# On the Ramanujan AGM fraction

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Abstract. The Ramanujan AGM fraction is a construct

$$\mathcal{R}_\eta(a,b) = \cfrac{a}{\eta + \cfrac{b^2}{\eta + \cfrac{4a^2}{\eta + \cfrac{9b^2}{\eta + \ddots}}}}$$

enjoying attractive algebraic properties such as a striking arithmetic-geometric mean (AGM) relation and elegant connections with elliptic-function theory. But the fraction also presents an intriguing computational challenge. Herein we show how to rapidly evaluate $\mathcal{R}$ for any triple of positive reals $a, b, \eta$, the problematic scenario being when $a \approx b$, although even in such cases certain transformations allow rapid evaluation. In this process we find, for example, that when $a = b =$ rational, $\mathcal{R}_\eta$ is essentially an $L$-series that can be cast therefore as a finite sum of fundamental numbers. We ultimately exhibit an algorithm that yields $D$ good digits of $\mathcal{R}$ in $O(D)$ iterations where the implied big-$O$ constant is independent of the positive-real triple $a, b, \eta$. Finally, we address the evidently profound theoretical and computational dilemmas that arise when the parameters are allowed to become complex, finding means to extend the AGM relation for complex parameter domains.

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# 1 Introduction

In Entry 12 of Chapter 18 of Ramanujan's Second Notebook [?] one finds the beautiful construct

$$\mathcal{R}_\eta(a,b) = \cfrac{a}{\eta + \cfrac{b^2}{\eta + \cfrac{4a^2}{\eta + \cfrac{9b^2}{\eta + \ddots}}}} \tag{1.1}$$

which we interpret—in most but not all of the present treatment—for real $a, b, \eta > 0$. Remarkably, for the indicated parameter space, $\mathcal{R}$ satisfies an AGM relation

$$\mathcal{R}_\eta\left(\frac{a+b}{2}, \sqrt{ab}\right) = \frac{\mathcal{R}_\eta(a,b) + \mathcal{R}_\eta(b,a)}{2}. \tag{1.2}$$

This relation is one of many expedients we shall develop with a view to computation of $\mathcal{R}_\eta$. It will turn out that the computationally difficult cases can be summarized in the phrase "$b$ is near to $a$," including the case $a = b$. What we shall eventually exhibit is a computational algorithm that is uniformly of geometric/linear convergence across the entire positive quadrant $a, b > 0$. Along the way, we find attractive identities, such as the expression of any $\mathcal{R}_\eta(r, r)$, where $r$ is rational, as a finite series of fundamental numbers. Finally, we consider complex $a, b$ and note the considerable difficulties in such analysis; accordingly we prove results and posit various conjectures pertaining to fraction convergence and the domain of validity for the AGM relation (1.2).

We would be remiss in not adding this perspective: The present research began when the authors realized—via numerical experimentation—that $\mathcal{R}_1(1, 1)$ "seemed to be" the number $\log 2$. Such is the value of experiment: One can be led thereby into deep waters.

# 2 Preliminaries

An initial observation is that

$$\mathcal{R}_\eta(a,b) = \mathcal{R}_1(a/\eta, b/\eta),$$

as can be formally inferred by cancellation of the $\eta$ elements down through the fraction form. Such manipulations are valid when the continued fraction $\mathcal{R}_\eta$ converges. One way to prove convergence—at least for positive, real $a, b$ is to put the entity $a/\mathcal{R}_1$ in RCF (reduced continued fraction) form, meaning

$$\mathcal{R}_1(a,b) = \frac{a}{[A_0; A_1, A_2, A_3, \dots]} \tag{2.1}$$

$$:= \cfrac{a}{A_0 + \cfrac{1}{A_1 + \cfrac{1}{A_2 + \cfrac{1}{A_3 + \ddots}}}}$$

where the elements $A_i$ are all positive real. (We take the more restricted, classical mnemonic SCF (simple continued fraction) to denote the instance of an RCF where each $A_i$ is a positive integer; in our present case, though, the $A_i$ are not generally integers.) Inspection of Ramanujan's pattern for $\mathcal{R}$ reveals that the RCF elements can be given explicitly:

$$A_n = \frac{n!^2}{(n/2)!^4} 4^{-n} \frac{b^n}{a^n} \quad \sim \quad \frac{2}{\pi n} \frac{b^n}{a^n} \qquad , \; n \text{ even},$$

$$A_n = \frac{((n-1)/2!)^4}{n!^2} 4^{n-1} \frac{a^{n-1}}{b^{n+1}} \quad \sim \quad \frac{\pi}{2abn} \frac{a^n}{b^n} \qquad , \; n \text{ odd},$$

where we have indicated also the asymptotic behavior of the $A_n$. This element representation leads immediately to

**Theorem 2.1:** For any positive real pair $a, b$ the fraction $\mathcal{R}_1(a, b)$ converges (to a finite limit).

**Remark:** In continued fraction theory, there is the concept of converging on the extended complexes, including $\infty$ (see Section 9), so we have signified a finite limit in Theorem 2.1.

