\|}$ contradicts Theorem 2.2.4.

■

To proceed to the proof of Brouwer's Theorem we need one more piece of machinery. The unit ball in $\mathbf{R}^n$ can be projected onto the lower hemisphere of the unit sphere in $\mathbf{R}^{n+1}$ with a stereographic projection. Write $\mathbf{R}^{n+1} = \{(x, x_{n+1}) : x \in \mathbf{R}^n, x_{n+1} \in \mathbf{R}\}$. Then for $x \in \mathbf{R}^n$ this projection is defined by

$$S_+(x) = \left(\frac{2x}{\|x\|^2 + 1}, \frac{\|x\|^2 - 1}{\|x\|^2 + 1}\right).$$

Hence $S_+(x)$ is the intersection with $\mathbf{S}^n$ of the ray starting at $(0, 1) \in \mathbf{S}^n$ and passing through $x \in \mathbf{B}^n$. Clearly, this mapping is $C^1$. The image of $\mathbf{B}^n$ under $S_+$ is the lower hemisphere of $\mathbf{S}^n$, which we denote by $S^n_-$.

Similarly, we can define the projection $S_-$ from $(0, -1)$ of $\mathbf{B}^n$ onto $S^n_+$, the upper hemisphere of $\mathbf{S}^n$.

$$S_-(x) = \left(\frac{2x}{\|x\|^2 + 1}, \frac{1 - \|x\|^2}{1 + \|x\|^2}\right).$$

Now we are ready to prove Brouwer's theorem.

Let $f : \mathbf{B}^{2k} \to \mathbf{B}^{2k}$ be a continuous function, and suppose $f$ leaves no point fixed. We will use $f$ to define a nonzero $C^1$ vector field on $\mathbf{B}^{2k}$ that points directly outward at points on $\mathbf{S}^{2k-1}$.

Suppose $F$ is such a vector field. For each $x \in \mathbf{B}^n$, $\{x + tF(x) : 0 \leq t \leq 1\}$ will be a segment in $\mathbf{R}^n$. The image of this set under $S_+$ will be an arc on $\mathbf{S}^n$ with initial point $S_+(x)$ lying in $S^n_-$. On $S^{2k}_-$ we can define a continuous nonzero field of tangent vectors by setting

$$T_-(y) = \frac{d}{dt} S_+(x + tF(x))|_{t=0},$$

for each $y \in \mathbf{S}^{2k}$, where $y = S_+(x)$. $T_-(y)$ is the tangent of the arc $S_+(x+tF(x) : 0 \leq t \leq 1)$. Note that since $F$ points outward on $S^{2k-1}$, the projections of the segment $\{x + tF(x) : 0 \leq t \leq 1\}$ will be "vertical", and so the tangent of the projected arc will be $(\overline{\mathbf{0}}, 1)$.

Similarly, we can define $T_+(y) = \frac{d}{dt} S_-(x + tF(x))|_{t=0}$ for $y \in S^{2k}_+$. Again, for $y$ on the equator, $T_+(y) = (0, 1)$.

Now we define $T : \mathbf{S}^{2k} \to \mathbf{R}^{n+1}$ by

$$T(y) = \begin{cases} T_-(y) & \text{for } y \in S^{2k}_-, \\ T_+(y) & \text{for } y \in S^{2k}_+. \end{cases}$$

Then $T$ is a continuous nonvanishing vector field tangent to $\mathbf{S}^{2k}$. This contradicts the Hairy ball theorem.

The required field $F : \mathbf{B}^{2k} \to \mathbf{R}^n$ can be defined by

$$F(x) = x - \left( \frac{1 - \langle x, x \rangle}{1 - \langle x, f(x) \rangle} \right) f(x).$$

Clearly $F$ is $C^1$, and points outward for $x \in \mathbf{S}^{2k-1}$. To see that $F$ is nonzero, suppose the contrary. Then for some $x$, $f_x$ is a scalar multiple of $x$, and thus $\langle x, f(x) \rangle x = \langle x, x \rangle f(x)$. It follows from the definition of $F$ that $x = f(x)$, contradicting the hypothesis that $f$ has no fixed points.

To complete the proof, suppose $f : \mathbf{B}^{2k-1} \to \mathbf{B}^{2k-1}$ is continuous and fixed point free. Then so is the function $g : \mathbf{B}^{2k} \to \mathbf{B}^{2k}$ defined by $(x, x_{2k}) \mapsto (f(x), 0)$, contradicting the above.

### 2.2.2 An Alternate Change of Variables Formula

The standard change of variables formula in multiple integrals states that

$$\int_{S_1} f(x)dx = \int_S (f \circ g)(x)|\det \nabla g(x)|dx,$$

where $S_1, S \subseteq \mathbf{R}^n$ are open , $f : S_1 \to \mathbf{R}$ is continuous, and $g : S \to S_1$ is a one–to–one map such that both $g$ and $g^{-1}$ are continuously differentiable. In this section we present a recent proof by Peter Lax [40] of a modified version of this theorem that yields Brouwer's theorem as a nice corollary.

In what follows, except when explicitly stated, the function $g : \mathbf{R}^n \to \mathbf{R}^n$ is twice differentiable and equal to the identity function outside some sphere, say $\mathbf{S}_r^{n-1}$, the sphere centred at the origin, with radius $r$. The function $f : \mathbf{R}^n \to \mathbf{R}$ is differentiable, with compact support. The fixed constant $c > r$ is chosen such that $f$ is zero outside the $c$–cube $\{x \in \mathbf{R}^n : -c \leq x_i \leq c\}$. Also the function $h : \mathbf{R}^n \to \mathbf{R}$ is defined by

$$h(x_1, x_2, \ldots, x_n) = \int_{-\infty}^{x_1} f(z, x_2, \ldots, x_n)dz.$$

We note that $\frac{\partial h}{\partial x_1} = f$ by the fundamental theorem of calculus, and differentiability of $h$ follows from that of $f$.

**Lemma 2.2.6**

$$(f \circ g) \det \nabla g = \det (\nabla(h \circ g), \nabla g_2, \ldots \nabla g_n). \tag{2.2}$$

Proof. By the chain rule,

$$\nabla(h \circ g) = \nabla h \circ \nabla g = \sum_{i=1}^{n} \frac{\partial h}{\partial x_i}(g) \nabla g_i.$$

Thus the first column of the right hand side of (2.2) can be written as a linear combination of the remaining $n-1$ columns. For $2 \leq i \leq n$, subtract $\dfrac{\partial h}{\partial x_i}(g)\nabla g_i$ from the first column of the determinant, and then factor out the scalar $\dfrac{\partial h}{\partial x_1}(g)$. We are left with $\dfrac{\partial h}{\partial x_1}(g) \det (\nabla g_1, \nabla g_2, \ldots \nabla g_n)$. From the definition of $h$, we see this is exactly $(f \circ g) \det \nabla g$. ■

We state the following classical identity without proof.

**Lemma 2.2.7** *Let* $\{M_i\}_{i=1}^n$ *be the set of cofactors obtained by expanding the Jacobian determinant* $\det \nabla g$ *along its first column. Then* $\sum_{i=1}^n \frac{\partial}{\partial x_i} M_i \equiv 0$.

The following is a preliminary version of Lax's change of variables formula.

**Theorem 2.2.8** *Let* $f$ *and* $g$ *be stated as above. Then*

$$\int f(x)dx = \int (f \circ g)(x) \det \nabla g(x) dx.$$

PROOF. Since $f \equiv 0$ outside the $c$–cube $\{x \in \mathbf{R}^n : -c \leq x_i \leq c\}$ it is sufficient to restrict our integration to this set. We will use Lemma 2.2.6, and show $\int f(x)dx$ is equal to the integral of $\det(\nabla h \circ g, \nabla g_2, \ldots, \nabla g_n)$. To this end, expand the determinant along its first column. The integrand becomes

$$\int \frac{\partial}{\partial x_1}(h \circ g) M_1 + \cdots + \frac{\partial}{\partial x_n}(h \circ g) M_n.$$

Since $g$ is twice differentiable, each of its partials are continuous. Thus the cofactors $M_i$ are continuous, and hence each term has finite integral over the $c$–cube. Thus we may do the integration term by term. We will use the notation

$$\begin{aligned} dx/dx_i &:= dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_n, \\ c_i &:= (x_1, \ldots, x_{i-1}, c, x_{i+1}, \ldots, x_n). \end{aligned}$$

Then using Fubini's theorem, we have

$$\int M_i(x) \frac{\partial}{\partial x_i}(h \circ g)(x) dx = \int \left( \int_{x_i=-c}^{x_i=c} M_i(x) \frac{\partial}{\partial x_i}(h \circ g)(x) dx_i \right) dx/dx_i.$$

The integration by parts formula can be applied to the inner one–dimensional integral to obtain

$$\int (h \circ g)(x) M_i(x) |_{x_i=-c}^{x_i=c} dx/dx_i - \int (h \circ g)(x) \frac{\partial}{\partial x_i} M_i(x) dx. \tag{2.3}$$

Recall that $g$ is the identity outside the $c$–cube. Thus $h \circ g(c_i) = h(c_i)$, which is zero for $i \geq 2$. Similarly, $(h \circ g)(-c_i) = 0$, for all $i$. Thus the first term in (2.3) is zero unless $i = 1$, and in that case the integral is $\int (h \circ g)(c_1) dx/dx_1$, since $M_1(c_1) = 1$.

Now summing (2.3) over $1 \leq i \leq n$, we obtain

$$\int (h \circ g)(c_1) dx/dx_1 - \int (h \circ g)(x) \sum_{i=1}^n \frac{\partial}{\partial x_i} M_i(x) dx.$$

Lemma 2.2.7 says the second term is zero. In the first term, $(h \circ g)(c_1) = h(c_1) = \int_{-\infty}^{c} f(z, x_2, \ldots, x_n)dz$.

Thus what remains is exactly $\int f(x)dx$, and we are done.

■

We now extend the theorem to functions $f$ that are continuous with compact support, and $g$ that are differentiable, and equal to the identity outside $\mathbf{S}_r^{n-1}$. In the following lemmata, $\phi_\epsilon : \mathbf{R}^n \to \mathbf{R}$ is a smooth nonnegative spherically symmetric function whose support lies in $B_\epsilon$, and with $\int_{\mathbf{R}} \phi_\epsilon dy = 1$. Also, the convolution of $\phi_\epsilon$ with a function $f$ is defined by

$$(\phi_\epsilon * f)(x) = \int_{B_\epsilon} \phi_\epsilon(y) f(x-y) dy.$$

**Lemma 2.2.9** *Let $f : \mathbf{R}^n \to \mathbf{R}$ be continuous with compact support. Define $f^\epsilon = \phi_\epsilon * f$. Then $f^\epsilon$ is differentiable, with compact support, and $f^\epsilon \to f$ uniformly.*

Proof. We omit the proof that $f^\epsilon$ is differentiable. For references, see [22]. The function $f^\epsilon$ has compact support. Let the support of $f$ be contained in $B_r$. Then for $y \in B_\epsilon$, $f(x-y)$ will be zero for $x$ outside $B_{r+\epsilon}$. To see that $f^\epsilon \to f$ uniformly, let $\epsilon' > 0$ be given. By uniform continuity of $f$ on $B_{r+\epsilon'}$, fix $\delta > 0$ so that $|y| < \delta$ implies $|f(x-y) - f(x)| < \epsilon'$. Choose $\epsilon < \min\{\delta, \epsilon'\}$. Then for any $x \in \mathbf{R}^n$, we have

$$\begin{aligned}
|f^\epsilon(x) - f(x)| &= \left| \int_{B_\epsilon} \phi_\epsilon(y) f(x-y) dy - f(x) \right| \\
&= \left| \int_{B_\epsilon} \phi_\epsilon(y)(f(x-y) - f(x)) dy \right| \\
&\leq \int_{B_\epsilon} \phi_\epsilon(y) |f(x-y) - f(x)| dy \\
&< \epsilon' \int_{B_\epsilon} \phi_\epsilon(y) dy = \epsilon'.
\end{aligned}$$

■

**Lemma 2.2.10** *Let $g_i : \mathbf{R}^n \to \mathbf{R}$ be differentiable with $1 \leq i \leq n$. Suppose $g_i(x) = x_i$ for $x$ outside $\mathbf{S}_r^{n-1}$. Then $g_i^\epsilon = \phi_\epsilon * g_i$ is twice differentiable, and $g_i^\epsilon(x) = x_i$ for $x$ outside $\mathbf{S}_{r+\epsilon}^{n-1}$. Further, $g_i^\epsilon \to g_i$ uniformly.*

PROOF. Again we refer to [22] for the proof of twice differentiability. Let $\epsilon' > 0$ be given. By uniform continuity of $g_i^\epsilon$ on $B_{r+2\epsilon'}$, fix $\delta > 0$ so that $|x-y| < \delta$ gives $|g_i(x) - g_i(y)| < \epsilon'$. Choose $\epsilon < \min\{\delta, \epsilon'\}$. Then for $x \in \mathbf{R}^n$, we have the following.

$$\begin{aligned} |g_i^\epsilon(x) - g_i(x)| &= \left| \int_{B_\epsilon} \phi_\epsilon(y) g_i(x-y) dy - g_i(x) \right| \\ &= \left| \int_{B_\epsilon} \phi_\epsilon(y)(g_i(x-y) - g_i(x)) dy \right| \end{aligned}$$

If $|x| > r + \epsilon$, then $|x - y| > r$, and we have $g_i(x-y) - g_i(x) = -y_i$. Thus the above becomes

$$|g_i^\epsilon(x) - g_i(x)| = \left| \int_{B_\epsilon} \phi_\epsilon(y) y_i dy \right| = 0,$$

because $\phi_\epsilon$ is spherically symmetric.