**Proof:** It is know that an RCF with positive real elements converges to a finite limit iff $\sum A_i$ diverges (this is the Seidel–Stern theorem [**?**, **?**]). In our case, such divergence is evident for any choice of real $a, b > 0$. □

Indeed, the divergence of $\sum A_i$ is only logarithmic for $a = b$, and this is a true indication of slow convergence (we wax more quantitatively later). Sure enough, our interest in the computational aspect started with the question of how to rapidly evaluate

$$\mathcal{R}(a) := \mathcal{R}_1(a, a)$$

for positive real a and thereby to prove some suspected identities. We shall later encounter a different continued fraction for $\mathcal{R}(a)$, as well as other computationally efficient constructs.

# 3 Sech-elliptic forms

Using connections between standard Jacobi theta functions $\theta_2, \theta_3$ and elliptic integrals we can establish various results, observing that the wonderful sech identities to follow stem from classical work of Rogers, Stieltjes, Preece, and of course Ramanujan [**?**] in which one may find the earlier work detailed. We start with

**Theorem 3.1:** For real $y, \eta > 0$ and $q := e^{-\pi y}$ we have

$$\eta \sum_{k \in D} \frac{\operatorname{sech}(k\pi y/2)}{\eta^2 + k^2} = \mathcal{R}_\eta(\theta_2^2(q), \theta_3^2(q)),$$

$$\eta \sum_{k \in E} \frac{\operatorname{sech}(k\pi y/2)}{\eta^2 + k^2} = \mathcal{R}_\eta(\theta_3^2(q), \theta_2^2(q)),$$

where $D, E$ denote respectively the odd, even integers. Accordingly, the Ramanujan AGM identity (1.2) holds for positive triples $\eta, a, b$. □

**Remark:** The proof following for the AGM conclusion has been interpreted for certain complex $a, b$ sometimes incorrectly in the literature (see Section 9). For the moment, we are stating the last, AGM part of Theorem 3.1 for positive reals $a, b$, with attention paid to complex parameters later in the present work.

**Proof:** The sech relations are proved—with somewhat different but equivalent notation—in Berndt's treatment [**?**, Vol II, Ch. 18] of Ramanujan's Notebooks. As for the AGM identity, observe that for $0 < b < a$ the assignments

$$\theta_2(q)^2/\theta_3(q)^2 := b/a$$

$$\eta := \theta_2(q)^2/b$$

are possible (since $b/a \in [0, 1)$, see [**?**]) implicitly define $q, \eta$, and together with the Jacobi identities

$$\theta_2(q)^2 + \theta_3(q)^2 = \theta_3(\sqrt{q})^2,$$
$$2\theta_2(q)\theta_3(q) = \theta_2(\sqrt{q})^2$$

and the sech sums above yield

$$\mathcal{R}_1\left(\theta_3(q)^2/\eta, \theta_2(q)^2/\eta\right) + \mathcal{R}_1\left(\theta_2(q)^2/\eta, \theta_3(q)^2/\eta\right) = 2\mathcal{R}_1\left(\theta_3(\sqrt{q})^2/(2\eta), \theta_2(\sqrt{q})^2/(2\eta)\right)$$

and so the AGM identity (1.2) holds for any positive reals $a > b$. The case $0 < a < b$ is handled symmetrically, starting with $\theta_2(q)^2/\theta_3(q)^2 := a/b$. □

These sech series can be used in turn to establish two evaluation series involving the standard elliptic integral $K$:

**Theorem 3.2:** In what follows we intend $\mathrm{K} := \mathrm{K}(k)$, $\mathrm{K}' := \mathrm{K}(k')$ with $k' := \sqrt{1 - k^2}$. For real $0 < b < a$ and $k := b/a$ we have

$$\mathcal{R}_1(a, b) = \frac{\pi a \mathrm{K}}{2} \sum_{n \in Z} \frac{\operatorname{sech}\left(n\pi \frac{\mathrm{K}'}{\mathrm{K}}\right)}{\mathrm{K}^2 + \pi^2 a^2 n^2}. \tag{3.1}$$