If $|x| < r + \epsilon$, then $|x - y| \leq r + 2\epsilon$. Since $\epsilon$ was chosen smaller than $\delta$, we may continue from above with

$$\begin{aligned} |g_i^\epsilon(x) - g_i(x)| &\leq \int_{B_\epsilon} \phi_\epsilon(y) |g_i(x-y) - g_i(x)| dy \\ &\leq \epsilon' \int_{B_\epsilon} \phi_\epsilon(y) dy = \epsilon'. \end{aligned}$$

Thus $g_i^\epsilon \to g_i$ uniformly. From the proof we also see that $g_i^\epsilon(x) = x_i$ for $|X| \geq r + \epsilon$.

■

Now suppose that $f : \mathbf{R}^n \to \mathbf{R}$ is continuous, with compact support, and $g : \mathbf{R}^n \to \mathbf{R}^n$ is differentiable and equal to the identity outside $\mathbf{S}_r^{n-1}$. Applying Lemma 2.2.10 to each component of $g$ gives us a twice differentiable function $g^\epsilon = (g_1^\epsilon, g_2^\epsilon, \ldots g_n^\epsilon)$ that is equal to the identity outside the sphere of radius $r + \epsilon$.

Applying Theorem 2.2.8 to $f^\epsilon$ and $g^\epsilon$, we get

$$\int (f^\epsilon \circ g^\epsilon)(x) \det \nabla g^\epsilon(x) dx = \int f^\epsilon(x) dx.$$

Since $f^\epsilon \to f$ and $g^\epsilon \to g$ uniformly, we may let $\epsilon \to 0$ to obtain the desired result. We have the following theorem.

**Theorem 2.2.11 (Change of Variables)** *Let $f : \mathbf{R}^n \to \mathbf{R}$ be continuous with compact support, and $g : \mathbf{R}^n \to \mathbf{R}^n$ be a continuous map that is the identity outside $\mathbf{S}_r^{n-1}$, for some $r > 0$. Then*

$$\int f(x) dx = \int (f \circ g)(x) \det \nabla g(x) dx.$$

We remark that this change of variables formula was proven without requiring the absolute value of the Jacobian determinant of $g$ in the integrand. This is consistent with the standard version of the formula. It can be shown that if $g$ is differentiable and the identity outside the unit sphere, then $\det \nabla g > 0$.

From the following corollaries we deduce Brouwer's theorem.

**Corollary 1** *Let $g : \mathbf{B}^n \to \mathbf{R}^n$ be differentiable and equal to the identity function outside $\mathbf{S}^{n-1}$. Then $g$ is onto.*

PROOF. First note that since $\mathbf{B}^n$ is compact, and $g$ is continuous, then the image $g(\mathbf{B}^n)$ is closed. Suppose $g$ is not onto, and pick $y$ not in $g(\mathbf{B}^n)$. Since $g$ is the identity outside $\mathbf{S}^{n-1}$, we know $y \in \mathbf{B}^n_r$. By closedness of $g(\mathbf{B}^n)$, there must be a neighbourhood of $y$ also not in $g(\mathbf{B}^n)$. Let $B_0$ be a ball contained in this neighbourhood, centred at $y$. Choose $c > 0$ less than the radius of $B_0$. Let $f : \mathbf{R}^n \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} c - \|x - y\|, & \text{if } \|x - y\| \leq c, \\ 0, & \text{otherwise.} \end{cases}$$

Then $f$ is continuous and nonnegative. Thus $\int f(x)dx > 0$. However, by Theorem 2.2.11, $\int f(x)dx = \int (f \circ g)(x) \det \nabla g(x)dx$. Since the support of $f$ lies inside $B_0$, and $B_0$ is not the image of $g$, this integral must equal zero.

■

**Corollary 2** *Let $g : \mathbf{B}^n \to \mathbf{R}^n$ be continuous and equal to the identity function on $\mathbf{S}^{n-1}$. Then $g(\mathbf{B}^n)$ contains $\mathbf{B}^n$.*

PROOF. Extend $g$ to be the identity map outside $\mathbf{B}^n$. Using the same process as above, approximate $g$ by a sequence of differentiable functions $g^\epsilon$ that are the identity outside $\mathbf{S}^{n-1}_{1+\epsilon}$. By corollary 1, each of these is onto, and so $\mathbf{B}^n \subseteq g^\epsilon(\mathbf{B}^n)$. Since $g^\epsilon(\mathbf{B}^n) \to g(\mathbf{B}^n)$, and $g(\mathbf{B}^n)$ is compact, we must have $\mathbf{B}^n \subseteq g(\mathbf{B}^n)$.

■

Brouwer's theorem now follows by a standard argument. Suppose $f : \mathbf{B}^n \to \mathbf{B}^n$ is fixed point free. Define $g : \mathbf{B}^n \to \mathbf{S}^{n-1}$ by setting $g(x)$ to be the intersection of the ray from $f(x)$

through $x$, with the sphere. Formally, define $g(x) = x + \alpha(x)(x - f(x))$, where

$$\alpha(x) = \frac{\langle x, f(x)\rangle - \|x\|^2 + \sqrt{(\langle x, f(x)\rangle - \|x\|^2)^2 - \|x - f(x)\|^2(\|x\|^2 - 1)}}{\|x - f(x)\|^2}.$$

Since $x \neq f(x)$, $g$ is continuous. Then we have defined a continuous function $g$ on $\mathbf{B}^n$ that is the identity on the sphere, whose image lies strictly in $\mathbf{S}^{n-1}$, violating corollary 2.

### 2.2.3 Garcia's Proof

The following proof was found in a text on mathematical economics [27]. In the text, the author relates an anecdote in which he complains to a colleague that there is no simple proof of Brouwer's theorem. The colleague, Adriano Garcia, responded by giving the following proof, based on Green's theorem. Much of the work done in this section is needed to define a rather tedious function, which Garcia refers to as the discriminant.

#### The Existence of the Discriminant

We need the discriminant function $\delta$ to be real valued, and to act on the set of continuous functions $f : \mathbf{R}^n \to \mathbf{R}^n$. To begin, let $A$ be a neighbourhood of $\mathbf{S}^{n-1}$, and let $Z$ be the set of twice continuously differentiable functions $f : A \to \mathbf{R}^n$ such that $\|f(x)\| = 1$ on $\mathbf{S}^{n-1}$. Recall that the Jacobian of a function $f$, is the matrix of partial derivatives, the $i$th row being the derivative of the $i$th component of $f$. Next, let $J_i(f)$ denote the Jacobian matrix of $f$, with the $i$th column replaced with $f = (f_1, \ldots, f_n)$, and define the operator $G$ on the set $Z$ by

$$G(f) = (\det J_1(f), \det J_2(f), \ldots, \det J_n(f)).$$

Now, on $Z$ we define the real valued function $\delta$ by

$$\delta(f) = \int_{\mathbf{S}^{n-1}} G(f) \cdot \mathbf{n}\, dS.$$

The function $\delta$ has the following properties.

**Property 2.2.12** *Let $f \in Z$, and suppose $f$ has a normed continuously twice differentiable extension to all of $\mathbf{B}^n$. Then $\delta(f) = 0$.*

**Property 2.2.13** *Let $f$,$g \in Z$. If $f(x) + g(x) \neq 0$ for all $x \in \mathbf{S}^{n-1}$, then $\delta(f) = \delta(g)$.*

To prove these facts we will have to know more about the behaviour of the operator $G$.

**Lemma 2.2.14** *Let $f : \mathbf{R}^n \to \mathbf{B}^n$ be twice continuously differentiable in a neighbourhood of $\mathbf{S}^{n-1}$. Then $\nabla \cdot G(f) = n \det J(f)$.*

PROOF. Recall that

$$\partial_k(detJ_k) = \sum_{m=1}^{n} \det(J_{km}),$$

where $J_{km}$ is the matrix $J_k$ with $\partial_k$ applied to its $mth$ column. Note since $\partial_k\partial_m f = \partial_m\partial_k f$, we see that $J_{km}$ and $J_{mk}$ will be the same but with columns $m$ and $k$ interchanged. Thus $\det J_{km} = -\det J_{mk}$. Also it is clear that $J_{kk} = J(f)$. Then we obtain

$$\begin{aligned} \nabla \cdot (G(f)) &= \sum_{k=1}^{n} \partial_k(\det J_k) \\ &= \sum_{k=1}^{n} \sum m = 1^n \det J_{km} \\ &= n \det J(f). \end{aligned}$$

■

**Lemma 2.2.15** *If $f$ is twice continuously differentiable with $\|f(x)\| = 1$, then $\nabla \cdot G(f) = 0$.*

PROOF. This follows from the previous lemma by noting that for such an $f$, and for $0 \leq i \leq n$, we have $\partial_i |f|^2 = 2f \cdot \partial_i f = 0$. Thus each of the $n$ columns of the Jacobian of $f$ are orthogonal to $f$, and so are linearly dependent. Hence $\det J(f) = 0$.

■

We can now deduce property 2.2.12 by applying Green's theorem to $\delta(f)$.

$$\delta(f) = \int_{\mathbf{S}^{n-1}} G(f) \cdot \mathbf{n} \, dS = \int_{\mathbf{B}^n} \nabla \cdot G(f) \, dV = 0$$

Let $f$,$g \in Z$, and suppose $f(x) + g(x) \neq 0$ on $\mathbf{S}^{n-1}$. Fix $\epsilon > 0$ such that $f$ and $g$ are defined for $1 - \epsilon < |x| < 1 + \epsilon$. Set $1 - \epsilon < \alpha < \beta < 1$. Let $\phi : \mathbf{R} \to [0, 1]$ be a twice continuously differentiable function defined on $[1 - \epsilon, 1 + \epsilon]$ such that $\phi(x) = 0$ on $[1 - \epsilon, \alpha]$, and $\phi(x) = 1$ on $[\beta, 1 + \epsilon]$. Now we take the convex combination $(1 - \phi(|x|)f(x) + \phi(|x|)g(x)$ of $f$ and $g$, and call its normalization $h(x)$. Then $h$ must also be in the set $Z$. Thus $\nabla \cdot G(h) = 0$. Fix $\gamma \in (1 - \epsilon, \alpha)$.

$$\int_{\gamma<|x|<1} \nabla \cdot G(h)\, dV = \int_{\gamma<|x|<1} \nabla \cdot G(f)\, dV$$

From which we obtain, after applying Green's theorem,

$$\int_{|x|=1} G(h) \cdot \mathbf{n}\, dS - \int_{|x|=\gamma} G(h) \cdot \mathbf{n}\, dS = \int_{|x|=1} G(f) \cdot \mathbf{n}\, dS - \int_{|x|=\gamma} G(f) \cdot \mathbf{n}\, dS.$$

But $h = f$ on $|x| = \gamma$, and $h = g$ on $|x| = 1$. From the definition of $\delta$ we obtain the result. That is, $\delta(f) = \delta(g)$.

To reach our goal of using $\delta$ to prove Brouwer's theorem we shall need to extend our definition to an arbitrary function $f$ that is continuous and nonzero on $\mathbf{S}^{n-1}$.

**Definition 2.2.16** *Let $f$ be continuous and nonzero on $\mathbf{S}^{n-1}$. Let $g \in Z$ be such that $f(x) \cdot g(x) > 0$ for all $x \in \mathbf{S}^{n-1}$. Then we define $\delta(f) = \delta(g)$.*

It is clear that $\delta$ is well defined. If $g_1$ and $g_2$ are two functions in $Z$ whose inner products with $f$ are positive on the unit sphere, then so too is $(g_1(x) + g_2(x)) \cdot f(x) > 0$, and so $g_1(x) + g_2(x) \neq 0$. By property 2.2.13, $\delta(g_1) = \delta(g_2)$. It should also be evident that such a function $g$ actually exists. Given $f$, we can extend $f$ to a neighbourhood of $\mathbf{S}^{n-1}$ by setting

$$f(x) = |x| f\left(\frac{x}{|x|}\right).$$

Then $f$ can be approximated by twice continuously differentiable functions using the Weierstrauss Approximation Theorem. Normalizing the resultant function yields a member of $Z$ with the desired property.

With the function $\delta$ defined, deducing Brouwer's theorem is fairly straightforward. Let $f : \mathbf{B}^n \to \mathbf{B}^n$ be continuous. Define $h(x) = x - f(x)$.

Now, let $x \in \mathbf{S}^{n-1}$, and suppose $(1-t)x + th(x) = 0$ for some $t \in (0,1)$. Then taking the inner product with $x$ tells us $(1-t) + t(x \cdot h(x)) = 0$. That is,

$$h(x) \cdot x = \frac{t-1}{t}, \text{ for some } t \in (0,1].$$

But $x \cdot h(x) = x \cdot (x - f(x)) = 1 - x \cdot f(x) > 0$. This contradiction tells us that $x$ and $h(x)$ nowhere point in opposite directions on the unit sphere. Hence by property 2.2.13, $\delta[h] = \delta[id] \neq 0$. Then by property 2.2.12, any continuous extension of $h$ to the whole of the unit ball must vanish somewhere. That is, $f$ must have a fixed point.