On the other hand, for $0 < a < b$ and $k := a/b$ we have

$$\mathcal{R}_1(a,b) = 2\pi b \mathrm{K} \sum_{n \in D} \frac{\operatorname{sech}\left(n\pi \frac{\mathrm{K}'}{2\mathrm{K}}\right)}{4\mathrm{K}^2 + \pi^2 b^2 n^2}. \tag{3.2}$$

**Proof:** The two series here follow from the assignments $\theta_3^2(q)/\eta := \max(a,b)$, $\theta_2^2(q)/\eta := \min(a,b)$ and classical relations

$$e^{-\pi K'/K} = q, \qquad K = \frac{\pi}{2}\theta_3(q)^2$$

inserted into the appropriate sech identities from Theorem 3.1. □

The sech-elliptic series (3.1), (3.2) do allow for rapid computation of $\mathcal{R}_1(a,b)$ when $b$ is *not* too near to $a$. Indeed, to get $D$ good digits for $\mathcal{R}_1$ one requires $O(D\mathrm{K}/\mathrm{K}')$ summands. So, yet another motive for the present analysis was the problem of slow convergence of the sech-elliptic form for $b \approx a$.

Note that we also have attractive evaluations such as

$$\mathcal{R}_1\left(1, \frac{1}{\sqrt{2}}\right) = \frac{\pi}{2}\mathrm{K}(1/\sqrt{2}) \sum_{n \in \mathcal{Z}} \frac{\operatorname{sech}(n\pi)}{\mathrm{K}^2(1/\sqrt{2}) + n^2\pi^2},$$

where we remind ourselves that $\mathrm{K}(1/\sqrt{2}) = \Gamma^2(1/4)/(4\sqrt{\pi})$ [?], with similar series for $\mathcal{R}_1(1, k_N)$ at the $N$-th singular value as discussed in [?]. A similar relation for $\mathcal{R}_1\left(\frac{1}{\sqrt{2}}, 1\right)$ obtains via (3.2), and via the AGM relation (1.2) yields in turn the oddity

$$\mathcal{R}_1\left(\frac{1+\sqrt{2}}{2\sqrt{2}}, \frac{1}{2^{1/4}}\right) = \pi \mathrm{K}(1/\sqrt{2}) \sum_{n \in calZ} \frac{\operatorname{sech}(n\pi/2)}{4\mathrm{K}^2(1/\sqrt{2}) + n^2\pi^2}.$$

# 4 Relations for $\mathcal{R}(a)$

Recalling that $\mathcal{R}(a) := \mathcal{R}_1(a,a)$ we next derive relations for the problematic cases $b = a$. Interpreting (3.1) as a Riemann-integral relation in the limit $b \to a^-$, we have (for $a > 0$) a slew of relations involving the digamma function $\psi := \Gamma'/\Gamma$ [?, ?] and the Gauss-hypergeometric $F$, here presented in an order that can be serially derived:

$$\begin{aligned}
\mathcal{R}(a) &= \int_0^\infty \frac{\operatorname{sech}\left(\frac{\pi x}{2a}\right)}{1+x^2}\,dx = 2\,a \sum_{k1}^{\infty} \frac{(-1)^{k+1}}{1+(2\,k-1)\,a} \\
&= \frac{1}{2}\left(\psi\left(\frac{3}{4}+\frac{1}{4a}\right) - \psi\left(\frac{1}{4}+\frac{1}{4a}\right)\right) \\
&= \frac{2a}{1+a}\,F\left(\frac{1}{2a}+\frac{1}{2}, 1; \frac{1}{2a}+\frac{3}{2}; -1\right) \\
&= 2\int_0^1 t^{1/a}(1+t^2)^{-1}\,dt = \int_0^\infty e^{-x/a}\operatorname{sech}(x)\,dx.
\end{aligned}$$

The first series representation or the $t$-integral can be used to establish a recurrence

$$\mathcal{R}(a) = \frac{2a}{1+a} - \mathcal{R}\left(\frac{a}{1+2a}\right),$$

while known relations for the digamma [?, ?] can be used—with some symbolic care—to derive

$$\mathcal{R}(a) = \frac{\pi}{2}\sec\frac{\pi}{2a} - 2\frac{a^2(1+8a-106a^2+280a^3+9a^4)}{1-12a+25a^2+120a^3-341a^4-108a^5+315a^6} + C(a) \tag{4.1}$$

where

$$C(a) = \frac{1}{2}\sum_{n\geq 1}(\zeta(2n+1)-1)\frac{(3a-1)^{2n}-(a-1)^{2n}}{(4a)^{2n}}$$

is a "rational-zeta" series as analyzed in [?]. Note that this representation of $\mathcal{R}(a)$, while allowing rapid convergence for some $a$, has sec poles, some of which are cancelled by the rational function. In any case we require $a > 1/9$ for convergence of the rational-zeta sum. However, as a computational matter, the recurrence relation above can generally be used to force convergence of such a rational-zeta series.