# Chapter 3

# Generalizations

In the past century Brouwer's theorem has seen many generalizations. The literature is vast; in this chapter we outline some of the major results. The first extensions were to specific function spaces. In 1922 Birkhoff and Kellogg showed that compact convex subsets of $C^n[0,1]$ and $L_2[0,1]$ have the fixed point property [7]. Cacciopoli showed the same is true in $C[0,1]$ [17]. Schauder made the leap to Banach spaces, and Tychonoff extended this further to locally convex topological vector spaces. The goal of these earlier efforts was to generalize the space $\mathbf{E}$ in which the theorem is set. Following this, results were proven for varying conditions on all three components of the fixed point theorem: the space $\mathbf{E}$, the set $X$, and the function $f : X \to X$. Many results were obtained that relaxed requirements on one or two of these components while asking more of the others. Some theorems based on certain boundary conditions do not require that $f$ be a self–map. Throughout most of the development, containment of the image set $f(X)$ in a compact set is necessary. However in the 50's and 60's, it was shown that we merely require the image to be somehow closer to compact than $X$ itself. At some points we pause from our discussion of what types of sets have the topological fixed point property to consider some results on classes of functions for which closed, bounded, convex sets that the fixed point property. The generalizations we present here will be given in the following template as to facilitate comparison.

**Theorem 3.0.17** *Let $X$ be a subset of a space $\mathbf{E}$, and $f : X \to \mathbf{E}$.*

1. *set:*
2. *space:*
3. *function:*

*Then [conclusion].*

## 3.1 Extensions to Infinite Dimensions

Having proved Brouwer's theorem, the fact that a nonempty compact convex set in Banach Space has the topological fixed point property is easy to acquire. Whereas Brouwer's theorem has been derived through a variety of approaches, as described in the second chapter, the proofs of the infinite dimensional version are generally variations on a theme. We start with a continuous function on a compact convex set, and approximate this set with one that can be embedded in $\mathbf{R}^n$. Then Brouwer's theorem is applied to get a fixed point $x_n$. Letting $n \to \infty$ gives us a sequence $\{x_n\}$ in our original space, whose limit point turns out to be the desired fixed point. The different proofs in the literature vary in the method by which the approximation to finite dimensions is done.

### 3.1.1 Schauder's Fixed Point Theorem

**Theorem 3.1.1 (Schauder 1927)** *Let $X$ be a subset of $\mathbf{E}$, and $f : X \to \mathbf{E}$.*

1. *$X$ : compact, convex.*
2. *$\mathbf{E}$ : Banach space.*
3. *$f$ : continuous self–map.*

*Then $f$ has a fixed point in $X$.*

Proof. Suppose $f : X \to X$ is a continuous function, and fix $\epsilon > 0$. We will construct a function $g : X \to X$ that almost fixes every point of $X$. To start, by compactness of $X$, there exists a finite $\epsilon$–net, $\{a_1, \ldots, a_n\}$ in $\mathbf{E}$, that covers $X$. For each $i \in \{1, \ldots, n\}$, we define the following function.

$$m_i(x) = \begin{cases} \epsilon - |x - a_i|, & |x - a_i| \leq \epsilon \\ 0, & |x - a_i| \geq \epsilon. \end{cases}$$

Note that for each $x \in X$, there exists an $i$ such that $m_i(x)$ is nonzero. Thus we can define $g : X \to X$ by

$$g(x) = \frac{\sum_{i=1}^{n} m_i(x) a_i}{\sum_{i=1}^{n} m_i(x)}.$$

For any $x \in X$, we have

$$
\begin{aligned}
|g(x)-x| &= \left|\frac{\sum_{i=1}^n m_i(x)a_i}{\sum_{i=1}^n m_i(x)} - x\right| \\
&= \left|\frac{\sum_{i=1}^n m_i(x)(a_i - x)}{\sum_{i=1}^n m_i(x)}\right| \\
&\leq \epsilon.
\end{aligned}
$$

Let $X_n = \overline{\text{conv}}\{a_1, \ldots, a_n\}$, and consider $(g \circ f) : X_n \to X_n$. As it is the convex hull of a finite set of points, $X_n$ can be embedded in $\mathbf{R}^n$. Thus by Brouwer's theorem, there is a point $x_n$ that is fixed by $g \circ f$.

$$
\begin{aligned}
|x_n - f(x_n)| &\leq |x_n - g \circ f(x_n)| + |g \circ f(x_n) - f(x_n)| \\
&= |g \circ f(x_n) - f(x_n)| \\
&\leq \epsilon
\end{aligned}
$$

The sequence $\{x_n\}$ is contained in $X$, a compact set. Thus we have a convergent subsequence $\{x_{n_k}\}$, with limit $x$. By completeness, $x$ is in $X$. Continuity of $f$ tells us that $f(x_{n_k})$ converges to $f(x)$. Now the above inequality tells us $f(x) = x$.

■

The following, due to Kakutani, illustrates the necessity of compactness in Schauder's theorem. We cannot merely require $X$ to be closed and bounded.

**Example 2** *Let $\ell_2(\mathbf{Z})$ be the space of functions on $\mathbf{Z}$, with norm given by $|x| = \max_{\mathbf{Z}}(|x_n|)$, where $x_n$ is the nth coordinate of $x$.*

Consider the unit ball in $\ell_2(\mathbf{Z})$. If we write $b^n$ to be the element in this space that is 1 in the $n$th coordinate, and zero everywhere else, for any $x \in \ell_2(\mathbf{Z})$ we can write

$$x = \sum_{\mathbf{n} \in \mathbf{Z}} x_n b^n.$$

Define $R : \ell_2(\mathbf{Z}) \to \ell_2(\mathbf{Z})$ by $(R(x))_n = x_{n-1}$ for each integer $n$. Then $R$ is the right-shift operator, and is clearly continuous. Next, define $f : \mathbf{B} \to \mathbf{B}$ by $f(x) = (1 - |x|)b^0 + R(x)$.

Continuity of $f$ follows from the continuity of the norm, and the continuity of $R$. To see that $f$ is indeed a self–map of the unit ball, let $x \in \mathbf{B}$. Then we have

$$\begin{aligned} |f(x)| &= |(1-|x|)b^0 + R(x)| \\ &\leq |(1-|x|)b^0| + |R(x)| \\ &= |1-|x|| + |x| \\ &= 1. \end{aligned}$$

Now suppose $f$ has a fixed point. Then there is an $x$ such that $x - R(x) = (1-|x|)b^0$. But the left hand side can be a multiple of $b^0$ only when $x = 0$. In this case we get $0 = x - R(x) = b^0$, a contradiction. Thus $f$ is a fixed point free mapping of the unit ball in $\ell_2(\mathbf{Z})$.

The above result was actually Schauder's first extension of Brouwer's theorem to infinite dimensional space. He also proved the theorem for $X$ closed and convex, but with $f$ a compact mapping. Clearly a continuous mapping on a compact set is compact, but if $f : X \to X$ is compact, we need not have compactness of $X$. For instance, the retraction of the open unit ball in $\mathbf{R}^2$ onto the origin is a compact map. This result is sometimes called "Schauder's second theorem".

### 3.1.2 Tychonoff's Fixed Point Theorem

In 1935 Tychonoff proved the following extension of Brouwer's theorem.

**Theorem 3.1.2** *Let $X$ be a subset of a space $\mathbf{E}$, and $f : X \to \mathbf{E}$.*

1. *$X$ : compact, convex.*
2. *$\mathbf{E}$ : locally convex topological vector space.*
3. *$f$ : continuous self–map.*

*Then $f$ has a fixed point in $X$.*

Recall that given two open covers $U = \{U_\alpha\}_{\alpha \in A}$, $V = \{V_\beta\}_{\beta \in B}$ of a space $X$, the second is a refinement of the first if each member $V_\beta$ is contained in some $U_\alpha$. Also, $U$ is called nbd-finite provided for each $x \in X$, there is a neighbourhood of $x$ that intersects only a finite number of the $U_\alpha$. A Hausdorff space $X$ is called paracompact when every open cover of $X$ admits a nbd-finite refinement. Given $Y \subset X$, the star of $Y$ with respect to the cover $U$ is the set $St(Y, U) = \bigcup\{U_\alpha : Y \cap U_\alpha \neq \emptyset\}$. A covering $V$ is called a star–refinement of $U$

provided the cover $\{St(V_\beta, V)\}_{\beta \in B}$ is a refinement of $U$. We remark that every open cover of a paracompact space admits an open star–refinement.

**Lemma 3.1.3** *Let* $\mathbf{E}$ *be a Hausdorff topological space,* $X \subset \mathbf{E}$*, and* $f : X \to X$ *continuous. Then* $f$ *has a fixed point if and only if for each open cover* $\{U_\alpha\}_{\alpha \in A}$ *of* $X$*, there exists* $x \in X$ *and* $U_\alpha$ *such that both* $x$ *and* $f(x)$ *lie in* $U_\alpha$*.*

Proof. The "only if" part if the implication is certainly clear. Next assume that for every open cover $\{U_\alpha\}$ of $X$ there is an $x \in X$ and $\alpha$ such that both $x$ and $f(X)$ lie in $U_\alpha$. Suppose $f : X \to X$ is continuous with no fixed points. Since $\mathbf{E}$ is Hausdorff, we can find open neighbourhoods $W_x$ and $V_{f(x)}$ that separate $x$ and $f(x)$. Then the open cover $\{W_x\}_{x \in X}$ violates our assumption.

■

With the lemma and the language above we can now prove Tychonoff's theorem.

Proof. Let $\{U_k\}$ be an open cover of $X$. Without loss of generality, assume each $U_\alpha$ is convex. We will approximate $X$ with a finite dimensional set $X_n$, and $f$ with a simplicial approximation $F$, and from the fixed point of $F$, we obtain an $x$ such that $f(x)$ and $x$ lie in the same $U_\alpha$ for some $\alpha$.

To this end, let $\{W_i\}$ be a star–refinement of $\{U_\alpha\}$, and $W_1, \ldots, W_n$ a finite subcover of this refinement. For each $k$, $0 \leq k \leq n$, select and element from $W_k$ and label it $x_k$. Now define $X_k = \overline{\text{co}}\{x_1, \ldots, x_k\}$, and set $m = \dim X_k \leq n$. Subdivide $W_k$ into a series of $m$–simplexes intersecting only in their faces, small enough that each $m$–simplex is completely contained in $\overline{f^{-1}(W_k)}$ for some $k$. Let $\{v_0, \ldots v_s\}$ be the combined vertex sets of our simplexes. For each $v_i$, $f(v_i)$ is contained in one or more $W_k$. Pick one of these $k$, and define $F(v_i) = x_k$. Next extend $F$ linearly to all $x \in X_n$. Then $F$ is a continuous self-map on $X_n$, and thus by Brouwer's theorem, has a fixed point, say $x \in X_n$.

To complete our proof we show the existence of an $\alpha$ such that both $x$ and $f(x)$ lie in $U_\alpha$. Indeed, let $x \in f^{-1}(W_k)$. Then $f(x) \in W_k \subseteq U_\alpha$, where $U_\alpha$ contains $\text{St}(W_k, W)$. Now $x$ lies in some $m$–simplex, so let $x = \sum \lambda_i y_i$. Then $F(x) = \sum \lambda_i F(y_i)$, where each $F(y_i)$ gets mapped to an $x_j$, where $f(y_i) \in W_j$. Since $f(y_i) \in W_k$, this $W_j$ must be a neighbourhood of $W_k$. That is, $f(y_i) \in \text{st}(W_k, W) \subseteq U_\alpha$. Thus by convexity, $F(x) \in U_\alpha$. Since $x$ was a fixed point of $F$, we are done.

■

Eventually Schauder's second theorem and Tychonoff's theorem were unified to produce the so–called "Schauder–Tychonoff" fixed point theorem. The following was proven by Hukuhara in 1950.

**Theorem 3.1.4** *Let $X$ be a subset of a space $\mathbf{E}$, and $f : X \to \mathbf{E}$.*

1. *$X$ : convex.*
2. *$\mathbf{E}$ : locally convex topological vector space.*
3. *$f$ : compact, continuous self–map.*

*Then $f$ has a fixed point in $X$.*

## 3.2 Fixed Point Properties for Closed Bounded Convex Sets

As noted, compactness is important in establishing the topological fixed point theorem in infinite dimensions. In infinite dimensions, the property fails for closed bounded sets. In this section we pause from our consideration of continuous functions to consider some classes of functions for which closed bounded convex sets have fixed points.

### 3.2.1 Theorems with Boundary Conditions

The first of the theorems in this genre was by KKM in 1929. They proved that a continuous function on the unit ball in $\mathbf{R}^n$ that mapped the unit sphere into the unit ball must have a fixed point. The following theorem is due to Rothe.

**Theorem 3.2.1** *Let $X$ be a subset of a space $\mathbf{E}$, and $f : X \to \mathbf{E}$.*

1. *$X$ : closed unit ball.*
2. *$\mathbf{E}$ : Banach space.*
3. *$f$ : compact, continuous, and such that $f(\partial X) \subset X$.*

*Then $f$ has a fixed point in $X$.*

Proof. Suppose $f : \mathbf{B} \to \mathbf{E}$ is a compact continuous function such that $f(\mathbf{S}) \subset \mathbf{B}$. Let $r : \mathbf{E} \to \mathbf{B}$ be the projection of $\mathbf{E}$ onto $\mathbf{B}$. Since the continuous image of a compact set is again compact, we have that $(r \circ f) : \mathbf{B} \to \mathbf{B}$ is a compact continuous map. Thus by Schauder's theorem, there is an $x \in \mathbf{B}$ such that $r(f(x)) = x$. If $x \in \text{int}(B)$, then $f(x) = x$ by the definition of $r$. Else, $x$ is on the boundary of the ball, and by supposition, $f(x) \in \mathbf{B}$.

Thus $r(f(x)) = f(x)$, and again $x$ is the desired fixed point of $f$.

■

We note that Rothe's theorem does not hold if we simply require that $f$ be a compact mapping on the boundary of $X$. The following example illustrates this.

**Example 3** *Let $X$ be the ball of radius 2, centered at the origin, in the space $\ell_2$. Define $f : X \to X$ as follows. For $x = (x_1, x_2, \ldots, x_n, \ldots)$,*

$$f(x) = \begin{cases} (\sqrt{1 - \|x\|^2}, x_1, x_2, \ldots, x_n, \ldots), & \|x\| \le 1, \\ (2 - \|x\|) f(\frac{x}{\|x\|}), & 1 \le \|x\| \le 2. \end{cases}$$

We see that $f$ maps the boundary of $X$ to the origin, and thus is a compact mapping on this set. However, a similar argument to that in example 2 shows that $f$ has no fixed points.

Since Rothe proved this result, many authors have investigated various boundary conditions. It is possible to replace the condition $f(\partial X) \subset X$ with any of: i) $\langle f(x), x\rangle \le \|x\|^2$ (Krasnoselskii), ii) $\|x - f(x)| \ge \|f(x)\|$ (Petryshyn), or iii) $\|x - f(x)\|^2 \ge \|f(x)\|^2 - \|x\|^2$ (Altman), for $x$ on the unit sphere. For a discussion of these boundary conditions and how they relate to each other, see Istratescu [35].

### 3.2.2 Conditions on Compactness

Another avenue for the generalization of Brouwer's theorem is through the requirements on compactness. It has already been shown that we do not need the set $X$ itself to be compact, as long as the image of $X$ is contained in a compact subset of $\mathbf{E}$. It is interesting that what we need is that $f$ in some sense takes $X$ a little closer to being compact. We will consider two types of mappings that satisfy this condition.

**Definition 3.2.2** *Let $X$ be a complete metric space, and $M$ the family of bounded nonempty subsets of $X$. The Kuratowski measure of noncompactness is the function $\alpha : M \to \mathbf{R}^+$ defined by*

$$\alpha(A) = \inf \left\{ \begin{array}{lcl} \epsilon > 0 & : & A \text{ can be covered by a finite number} \\ & & \text{of sets in } M \text{ with diameter} < \epsilon \end{array} \right\}.$$

**Example 4** *In infinite dimensional Banach space,* $\alpha(\mathbf{B}) = 2$.

Clearly $\alpha(\mathbf{B}) \leq 2$. Let $\{e^i\}_{i \in I}$ be a basis for the space where $e^i_j = 1$ if $j = i$, and zero otherwise. Each $e^i$ lies in $\mathbf{B}$. the smallest ball containing $n$ basis vectors $e^{i_1}, \ldots, e^{i_n}$ would be centered at $\displaystyle\frac{1}{n}\sum_{t=1}^{n} e^{i_t}$, with radius $\sqrt{1 - \frac{1}{n}}$. Then each ball of diameter $2\sqrt{1-\frac{1}{n}}$ can cover at most $n$ basis vectors. Thus it would take an infinite number of these balls to cover $\mathbf{B}$. Since any set of diameter $2\sqrt{1-\frac{1}{n}}$ is contained in a ball of diameter $2\sqrt{1-\frac{1}{n}}$, any covering of $\mathbf{B}$ of sets with diameter $2\sqrt{1-\frac{1}{n}}$ would have to be infinite. Thus $\alpha(\mathbf{B}) > 2\sqrt{1-\frac{1}{n}}$ for all $n$. Thus $\alpha(\mathbf{B}) = 2$.

There are other measures of noncompactness. For example, in the above definition we could have taken the infimum over all balls of radius less than $\epsilon$, rather than sets of diameter less than $\epsilon$. Then we would have obtained the Hausdorff measure of noncompactness. Any measure would do in what follows. We have the following properties.

**Properties of the Measure of Noncompactness**

1. $\alpha(A) = 0 \Leftrightarrow \overline{A}$ is compact.

2. $A \subseteq B \Rightarrow \alpha(A) \leq \alpha(B)$

3. $\alpha(A) = \alpha(\overline{A})$

4. $\alpha(A \cup B) \leq \max\{\alpha(A), \alpha(B)\}$

5. $\alpha(A + B) \leq \alpha(A) + \alpha(B)$

6. $\alpha(cA) = |c|\alpha(A)$

7. $\alpha(\text{conv}(A)) = \alpha(A)$

Proof. For (1), assume $\alpha(A) = 0$, and suppose $\overline{A}$ is not compact. Then there exists a sequence $\{x_n\}$ with no convergent subsequence. Thus we can find an $\epsilon > 0$ such that no $\epsilon$–ball contains a subsequence of $\{x_n\}$. Then no finite collection of sets with diameter less than $2\epsilon$ can cover $\overline{A}$, violating our assumption. Next, (2) follows clearly since any covering of $B$ is also a covering of $A$. Property (3) is also immediate. From (2) we see $\alpha(A) \leq \alpha(\overline{A})$. The reverse inequality follows since if $\{U_\alpha\}$ covers $A$, then $\{\overline{U_\alpha}\}$ covers $\overline{A}$, and $\text{diam}(U_\alpha) = \text{diam}(\overline{U_\alpha})$. Property (4) is clear. For (5) we need just note that if $\{U_i\}$ and $\{V_j\}$ are finite covers of $A$ and $B$ respectively, and $\text{diam}(U_i) \leq \epsilon_1$, $\text{diam}(V_j) \leq \epsilon_2$, then $\text{diam}(U_i + V_j) \leq \epsilon_1 + \epsilon_2$. (6) is also evident.

∎

Before we can prove property 7, we need the following lemma.

**Lemma 3.2.3** *If $C_1, C_2$ are bounded and convex, then $\alpha(conv(C_1 \cup C_2)) \leq \max\{\alpha(C_1), \alpha(C_2)\}$.*

Proof. Let $\epsilon > 0$ be given. By boundedness of $C_1$ and $C_2$, choose $K \in \mathbf{R}$ such that $\|x\| < K$ for $x \in C_1 \cup C_2$. Select a partition $\{t_i\}_{i=1}^n$ of $[0,1]$ such that $t_i - t_{i-1} \leq \frac{\epsilon}{2K}$. Then we have

$$\text{conv}(C_1 \cup C_2) \subseteq \bigcup_{1 \leq i \leq n} \{t_i C_1 + (1-t_i)C_2 + \epsilon \mathbf{B}\}.$$

Thus,

$$\begin{aligned} \alpha(\text{conv}(C_1 \cup C_2)) &\leq \max\{\alpha(t_i C_1 + (1-t_i)C_2 + \epsilon\mathbf{B})\} \\ &\leq \max\{t_i \alpha(C_1) + (1-t_i)\alpha(C_2) + \epsilon\alpha(\mathbf{B})\} \\ &= \max\{\alpha(C_1), \alpha(C_2)\} + 2\epsilon. \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, the result is proven.

∎

We can now prove property 7.

Proof. Since $A \subseteq \text{conv}(A)$, we need only show $\alpha(\text{conv}(A)) \leq \alpha(A)$. Let $\epsilon > 0$ be given, and $U_1, \ldots, U_n$ be a finite cover of $A$ with $\text{diam}U_i < \alpha(A) + \epsilon$. Define $A_1 = U_1$, and $A_i = \text{conv}\,(A_{i=1} \bigcup U_i)$. Then $\text{conv}(A) \subseteq \text{conv}\,(\bigcup_{i=1}^n U_i) \subseteq A_n$, and so $\alpha(\text{conv}(A)) \leq \alpha(A_n)$. But by Lemma 3.2.3,

$$\alpha(A_n) \leq \max\{\alpha(U_i)\} \leq \text{diam}(U_i) \leq \alpha(A) + \epsilon.$$

Since $\epsilon > 0$ was arbitrary, the result is proven.

∎

We can use $\alpha$ to define a class of functions for which closed, bounded, convex sets have the fixed point property.

**Definition 3.2.4** *Let $X$ be a complete metric space. A continuous function $f : X \to X$ is called an $\alpha$–set contraction provided there exists a $k \in [0, 1)$ such that for all $A \subseteq X$, we have*

$$\alpha\left(f(A)\right) \leq k\alpha(A).$$

Clearly any contraction mapping is an $\alpha$–set contraction. Further, any completely continuous compact function is an $\alpha$–set contraction. This follows since if $A$ is a bounded subset of $\mathbf{E}$, then $f(A)$ is precompact. Thus $\alpha(f(A)) = 0 \leq k\alpha(A)$ for any $k \in [0, 1)$.

We can now prove the following theorem by Darbo [19].

**Theorem 3.2.5 (Darbo 1955)** *Let $X$ be a subset of a space $\mathbf{E}$, and $f : X \to \mathbf{E}$.*

1. *$X$ : closed, bounded, convex.*
2. *$\mathbf{E}$ : Banach space.*
3. *$f$ : an $\alpha$–set contraction, self–map on $X$.*

*Then $f$ has a fixed point in $X$.*

Proof. Let $X_0 = X$, and define $X_n = \overline{\text{conv}}(f(X_{n-1}))$ for $n \geq 1$. By convexity of $X$, $X_1 \subseteq X_0$. An induction argument shows that $X_n \subseteq X_{n-1}$ for each $n$. Thus $\{X_n\}_{n=0}^{\infty}$ is a decreasing sequence of closed convex sets. Thus $X_\infty = \bigcap_{n=0}^{\infty} X_n$ is nonempty, closed and convex. Also,

$$\begin{aligned} \alpha(X_n) &= \alpha(\overline{\text{conv}}(f(X_{n-1}))) \\ &= \alpha(f(X_{n-1})) \\ &\leq k\alpha(X_{n-1}), \end{aligned}$$

where $k$ is some fixed real in $[0, 1)$. From this, we obtain $\alpha(X_n) \leq k^n\alpha(X)$. Since $X$ is bounded, $\alpha(X) < \infty$. Taking limits gives $\alpha(X_\infty) = 0$, and so $X_\infty$ is compact. Further, since $\bigcap_{n=0}^{\infty} X_n = \bigcap_{n=0}^{\infty} \overline{\text{conv}}(f(X_n))$, it follows that $f$ is a self-map on $X_\infty$. Thus by Schauder's theorem $f$ has a fixed point.

■

Now we turn to another more general class of functions for which closed, bounded, convex sets have the fixed point property.

**Definition 3.2.6** *Let $X$ be a complete metric space. A function $f : X \to X$ is called condensing, or densifying, provided for each noncompact subset $A$ of $X$, we have $\alpha(f(A)) < \alpha(A)$.*

Sadovski used Zorn's lemma, and Schauder's theorem to prove a theorem analogous to, but stronger than Darbo's theorem. It is strange that Darbo's theorem is more frequently referenced in the literature. We have included both results for the sake of completeness. To obtain Sadovski's theorem, we need the following result.

**Theorem 3.2.7** *Let $X$ be compact, and $f : X \to X$. Then there exists a nonempty subset $M \subseteq X$ such that $M = \overline{f(M)}$.*

Proof. Let $Z = \{A \subseteq X : \overline{f(A)} \subseteq A, A \text{ closed}, A \neq \emptyset\}$, and partially order $Z$ by reverse inclusion. Then $Z \neq \emptyset$, since $X \in Z$. Consider a chain in $A$:

$$A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots$$

Then $\bigcap A_i$ is closed, and since $X$ is compact, $\bigcap A_i \neq \emptyset$. Also, since $\bigcap \overline{f(A_i)}$ is closed, it is clear that $\overline{\bigcap f(A_i)} \subseteq \bigcap \overline{f(A_i)} \subseteq \bigcap A_i$. Thus $\bigcap A_i \in Z$, and is an upper bound for our chain. Thus by Zorn's lemma, $Z$ has a maximal element, say $M$. Then necessarily, $\overline{f(M)} = M$. For if not, choose $x \in M \setminus \overline{f(M)}$. By closedness of $\overline{f(M)}$, we can find a neighbourhood $U_x$ of $x$ with $U_x \bigcap \overline{f(M)} = \emptyset$. Then $M \bigcap U_x^c$ is closed, and $\overline{f(M \bigcap U_x^c)} \subseteq \overline{f(M)} \subseteq M \bigcap U_x^c$, violating maximality of $M$ (under the reverse inclusion ordering).

■

It is easy to see that an $\alpha$–set contraction is a condensing function, but the converse need not be true. For example, see Istratescu (p.160) [35]. Thus the following theorem is indeed stronger than theorem 3.2.5.

**Theorem 3.2.8 (Sadovski 1967)** *Let $X$ be a subset of a space $\mathbf{E}$, and $f : X \to \mathbf{E}$.*

1. *$X$ : closed, bounded, convex.*
2. *$\mathbf{E}$ : Banach space.*
3. *$f$ : a condensing self–map.*

*Then $f$ has a fixed point in $X$.*

PROOF. Fix $x \in X$. Let $K = \bigcup_{n=1}^{\infty}\{f^n(x)\}$. Since $f$ is condensing, if $\alpha(K) \neq 0$ we have

$$\alpha(K) = \alpha(f(K) \cup \{f(x)\}) \leq \alpha(f(K)) < \alpha(K).$$

Thus $\alpha(K) = 0$, and so $\overline{K}$ is compact. Also $f(\overline{K}) \subseteq \overline{K}$. Thus by Lemma 3.2.7, there is a nonempty subset $M \subset \overline{K}$ such that $M = \overline{f(M)}$. We use $M$ to produce a compact convex set to which we can apply Schauder's theorem. To this end, set

$$C = \{B \subseteq X : M \subseteq B, B \text{ closed, convex, and invariant under } f\}.$$

Note that $C$ is nonempty, since $X \in C$. Next let $A = \bigcap_{B \in C} B$. Then $A = \overline{\text{conv}}(f(A))$, and assuming $\alpha(A) \neq 0$, we have

$$\alpha(A) = \alpha(\overline{\text{conv}}(f(A))) = \alpha(f(A)) < \alpha(A).$$

Thus $\alpha(A) = 0$, and $A$ is compact. Hence, by Schauder's theorem, $f$ has a fixed point.

■

## 3.3 Other Extensions

We end this chapter with a look at fixed point theorems involving more than one function. The geometric counterpart of the topological fixed point theorem is of course the famous Banach contraction principle. The following theorem due to Krasnoselskii considers the fixed points of the sum of a compact continuous mapping with a contraction.

**Theorem 3.3.1 (Krasnoselskii)** *Let $X$ be a subset of a space* $\mathbf{E}$*, and* $f, g : X \to \mathbf{E}$.

1. $X$ *: closed, convex.*
2. $\mathbf{E}$ *: Banach space.*
3. $f$ *and* $g$ *: continuous,* $f$ *compact,* $g$ *a contraction, and* $f(X) + g(X) \subseteq X$ .

*Then* $f + g$ *has a fixed point in* $X$.

**Lemma 3.3.2** *Let* $g : X \to \mathbf{E}$ *be a contraction, and id the identity map. Then* $id - g : X \to (id - g)(X)$ *is a homeomorphism, and if* $(id - g)(X)$ *is precompact, then so is* $X$.

Proof. To see that $(id - g)^{-1}$ is continuous. it is sufficient to note that

$$\|(id - g)(x) - (id - g)(y)\| \geq \|x - y\| - \|g(x) - g(y)\| \geq (1 - k)\|x - y\|,$$

for some $k \in [0, 1)$. Also, if $(id - g)(X)$ can be covered with a finite number of balls of radius $(1 - k)\epsilon$, then $X$ can be covered by a finite number of balls of radius $\epsilon$.

■

Proof. [of Theorem 3.3.1] Fix $a \in X$. The the function $x \mapsto g(x) + f(a)$ is a contraction on $X$, and hence has a unique fixed point $x_0$ in $X$. Then $x_0 - g(x_0) = f(a)$, and so $(id - g)^{-1} \circ f(a)$ is in $X$ for each $a$. By Lemma 3.3.2, $(id - g)^{-1} \circ f$ is continuous and compact, and so has a fixed point $z$. Then $f(z) + g(z) = z$.

■

In theorem 3.3.1, the requirement that $f(x) + g(y)$ be in $X$ for all $x, y \in X$ is quite strong. It is natural to ask whether we may merely require $f + g$ be a self map on $X$. The answer is yes, if we assume $X$ is bounded. For if $f$ is compact, then for $A \subseteq X$, $f(A)$ is precompact. Hence $\overline{f(A)}$ is compact, and $\alpha(f(A)) = 0$. Further if $g$ is a contraction, and $U_1, \ldots, U_n$ is a finite cover of $A$ with $\operatorname{diam}(U_i) < \epsilon$, then $g(U_1), \ldots, g(U_n)$ covers $g(A)$. For $g(x)$, $g(y) \in g(U_i)$ we have $\|g(x) - g(y)\| \leq k\|x - y\| < k\epsilon$, for some $k \in [0, 1)$. Thus $\operatorname{diam}(g(U_i)) < \operatorname{diam}(U_i)$, and so $\alpha(g(A)) < \alpha(A)$. In summary,

$$\alpha((f + g)(A)) = \alpha(f(A) + g(A)) \leq \alpha(f(A)) + \alpha(g(A)) < \alpha(A).$$

Thus $f + g : X \to X$ is condensing, and by Sadovski's theorem, $f + g$ has a fixed point.

Much theory has been developed on fixed point theorems for families of continuous functions. The following was obtained by Markov using the Tychonoff fixed point theorem. Kakutani gave a direct proof of the theorem, for a family of equicontinuous functions. Recall that a family of functions $F = \{f\}$ is said to be equicontinuous on a set $X$ if for every $x \in X$ and any family of neighbourhoods $\{V_f\}_{f \in F}$ of $f(x)$, there exists a neighbourhood $U$ of $x$ such that $y \in U$ implies $f(y) \in V_f$, for all $f \in F$.

A family of functions $F = \{f\}$ defined on a set $X$, is said to have a fixed point if there is a mutual fixed point for all $f \in F$. A family $F$ is said to be commuting if $f \circ g = g \circ f$ for all $f, g \in F$.

**Theorem 3.3.3 (Markov 1936)** *Let $X$ be a subset of a space $\mathbf{E}$, and $f : X \to \mathbf{E}$.*

1. *$X$ : compact, convex.*
2. *$\mathbf{E}$ : locally convex topological vector space.*
3. *$F = \{f\}$ : a commuting family of affine functions.*

*Then $F$ has a fixed point in $X$.*

Proof. For $f \in F$, let $\mathrm{Fix}(f)$ denote the set of fixed points of $f$. Since $f$ is affine, it is not hard to see that $\mathrm{Fix}(f)$ is closed and convex, and so $\mathrm{Fix}(f)$ inherits compactness from $X$. By Tychonoff's theorem, $\mathrm{Fix}(f)$ is nonempty. Next note that for $x \in \mathrm{Fix}(f)$, and $g \in F$, $g(x) = g(f(x)) = f(g(x))$, so $g(x) \in \mathrm{Fix}(f)$, and $g$ is a self-map of the compact convex set $\mathrm{Fix}(f)$. Thus, as above, the set of points in $\mathrm{Fix}(f)$ that are fixed by $g$ is compact, convex, and nonempty. This set is exactly $\mathrm{Fix}(g) \cap \mathrm{Fix}(f)$. Then by induction, the set $\{\mathrm{Fix}(f) : f \in F\}$ has the finite intersection property. Now by compactness of $X$, we see

$$\bigcap_{f \in F} \mathrm{Fix}(f) \neq \emptyset.$$

Hence $F$ has a fixed point.

■

Further results are known for nonaffine families of commuting mappings. In general, we know that an arbitrary family of commuting continuous functions need not have a common fixed point. See, for example, Boyce [15], or Huneke [34]. Both independently showed that there exist two commuting continuous functions on the unit interval that do not have a common fixed point. However, we know that if all but one member of the family of mappings is affine, then the theorem still holds. As well, it is not hard to see that any family of commuting continuous functions that contains just one contraction must have a common fixed point. Indeed, if $g$ is a contraction, then it must have a unique fixed point, say $x$. For $f \in F$, $g(f(x)) = f(g(x)) = f(x)$. Then by uniqueness, $f(x) = x$.

# Chapter 4

# Multifunctions and Kakutani's Theorem

Topological fixed point theory for multifunctions has its seeds in von Neumann's work in the proof of the fundamental theorem of game theory [45]. In an effort to simplify the arguments used, Kakutani proved an analog of Brouwer's theorem [36]. Since its conception, Kakutani's theorem has been applied numerous times in game theory, economic theory, and control theory. In particular, Nash used it to prove the existence of an equilibrium for finite games. Debreu [20] lists more than three hundred applications of the theorem in proving the existence of an economic equilibrium. As with Brouwer's theorem, various authors have sought generalizations of Kakutani's theorem. Some of the development has paralleled that of Brouwer.

Kakutani's theorem is a statement about fixed points in compact convex subsets of $\mathbf{R}^n$. The generalization to Banach space was given by Bohnenblust and Karlin [9]. Fan [23] and Glicksberg [30] independently extended this to locally convex topological vector spaces. Later, Himmelberg [33] dropped the compactness requirement on $X$, assuming that the image of the multifunction is contained in a compact subset of $X$. As with Brouwer's theorem, there have also been results based on boundary conditions. We give the analog to Rothe's theorem, as well as a boundary condition theorem by Browder [16] that does not require our function be a self map.

## 4.1 Kakutani's Fixed Point Theorem

Recall that a multifunction $F : X \to 2^X$ is a set valued map on $X$. As in Brouwer, we will consider sets that are compact and convex. The basic requirement for a multifunction in topological fixed point theorems is that it be upper semicontinuous, with nonempty, compact, convex images. For short, we say $F$ is a *cusco*. An element $x \in X$ is a fixed point for $F$ provided $x \in F(x)$. If every *cusco* mapping $X$ into $2^X$ has a fixed point, then we say $X$ has the Kakutani fixed point property (kfpp).

**Lemma 4.1.1** *The Kakutani fixed point property is preserved under retractions.*

PROOF. Suppose $X$ has the Kakutani fixed point property. If $r : X \to Y$ is a retraction, and $F : Y \to 2^Y$ is a *cusco*, then $G : X \to 2^Y$ defined by $G(x) = F(r(x))$ is a *cusco*, and has a fixed point that must also be a fixed point of $F$.

■

**Theorem 4.1.2 (Kakutani)** *Let $X$ be a compact convex subset in $\mathbf{R}^n$. If $F : X \to 2^X$ is a cusco, then $F$ has a fixed point.*

PROOF. In light of the above lemma, we may assume that $X$ is a $k$-simplex. Let $n > 0$ be given. Subdivide $X$ into a $k$-simplicial complex such that each $k$-subsimplex is contained in a ball of radius $\frac{1}{n}$. Define a single valued function $f_n : X \to X$ as follows. For each vertex $v$ in $X$, select $f_n(v)$ in $F(v)$. Then extend $f_n$ linearly over each $k$-subsimplex in the complex. So $f_n$ is continuous, and by Brouwer's theorem, has a fixed point, say $x_n$. Consider the sequence $\{x_n\}$ in $X$. We may assume $\lim_{n\to\infty} x_n = x$, since otherwise we can consider a convergent subsequence. We will show that $x$ is our desired point.

To this end, let $S_n$ be a $k$-simplex in our complex, containing $x_n$, and let $x_0^n, \dots, x_k^n$ be the vertices of $S_n$. As $n \to \infty$, the diameter of $S_n$ goes to zero. Thus $x_i^n \to x$ for each $i$. Let $f_n(x_i^n) = y_i^n$. So we can write $x_n = \sum_{i=0}^k \lambda_i^n y_i^n$, where $\sum_{i=0}^k \lambda_i^n = 1$, $\lambda_i^n \geq 0$. We can assume without loss of generality that $x_n \to x$, $\lambda_i^n \to \lambda_i$, and $y_i^n \to y_i$ all converge. Otherwise we could use convergent subsequences. Since for each $n$, $\sum_{i=0}^k \lambda_i^n = 1$, $\lambda_i^n \geq 0$, we must have $\sum_{i=0}^k \lambda_i = 1$, $\lambda_i \geq 0$. So $x = \sum_{i=0}^k \lambda_i y_i$ is a convex combination. Further, since $x_i \to x$, $y_i^n \to y_i$, $y_i^n \in F(x_i^n)$, by upper semicontinuity of $F$, we must have $y_i \in F(x)$ for each $i$. Thus by convexity of $F(x)$, we obtain $x \in F(x)$.

■

Not surprisingly, the kfpp is very closely related to the tfpp. If $X$ has the kfpp, then since any continuous single valued function can be viewed as a cusco, $X$ will also have the tfpp. Further, if $X$ is a compact subset of Euclidean space, then the two properties are equivalent. For in this case, Cellina's theorem says that there is an approximate continuous selection $f$ of $F$. That is, for each $\epsilon > 0$, there exists $f_\epsilon : X \to X$ continuous with

$$d_{Gr(F)}(x, f_\epsilon(x)) < \epsilon,$$

for any $x$ in $X$. So if $X$ has the tfpp, then each $f_\epsilon$ has a fixed point, say $x_\epsilon$. Letting $\epsilon$ go to zero gives a sequence $(x_\epsilon, x_\epsilon)$ that approaches the graph of $F$. Since $F$ is a cusco, this graph is closed, hence $\lim_{\epsilon\to 0}(x_\epsilon, x_\epsilon) = (x, x) \in Gr(F)$. That is, $x$ is a fixed point for $F$. It follows that the two properties are also equivalent for the class of convex sets. For if $X$ is convex and noncompact, then $X$ doesn't have the tfpp (or the kfpp).

## 4.2 Extended to Banach Space

As in the case for single valued mappings, a set that can be approximated by a sequence of sets having the Kakutani fixed point property will inherit this property. It is through this approximation process that we can extend Kakutani's theorem to Banach space. As stated, Bohnenblust and Karlin first showed the extension of Kakutani's theorem to Banach space. The following proof of this result can be found in Smart [52].

**Lemma 4.2.1** *Let* $\mathbf{E}$ *be a normed space, and* $X \subset \mathbf{E}$ *a compact and convex. Suppose that for every* $n \in \mathbf{N}$*, there exists* $f_n : X \to X$ *linear such that*

1. $\|f_n(x) - x\| < \epsilon_n$ *for all* $x \in X$*, where* $\epsilon_n \to 0$ *as* $n \to \infty$*.*
2. $f_n(X)$ *has the Kakutani fixed point property for each* $n$*.*

*Then* $X$ *has the Kakutani fixed point property.*

PROOF. Let $F : X \to 2^X$ be a *cusco*. Then $f_n \circ F$ is a *cusco* on $f_n(X)$, and so by assumption, has a fixed point. Let $x_n \in f_n(F(x_n))$, and write $x_n = f(y_n)$ for some $y_n \in F(x_n)$. Possibly after considering subsequences, let $x_n \to x$ in $f(X)$. Then by assumption, $\|x_n - y_n\| = \|f(y_n) - y_n\| \to 0$. Thus $y_n \to x$, and the sequence $(x_n, y_n)$ in $Gr(F)$ converges to $(x, x)$. Since $Gr(F)$ is closed, we obtain $x \in F(x)$. Thus $X$ has the Kakutani fixed point property.

■

In what follows, we use the fact that any compact convex subset of a Banach space is homeomorphic to a compact convex subset of the Hilbert cube, $H$. The following lemma is needed to show that $H$ has the Kakutani fixed point property.

**Theorem 4.2.2** *The Hilbert cube* $H = \{(x_n) \in \ell_2 : |x_n| \leq \frac{1}{n}\}$ *has the kfpp.*

PROOF. Let $p_n : \ell_2 \to \ell_2$ be the projection map defined by $p_n(x_1, x_2, \ldots) = (x_1, \ldots, x_n, 0, \ldots)$. The for each $x \in H$,

$$\|p_n(x) - x\|^2 \leq \sum_{k=n+1}^{\infty} \frac{1}{k^2},$$

which approaches zero, as $n \to \infty$. Also $p_n(H)$ can be embedded into $\mathbf{R}^n$ as a closed bounded convex set. Thus $p_n(H)$ has the kfpp for each $n$. Thus by lemma 4.2.1, $H$ has the kfpp.

■

**Theorem 4.2.3** *Let $X$ be a compact convex subset in Banach space. There is a linear homeomorphism of $X$ onto a compact convex subset of $H$.*

PROOF. Assume that $X$ is a subset of $\mathbf{B}$. Let $\{x_n\}$ be dense in span$(X)$, and for each $n \in \mathbf{N}$ choose a bounded linear function $F_n$ on $\mathbf{E}$ such that

$$f_n(x_n) = \frac{\|x\|}{n}, \text{ and } \|f_n\| = \frac{1}{n}.$$

Now define $f : X \to H$ by $f((x_i)) = (f_i(x))$. Then clearly $f$ is continuous. Also if $x \neq y$, then

$$\begin{aligned} |f_n(x) - f_n(y)| &\geq \|f_n(x_n)| - |f_n(x - y - x_n)| \\ &\geq \frac{\|x_n\|}{n} - \frac{\|(x-y) - x_n\|}{n} > 0, \end{aligned}$$

for $x_n$ sufficiently close to $x - y$. Thus $f$ is a linear homeomorphism onto $f(X)$. Since $X$ is compact and convex, so is $f(X)$.

■

Since the kfpp is preserved under linear homeomorphisms, and any compact convex subset of $H$ is a retract of $H$, we have the following theorem.

**Theorem 4.2.4** *Let $X$ be a compact convex subset of a Banach space, and $F : X \to 2^X$ a cusco. Then $F$ has a fixed point.*

As in the single-valued case, we can use this result to make a statement about a class of functions for which a closed convex set has the fixed point property.

**Theorem 4.2.5** *Let $X$ be a closed convex subset of a Banach space, and $F : X \to 2^X$ a cusco. If $F(X)$ is contained in a compact subset of $X$, then $F$ has a fixed point.*

PROOF. Let $Y = \overline{\text{conv}}\,(F(X)) \subseteq X$. Since $F(X)$ is contained in a compact subset of $X$, its closure must be compact. Thus by Mazur's theorem, $Y$ is compact. From theorem 4.2.4, $F : Y \to 2^Y$ has a fixed point.

■

Further generalizations of the Kakutani fixed point theorem has paralleled some of that of the single valued case. For references, see for instance, [47].

## 4.3 Theorems with Boundary Conditions

In this section we take a slight detour to consider some results in which the function considered is not strictly a self–map. The first theorem we consider is immediate from the material in the previous section. In the second part, we look at some interesting results by Browder that don't explicitly make any requirements that $F$ map $X$ into $2^X$. Of course, it is a necessary consequence of the conditions that Browder does insist on that $F(X) \cap X \neq \emptyset$. This theorem is of particular interest because it gives better insight into the characteristics of $F$ that give us a fixed point. As well, the preliminary results need to prove the theorems of Browder are of interest in their own right.

We begin with the multifunction analog of Rothe's theorem.

**Theorem 4.3.1** *Let $X$ be a closed convex subset of a Banach space $\mathbf{E}$, and $F : X \to 2^{\mathbf{E}}$ a cusco such that $F(X)$ is precompact. If $F(x) \subseteq X$ for each $x \in \partial X$, then $F$ has a fixed point.*

PROOF. First note that if $X$ has empty interior, then $X = \partial X$, and the result follows from theorem 4.2.5. In $\text{int}(X) \neq \emptyset$, then we can assume without loss of generality that

$0 \in \text{int}(X)$. By virtue of $F(X)$ being contained in a compact subset of $X$, choose $c \in \mathbf{R}^+$ such that $F(X) \subseteq cX$. Next let $g$ be the Minkowski functional, and let $r : cX \to X$ be the retraction defined by $r(x) = (\max\{g(x), 1\})^{-1}X$. Consider the function $G : cX \to 2^{cX}$ defined by

$$G(x) = F(r(x)).$$

Then $G$ is a *cusco*, and so there is an $x_0 \in cX$ with $x_0 \in G(x_0)$. Note that if $x \notin X$, then $r(x) \in \partial X$, and so $G(x) \subseteq X$. Thus any fixed point of $G$ must be in $X$. It follows that $x_0$ is a fixed point of $F$.

■

Now we turn to our first result by Browder. The previous multifunction results have required $F$ to be *usc*. In the following, we ask that the inverse image of open sets under $F$ be open. In this case, $F$ is said to be lower semicontinuous.

**Theorem 4.3.2** *Let $X$ be a nonempty compact convex subset of a topological vector space $\mathbf{E}$, and $F : X \to 2^X$ such that $F(x)$ is nonempty and convex for each $x \in X$. Suppose that $F^{-1}(y) = \{x \in X : y \in F(x)\}$ is open for each $y \in X$. Then $F$ has a fixed point.*

Proof. The set $\{F^{-1}(y) : y \in X\}$ is an open cover of our compact set $X$. Thus choose a finite subcover $\{F^{-1}(y_i)\}_{i=1}^n$, and let $p_i : X \to \mathbf{R}$, $1 \leq i \leq n$, be a partition of unity subordinate to this cover. Define $p : X \to X$ by

$$p(x) = \sum_{i=1}^{n} p_i(x) y_i.$$

By convexity of $X$, $p$ is well-defined. Also, for each $i$ such that $p_i(x) \neq 0$, we have $x \in F^{-1}(y_i)$. Thus $y_i \in F(x)$. Since $\sum_{i=1}^{n} p_i(x) = 1$,and $p_i(x) \geq 0$, we see that $p(x)$ is a convex combination of points in $F(x)$. Since $F(x)$ is convex, $p(x) \in F(x)$. But $p$ is a continuous self-map on the finite dimensional set, so by Brouwer's theorem $p$ has a fixed point. This point is the desired fixed point for $X$.

■

**Lemma 4.3.3** *Let $X$ be a compact subset of a topological vector space $\mathbf{E}$, and $f : X \to \mathbf{E}^*$ continuous. Then for a fixed $y \in X$, the function $g : X \to \mathbf{R}$ defined by $g(x) = \langle f(x), x - y \rangle$ is continuous.*

PROOF. Fix $y \in X$. By compactness of $X$, let $K = \sup_{x \in X} \{\|x - y\|\} < \infty$. Fix $z \in X$, and let $\epsilon > 0$ be given. Since $f$ is continuous, choose $\delta_1 > 0$ so that $\|x - z\| < \delta_1 \Rightarrow \|f(x) - f(z)\| < \frac{\epsilon}{2k}$. Since $f(x) \in \mathbf{E}^*$, let $\delta_2 > 0$ be such that $\|x - z\| < \delta_2 \Rightarrow |\langle f(z), z - x\rangle| < \frac{\epsilon}{2}$. Then choose $\delta = \min\{\delta_1, \delta_2\}$. Continuity of $g$ follows from

$$
\begin{aligned}
\|g(z) - g(x)\| &= \|\langle f(z), z - y\rangle - \langle (x), x - y\rangle\| \\
&\leq \|\langle f(z), z - y\rangle - \langle f(z), x - y\rangle + \langle f(z), x - y\rangle - \langle f(x), x - y\rangle\| \\
&\leq \|\langle f(z), z - x\rangle\| + \|\langle f(z) - f(x), x - y\rangle\|.
\end{aligned}
$$

■

**Lemma 4.3.4** *Let* $\mathbf{E}$ *be a locally convex topological vector space, and* $X$ *a nonempty subset of* $\mathbf{E}$*. Let* $F : X \to 2^{\mathbf{E}}$ *be a cusco, and suppose* $F$ *has no fixed points. Then there exists a continuous function* $f : X \to \mathbf{E}^*$ *such that for* $x \in X$*,* $f(x)$ *is strictly positive on* $x - F(x)$*.*

PROOF. For $x^* \in \mathbf{E}^*$, define

$$N_{x^*} = \{x \in X | \langle x^*, y\rangle > 0 \text{ for all } y \in x - F(x)\}.$$

Note that by assumption, $0 \notin x - F(x)$, for each $x \in X$. Also, $x - F(x)$ is closed and convex. Thus, for a fixed $x_0 \in X$, a familiar separation theorem tells us there is a $\delta > 0$ and $x^* \in \mathbf{E}^*$ with $\langle x^*, y\rangle > \delta$ for each $y \in x_0 - F(x_0)$. Further, by continuity of $x^*$, we can choose a neighbourhood $V$ of $x_0 - F(x_0)$ such that $\langle x^*, y\rangle > \frac{\delta}{2}$ for $y \in V$. By upper semicontinuity of $F$, there is a neighbourhood $U$ of $x_0$ such that $x \in U$ gives $x - F(x) \subseteq V$. Thus $U \subseteq N_{x^*}$. Hence $x_0 \in \text{int}N_{x^*}$, and $\{\text{int}N_{x^*} : x^* \in \mathbf{E}^*\}$ is an open cover of $X$. By compactness, let $\{N_{x_1^*}, \ldots, N_{x_n^*}\}$ be a finite cover of $X$, and choose $\{\beta_1, \ldots \beta_n\}$ a partition of unity subordinate to this cover. Define $f : X \to \mathbf{E}^*$ by

$$f(x) = \sum_{i=1}^{n} \beta_i(x) x_i^*.$$

Then $f$ is continuous, and for $\beta_i(x) \neq 0$, we have $\langle x_i^*, y\rangle > 0$ for all $y \in x - F(x)$. Thus $f(x)$ is strictly positive on $x - F(x)$.

■

The following lemma is typical in Browder's treatment of fixed point theorems for multifunctions. A more detailed account can be found in [16], and references therein. In this vein, Browder relates the absence of a fixed point of $F$ to the existence of a single valued function that is positive on $x - F(x)$ for some $x$.

**Lemma 4.3.5** *Let $X$ be a compact convex subset of a locally convex topological vector space $\mathbf{E}$, and $f : X \to \mathbf{E}^*$ continuous. Then there exists $x \in X$ such that $\langle f(x), x - y \rangle \geq 0$ for all $y \in X$.*

Proof. Suppose the contrary, and let $F(x) = \{y \in X : \langle f(x), x - y\rangle < 0\}$. Then by assumption, for any $x \in X$, $F(x) \neq 0$. Further, if $x_1, x_2 \in F(x)$, then for $0 \leq \lambda \leq 1$, we have

$$\langle f(x), x - (\lambda x_1 + (1-\lambda)x_2)\rangle = \lambda\langle f(x), x - x_1\rangle + (1-\lambda)\langle f(x), x - x_2\rangle < 0.$$

Thus $F(x)$ is convex. By Lemma 4.3.3, for a fixed $y \in X$, the function defined by $g(x) = \langle f(x), x - y\rangle$ is continuous. Thus $g^{-1}(-\infty, 0) = F^{-1}(y)$ is open. Now by theorem 4.3.2, $F$ has a fixed point. But $x \in F(x)$ means $0 > \langle f(x), x - x\rangle = 0$, a contradiction. ■

As stated, the theorems we present below make no explicit assumption that the intersection of $F(X)$ and $X$ is nonempty. Instead, strong assumptions on the behaviour of $F$ on the boundary of $X$ is required. Sometimes a function satisfying the hypothesis of the first theorem is called an "outward" mapping, and one satisfying the conditions of the second, an "inward" mapping.

**Theorem 4.3.6** *Let $\mathbf{E}$ be a locally convex topological vector space, and $X$ a nonempty compact convex subset of $\mathbf{E}$. Let $F : X \to 2^{\mathbf{E}}$ be a cusco, and suppose that for each $x \in \partial X$, there exists $y \in F(x)$, $z \in X$, $\lambda < 0$ such that $y - x = \lambda(z - x)$. Then $F$ has a fixed point.*

Proof. Suppose the contrary. Then by Lemma 4.3.4 , there is a continuous $f : X \to \mathbf{E}^*$ such that $f(x)$ is strictly positive on $x - F(x)$. By Lemma 4.3.5, there exists $x_0 \in X$ such that

$$\langle f(x_0), x_0 - x\rangle \geq 0, \text{ for all } x \in X. \tag{4.1}$$

There are two cases to consider.

First suppose $x_0 \notin \partial X$. Since $X$ is convex, for $y \in \mathbf{E}$, there is an $\epsilon > 0$ with $x = x_0 - \epsilon y \in X$. We have $0 \leq \langle f(x_0), x - x_0 \rangle = \epsilon \langle f(x_0), y \rangle$. Thus $\langle f(x_0), y \rangle \geq 0$ for all $y \in \mathbf{E}$. Replacing $y$ with $-y$, we obtain $\langle f(x_0), y \rangle = 0$, for all $y \in \mathbf{E}$. On the other hand, $f(x_0)$ is strictly positive on the nonempty set $x_0 - F(x_0)$, and so we have a contradiction.

Now suppose $x_0 \in \partial X$. Then by our assumption, choose $y \in F(x_0)$, $x \in X$, $\lambda < 0$ with $y - x_0 = \lambda(x - x_0)$. Since $f(x_0)$ is strictly positive on $x_0 - F(x_0)$, we have

$$0 < \langle f(x_0), x_0 - y \rangle = \lambda \langle f(x_0), x_0 - x \rangle \leq 0,$$

by (4.1) above, and since $\lambda < 0$. Thus again we have a contradiction. It follows that $F$ must have a fixed point.

■

The corresponding result for inward mappings is easily obtained from the previous theorem.

**Theorem 4.3.7** *Let* $\mathbf{E}$ *be a locally convex topological vector space, and* $X$ *a nonempty compact convex subset of* $\mathbf{E}$. *Let* $F : X \to 2^{\mathbf{E}}$ *be a cusco, and suppose that for each* $x \in X$, *there exists* $y \in F(x)$, $z \in X$, $\lambda > 0$ *such that* $y - x = \lambda(z - x)$. *Then* $F$ *has a fixed point.*

Proof. We reduce this to the previous case of theorem 4.3.6. Define $G : X \to 2^{\mathbf{E}}$ by $G(x) = 2x - F(x)$. Then $G$ is a *cusco*. Now, for $x \in \partial X$, choose $y \in F(X)$, $z \in X$, $\lambda > 0$ such that $y - x = \lambda(z - x)$. Then $2x - y \in G(x)$, and

$$(2x - y) - x = x - y = -\lambda(z - x), \text{ with } -\lambda < 0.$$

Thus $G$ satisfies the hypothesis for the previous theorem, and so has a fixed point, say $x$. Then $x \in 2x - F(x)$. It follows that $x$ is a fixed point for $F$.

■

# Chapter 5

# Applications

The application of topological fixed point theory is vast. In fields such as economics and game theory it is the only known method by which many important results are obtained. Brouwer can be used to show the existence of a winner in the game of Hex [28], and has been useful in other two player mathematical games. The theory is also fundamental in showing the existence of solutions for differential equations. As well, Brouwer's theorem and its relatives have served to simplify existing proofs of many well known results. In this chapter we highlight a few of these applications.

## 5.1 The KKM-Map Principle

In chapter two we saw that Sperner $\Rightarrow$ KKM $\Rightarrow$ Brouwer. The ease in which this chain was established caused much speculation as to their equivalence. The question was open for almost 50 years, when in 1974 Yoseloff showed that Brouwer $\Rightarrow$ Sperner. This is perhaps one of the most important relationships Brouwer's theorem has with any result. Together, the three theorems form a mathematical trinity that has had a great number of applications.

The KKM theorem was extended to topological vector spaces in 1952 by Ky Fan [23]. Its extensions and equivalent formulations have had such an effect as to have spawned a branch of research known as KKM theory. The theorem itself has become known as the KKM–map principle, and has been widely used as a tool for fixed point theory, minimax problems, dimension theory, and mathematical economics. As such, we consider Brouwer $\Rightarrow$ KKM to be an important application of the Brouwer fixed point theorem.

**Theorem 5.1.1 (KKM)** *Let $X$ be a nonempty subset of Euclidean space $E$. Suppose that for every $x \in X$, there exists a closed subset $M(x) \subseteq X$ such that*

$$\mathit{conv} F \subseteq \bigcup_{x \in F} M(x)$$

*for all finite subsets $F$ of $X$. Then, for any finite subset $F$ of $X$,*

$$\bigcap_{x \in F} M(x) \neq \emptyset.$$

*Hence, if for some $x \in X$, $M(x)$ is compact, we have*

$$\bigcap_{x \in X} M(x) \neq \emptyset.$$

Proof. Suppose the contrary. That is, suppose the hypothesis of the theorem, and that $F = \{x_1, x_2, \ldots, x_m\}$ is a finite set in $X$ with $\cap_{i=1}^{m} M(x_i) = \emptyset$.

Consider $f : \mathrm{conv}(F) \to E$ defined by

$$y \mapsto \frac{\sum_{i=1}^{m} d_{M(x_i)}(y) x_i}{\sum_{i=1}^{m} d_{M(x_i)}(y)}.$$

This map is well–defined since by assumption $\cap_{i=1}^{m} M(x_i) = \emptyset$. Thus we have $f$ is a self–map on $\mathrm{conv}(F)$. By Brouwer's theorem, $f$ has a fixed point, say $z$.

Define $F' = \{x \in F : z \notin M(x)\}$. Now,

$$z = \frac{\sum_{i=1}^{m} d_{M(x_i)}(z) x_i}{\sum_{i=1}^{m} d_{M(x_i)}(z)}.$$

Note that $z \notin M(x) \Leftrightarrow d_{M(x)}(z) \neq 0$. Therefore $z$ can be written as a convex combination of elements in $\mathrm{conv}(F')$. From the assumed property, $\mathrm{conv}(F') \subseteq \cup_{x \in F'} M(x)$. This implies $z \in M(x)$ for some $x \in F'$, a contradiction. Thus $\cap_{i=1}^{m} M(x_i) \neq \emptyset$.

The final statement of the theorem follows using the finite intersection property characterization of compactness.

■

Multifunctions satisfying the hypothesis of the KKM theorem, that is, functions $F : X \to 2^X$ with $\mathrm{conv}\{x_1, \ldots, x_n\} \subseteq \cup_{i=1}^{n} F(x_i)$, for finite subsets $\{x_1, \ldots, x_n\}$ of $X$, are called KKM–maps. For example, let $\mathbf{E}$ be a normed linear space, $X$ a convex subset, and $f : X \to \mathbf{E}$ a continuous function. Then it can be shown that $G : X \to 2^X$ defined by

$$G(x) = \{y \in X : \|f(y) - y\| \leq \|f(y) - x\|\}$$

is a KKM–map. Thus the KKM–map principle is useful in showing the existence of best approximations. A thorough account of fixed point theory and best approximations can be found in [51].

## 5.2 Solutions to Differential Equations

The usefulness of fixed point theorems in proving the existence of solutions for differential equations cannot be overstated. Often, the solution of an equation is found to coexist with the existence of a fixed point of some related function. For example, a solution to the equation $F(x) = 0$ is a fixed point of $F(x) + x$. If the set of valid solutions can be viewed as a compact convex subset of some space, then it may be possible to show the existence of a fixed point using topological fixed point theorems. Often in this area, the space is a function space, and so Schauder's theorem may be helpful.

As an example, we show the existence of a continuous $y : \mathbf{R} \to \mathbf{R}^n$ with

$$y'(t) = f(t, y(t))$$

$$y(a) = b,$$

where $f$ is a continuous function of $t$ and $y$, in a neighbourhood of $(a, b)$.

First note that by continuity, we can assume $|f(t, y)| \leq K$, for $|t-a| \leq \epsilon$, and $|y-b| \leq \epsilon$. Let $\delta < \min\{\epsilon, \frac{\epsilon}{K}\}$. Consider the space $\mathbf{E}$ of functions from $\mathbf{R}$ into $\mathbf{R}^n$ that are continuous on $|t-a| < \delta$. We equip $\mathbf{E}$ with the uniform norm. The set to which we will apply Schauder's theorem is

$$X = \{y \in \mathbf{E} : |y(t) - b| \leq \delta K, \text{ for all } t\}.$$

It is not hard to see that $M$ is closed and convex. next, define $h : X \to X$ by

$$h(y)(t) = b + \int_a^t f(s, y(s))ds.$$

Then for $y \in X$, $|h(y)(t) - b| \leq |t-a| \max\{|f(s, y(s))|\} \leq \delta K$. Thus, $h$ is indeed a self–map of $X$. Also $h$ is continuous, since

$$|h(y_1) - h(y_2)| = \sup_t \left| \int_a^t f(s, y_1(s)) - f(s, y_2(s)) \right|$$

approaches zero as $|y_1 - y_2| \to 0$, by uniform continuity of $f$. Next note that for any $y \in X$, $|h(y)| \leq |b| + \delta K$, and so $h(X)$ is uniformly bounded. Also note that $h(X)$ is equicontinuous, since

$$|h(y)(t_1) - h(y)(t_2)| \leq \left| \int_{t_1}^{t_2} f(s, y(s))ds \right| \leq K|t_1 - t_2|.$$

It follows that $h(X)$ is precompact, and so we can apply Schauder's (second) theorem to see that $h$ has a fixed point. This fixed point solves the equation.

Banach's contraction principle is also very important for showing existence of solutions to DE's. A nice description of various problems and generic methods for changing solution existence problems into fixed point problems can be found in [52].

## 5.3 The Jordan Curve Theorem

Many authors have employed Brouwer's theorem to give a simplified proof of a known result in mathematics. One such application is Maehara's proof of the Jordan Curve theorem [42]. A Jordan curve is the image under a homeomorphism of a circle. The Jordan curve theorem says that in $\mathbf{R}^2$, the compliment of a Jordan curve $J$ consists of two components, each of which have $J$ as its boundary. This result is notoriously difficult to prove. The first rigorous proof appeared in 1905 [55]. In 1984 Maehara used Brouwer's theorem to give a simplified version of a proof by Moise [44]. Still, a simplified proof of the Jordan curve theorem need not be brief. We present here the lemmas Maehara used to bypass some of the tedious arguments in Moise's proof.

Let $J$ be a Jordan curve in $\mathbf{R}^2$. By $E(a, b; c, d)$ we mean the rectangular region $\{(x, y) : a \leq x \leq b, c \leq y \leq d\}$.

**Theorem 5.3.1** *If $\mathbf{R}^2 - J$ is not connected, then each of its components has $J$ as its boundary.*

Proof. Let $U$ be a component of $\mathbf{R}^2 - J$. Then for any other component $W$, we have $(\overline{U} \bigcap U^c) \bigcap W = \emptyset$. Thus $\overline{U} \bigcap U^c$ is contained in $J$. If $\overline{U} \bigcap U^c \neq J$, the let $A \subseteq J$ be an arc with $\overline{U} \bigcap U^c \subseteq A$. Now, since $J$ is bounded, $\mathbf{R}^2 - J$ can have only one unbounded component. Thus we must have at least one bounded component.

First assume that $U$ is a bounded component. Choose $a \in U$, and let $D$ be a disk centred at $a$, containing $J$. By the Tietze Extension theorem, the function $id : A \to A$ has a continuous extension $R : D \to A$. We define $f : D \to D \setminus \{a\}$ by

$$f(x) = \begin{cases} r(x), & \text{if } x \in \overline{U}, \\ x, & \text{if } x \in U^c. \end{cases}$$

Note that $\overline{U} \bigcap U^c \subseteq A$, so $f$ is well-defined and continuous. Now let $p : D \setminus \{a\} \to S$ be the projection through $a$ onto the boundary $S$ of the disk, and $g : S \to S$ the antipodal map. Then the function $g \circ f : D \to D$ is continuous with no fixed point, violating Brouwer's theorem. Thus $\overline{U} \bigcap U^c = J$.

The case where $U$ is unbounded follows similarly, by letting

$$f(x) = \begin{cases} r(x), & \text{if } x \in U^c, \\ x, & \text{if } x \in \overline{U}. \end{cases}$$

∎

Next Maehara uses Brouwer's theorem to prove the following. Suppose in a rectangle we have two paths, one going from the left to the right side, and the other starting at the top and ending somewhere along the bottom. Then these paths must intersect.

**Theorem 5.3.2** *Let $f(t) = (f_1(t), f_2(t))$ and $g(t) = g_1(t), g_2(t))$, for $-1 \leq t \leq 1$, be paths in $E(a, b; c, d)$ such that*

$$f_1(-1) = a, f_1(1) = b, g_2(-1) = c, g_2(1) = d.$$

*Then there exists $t \in [-1, 1]$ such that $f(t) = g(t)$.*

Proof. Suppose the contrary. If $f(s) \neq g(t)$ for all $s, t \in [0, 1]$, then the function $h$ defined by

$$h(s,t) = \frac{(g_1(t) - f_1(s), f_2(s) - g_2(t))}{\max\{|f_1(s) - g_1(t)|, |f_2(s) - g_2(t)|\}}$$

is well–defined and continuous, and maps $E$ into its boundary. Thus by Brouwer's theorem, $h$h has a fixed point say $(s, t)$. But $h(s, t) = (s, t)$ implies $|s| = 1$ or $|t| = 1$. Examining these four possibility gives a contradiction in each case.

∎

Using these two results, Maehara proves the Jordan Curve theorem by first constructing a particular point $z \in \mathbf{R}^2 - J$, and proving that the component containing $z$ must be bounded, and then showing there can be no other bounded component than this. For details, see [42].

## 5.4 Existence of Equilibrium Points

Kakutani's theorem has had far reaching influences in game theory and economic theory where the goal is to show the existence of an equilibrium point. There are literally hundreds of citations of the theorem in the literature. Famous results by Nash, von Neumann, and Ky Fan have been shown to be consequences of Kakutani's fixed point theorem. Using Brouwer's theorem, Arrow and Debreu put to rest the centuries old question of the existence of an equilibrium in a competitive economy [4]. Here we present a famous lemma by von Neumann, from which he obtained a fundamental result in the theory of games.

**Theorem 5.4.1 (von Neumann)** *Let $X$ and $Y$ be nonempty compact convex subsets of $\mathbf{R}^n$. Suppose $A$ and $B$ are closed subsets of $X \times Y$ such that $A(y) = \{x \in X : (x, y) \in A\}$ and $B(x) = \{y \in Y : (x, y) \in B\}$ are nonempty closed convex subsets of $X$ and $Y$ respectively. Then $A \bigcap B$ is nonempty and compact.*

The original proof by Von Neumann was somewhat complicated and used a notion of integrals in Euclidean space. The theorem is very easy to prove using Kakutani's theorem. In fact, Kakutani published his theorem as a method for proving this lemma, from which it is not hard to obtain von Neumann's minimax theorem.

Proof. Define $F : X \times Y \to X \times Y$ by $F(x, y) = A(y) \times B(x)$. Compactness, convexity, and nonemptiness of $F(x, y)$ follow from that of its components parts. Also , it follows from the closedness of $A$ and $B$ that $F$ is *usc*. Thus by Kakutani's theorem, $F$ has a fixed point, say $(x, y) \in A(y) \times B(x)$. It follows that $(x, y) \in A \bigcap B$.

■

From the lemma, it is not hard to obtain the famous von Neumann minimax theorem. We state the theorem here, and refer to [36] for the proof.

**Theorem 5.4.2** *Let $K \subseteq \mathbf{R}^m$, $L \subseteq \mathbf{R}^n$ be closed and convex, and $f : K \times L \to \mathbf{R}$ continuous. Suppose for all $x \in K$ and $\alpha in \mathbf{R}$, $\{y \in L : f(x, y) \leq \alpha\}$ is convex, and for all $y \in L$, and $\beta in \mathbf{R}$, $\{x \in K : f(x, y) \geq \beta\}$ is convex. Then*

$$\max_{x \in K} \min_{y \in L} f(x, y) = \min_{y \in L} \max_{x \in K} f(x, y).$$

## 5.5 A Note on the Fundamental Theorem of Algebra

In a 1949 paper [2], B.H. Arnold wrote "...it has been known for some time that the fundamental theorem of algebra could be derived from Brouwer's fixed point theorem.". He continued to give a simple one page proof that a polynomial with degree $n \geq 1$ has at least one complex root. Unfortunately, he did this by applying Brouwer's theorem to a discontinuous function. This error was spotted and a correction was published less than two years later [3]. In 1951, M.K. Fort followed up with a brief proof, using the existence of continuous $n$th roots of a continuous non zero function on a disk in the complex plane to show that Brouwer can be used to prove the fundamental theorem [25]. In this author's opinion, except for an ill–fated choice of titles, this would have probably been the end of the incorrect proof by Arnold. Arnold's paper was boldly called "A Topological Proof of the Fundamental Theorem of Algebra", whereas Fort's paper was modestly titled "Some Properties of Continuous Functions". Since 1949, citations of Arnold's paper have popped up from time to time as proof of the existence of a topological proof of the fundamental theorem of algebra, including in a four hundred page volume on fixed point theory published in 1981, a talk given in 2000 at a meeting of the Association for Symbolic Logic, as well as in newsgroup discussions that are at least as recent as 1998. On the other hand, this author had a difficult time tracking down Fort's paper, even knowing before hand that it did exist. No amount of searching electronic databases with relevant key words would produce it, though Arnold's paper was invariably returned. Perhaps it would have been useful for Fort to rename his work "A Correct Topological Proof of the Fundamental Theorem of Algebra". It seems certain that Arnold would have preferred this to having his blunder quoted so long after he made his apologies.

# Appendix: A Compact Contractible Set Without the tfpp

It was long thought that contractibility and compactness were the key properties needed for a set in finite dimensions to inherit the fixed point property. A set with a hole in it could be rotated about this hole, leaving no points fixed. A set missing limit points could be shifted towards a limit point, fixing no elements. In 1932 K. Borsuk asked if a compact contractible set should always have the fixed point property. The question was open for more than 20 years, until Shin'ichi Kinoshita gave the following example [38].

We construct our mapping on a subset of $\mathbf{R}^3$, made up of the following sets, $A_1$, $A_2$, and $A_3$.

$$A_1 = \{(r, \theta, z) : 0 \le \theta \le 2\pi, 0 \le r < 1, z = 0\}$$

$$A_2 = \{(r, \theta, z) : 0 \le \theta \le 2\pi, r = 1, 0 \le z \le 1\}$$

$$A_3 = \{(r, \theta, z) : r = \frac{2}{\pi}\arctan(\varphi), \theta = \varphi \bmod 2\pi, 0 \le \varphi < \infty, 0 \le z \le 1\}$$

We equip $A$ with the relative topology from $\mathbf{R}^3$. $A$ is closed and bounded, thus compact. Let $g$,$h : A \to A$ be defined by $g(r, \theta, z) = (r, \theta, 0)$, and $h(r, \theta, z) = (0, \theta, z)$. Then $h \circ g$ testifies to the contractibility of $A$.

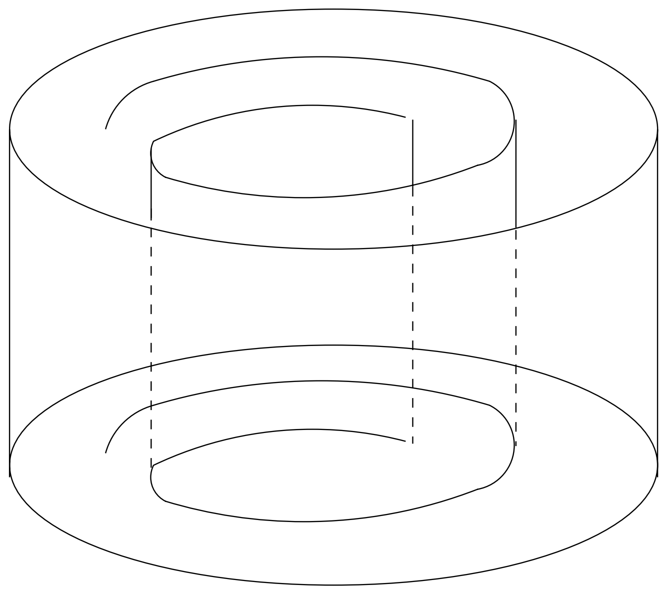

Figure 5.1: Kinoshita's example: $A = A_1 \cup A_2 \cup A_3$.

Kinoshita's example was based on the following idea. Imagine grasping the top of $A$ with on hand, the bottom with another hand. Twist the top counter-clockwise $\pi$ radians, the bottom clockwise $\pi$ radians. As you twist $A$ in this manner, in order to stay in $A_3$, elements in $A_3$ near $z = 1$ will increase in radius as they rotate. Elements in $A_3$ near $z = 0$ will shrink in radius as they are rotated. As $A_1$ is rotated, elements close to the boundary $r = 1$ will be rotated $\pi$ radians. Those closer to the center will be stretched towards the origin as they are rotated.

As described, this mapping will have problems at the origin. Thus in a neighbourhood around the origin we have to do some extra twisting and turning to guarantee the function has the desired properties. The following function formalizes this procedure.

On $A_1$ we define $f : A \to A$ by

$$f(r, \theta, z) = \begin{cases} (\frac{2}{\pi}\arctan(\tan(\frac{\pi r}{2} - \pi)), \theta - \pi, 0) & r \geq \frac{2}{\pi}\arctan(\pi) \\ (0, 0, 1 - \frac{1}{\pi}\arctan(\frac{\pi r}{2})) & 0 \leq r \leq \frac{2}{\pi}\arctan(\pi) \end{cases}.$$

Thus $f$ maps $(0, 0, 0)$ to $(0, 0, 1)$, the circle $r = \frac{2}{\pi}\arctan(\pi)$ to $(0, 0, 0)$, and the points inside this circle onto the segment $r = 0$, $0 < z \leq 1$. The points comprising the annulus outside this circle are mapped as described in the above description. That is, the annulus is rotated $\pi$ radians as its inner boundary circle is contracted to the origin. Note that this function leaves no point of $A_1$ fixed.

On $A_2$ $f$ is given by

$$f(1,\theta,z) = \begin{cases} (1,(\theta-\pi+2\pi z) \mod 2\pi, z+\frac{z}{2}) & 0 \le z \le \frac{1}{2} \\ (1,(\theta-\pi+2\pi z) \mod 2\pi, \frac{1}{2}+\frac{z}{2}) & \frac{1}{2} \le z \le 1 \end{cases}.$$

Thus the top of the cylinder is rotated counter clockwise $\pi$ radians, the bottom clockwise $\pi$ radians. At the same time, the circle $r = \frac{2}{\pi}\arctan(\pi)$,$z = 1$, is pulled upward to $z = \frac{3}{4}$, stretching the bottom half of the cylinder with it, and compressing the top half of the cylinder between $\frac{3}{4} \le z \le 1$. The twisting motion ensures no points in $A_2$ off the circle at $z = \frac{1}{2}$ are fixed, and the lifting motion ensures no point on this circle is left fixed.

On $A_3$, $f$ is a little more complicated. Outside the circle $r = \frac{2}{\pi}\arctan(\pi)$, $f$ behaves pretty much as described in the discussion. However, to keep continuity with regard to $f$ as defined on $A_1$, and to ensure no point on the segment $0 \le z \le 1$, $r = 0$ remains fixed, we have to do a little more twisting and turning.

For $0 \le \varphi \le \pi$, define $f$ on $A_3$ by

$$f(r,\theta,z) = \begin{cases} (\frac{2}{\pi}\arctan((\varphi+\pi)z, (\varphi+\pi)z \mod 2\pi, 1+\frac{\varphi}{\pi}(\frac{3}{2}z-1)) & 0 \le z \le \frac{1}{2} \\ (\frac{2}{\pi}\arctan((\varphi+\pi)z, (\varphi+\pi)z \mod 2\pi, 1+\frac{\varphi}{\pi}(\frac{1}{2}z-\frac{1}{2})) & \frac{1}{2} \le z \le 1 \end{cases}.$$

For $\pi \le \varphi < \infty$, $f$ is define by

$$f(r,\theta,z) = \begin{cases} (\frac{2}{\pi}\arctan(\varphi-\pi+2\pi z), (\varphi-\pi+2\pi z) \mod 2\pi, z+\frac{z}{2}) & 0 \le z \le \frac{1}{2} \\ (\frac{2}{\pi}\arctan(\varphi-\pi+2\pi z), (\varphi-\pi+2\pi z) \mod 2\pi, \frac{1}{2}+\frac{z}{2}) & \frac{1}{2} \le z \le 1 \end{cases}.$$

It is easily verified that on $A_1 \cap A_2$, $f|_{A_1} = f|_{A_2}$, and likewise $f$ is consistent on $A_1 \cap A_3$. Also $\lim_{r\to 1} f|_{A_3}(r,\theta,z) = f|_{A_2}(1,\theta,z)$. Since $f$ is continuous on each component of $A$, it is continuous on $A$, and by design has no fixed point.

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