The hypergeometric form above is of special interest, for there is the Gauss continued fraction for $F(\gamma, 1; 1+\gamma; -1)$, as discussed in [?]. Without belaboring the reader with details, we simply give a relevant RCF form, which form will later prove useful in convergence analysis:

$$F(\gamma,1;1+\gamma;-1) = [\alpha_1, \alpha_2, \ldots] \tag{4.2}$$

$$= \cfrac{1}{\alpha_1 + \cfrac{1}{\alpha_2 + \cfrac{1}{\alpha_3 + \cfrac{1}{\alpha_4 + \ddots}}}}$$

where $\alpha_1 = 1$ and

$$\alpha_n = ((n-1)/2)!)^{-2}\gamma(n-1+\gamma)\prod_{j1}^{(n-3)/2}(j+\gamma)^2 \quad ; n = 3,5,7,\ldots$$

$$\alpha_n = \frac{1}{\gamma}(n/2-1)!^2(n-1+\gamma)\prod_{j1}^{n/2-1}(j+\gamma)^{-2} \quad ; n = 2,4,6,\ldots$$

An interesting aspect of formal analysis is based upon the first sech-integral form above for $\mathcal{R}(a)$. Expanding said integral formally, and using a representation of the Euler number $E_{2n} = (-1)^n \int_0^\infty \mathrm{sech}(\pi x/2)x^{2n}\,dx$, one obtains

$$\mathcal{R}(a) \sim \sum_{n\geq 0} E_{2n}\, a^{2n+1},$$

where we are indicating an asymptotic series of *zero* radius of convergence. (It is a classic theorem of Borel [?, ?] that for every real sequence $(a_n)$ there exists a $C_\infty$ function $f$ on $R$ such that $f^{(n)}(0)a_n$.) It is possible to give expressions for the asymptotic error:

$$\left|\mathcal{R}(a) - \sum_{n1}^{N-1} E_{2n}\, a^{2n+1}\right| \le |E_{2N}|\; a^{2N+1},$$

[?, ?], but also interesting is to employ Padé approximants to the formal asymptotic series. It turns out that the oft-stated success of the Padé approach is exemplified well in our case. Indeed, if one takes the unique $(3,3)$ Padé form (meaning numerator and denominator of $\mathcal{R}(a)/a$) each have degree 3 in the variable $a^2$ we obtain

$$\mathcal{R}(a) \approx a\frac{1 + 90\,a^2 + 1433\,a^4 + 2304\,a^6}{1 + 91\,a^2 + 1519\,a^4 + 3429\,a^6}.$$

Even this simple approximant is remarkably good for small $a$; e.g., yielding $\mathcal{R}(1/10) \approx 0.09904494$ which is correct to the implied precision. For something like $\mathcal{R}(1/2)$ and the $(30,30)$ Padé approximant—so that numerator and denominator have degree 30 in $a^2$—one obtains 4 good digits. Though the convergence rate is slower for larger $a$, the method does give rapid means of, say, graphing the $R$ function to reasonable precision.

Having briefly discussed a formal expansion at $a = 0$, can one establish an asymptotic form for *large* $a$? The answer is yes—except that through a typical development for asymptotic forms we are rewarded with more, namely a convergent expansion for all $a > 1$. Using our second sech integral $\mathcal{R}(a) = \int_0^\infty e^{-x/a}\text{sech}x\,dx$ we can again use the Euler numbers and known Hurwitz-zeta evaluations of sech-power integrals for *odd* powers to obtain a convergent series valid at least for real $a > 1$:

$$\begin{aligned}\mathcal{R}(a) &= \frac{\pi}{2}\sec\left(\frac{\pi}{2a}\right) - 2\sum_{m\in D^+}\frac{\eta(m+1)}{a^m}\\ &= 2\sum_{k\ge 0}\eta(k+1)\left(\frac{-1}{a}\right)^k \qquad (4.3)\end{aligned}$$

where $D^+$ denotes the positive odd integers and $\eta(s) := 1/1^s - 1/3^s + 1/5^s - \dots$ (this standard alternating zeta function $\eta$ not to be confused with Ramanujan's $\eta$ parameter). Remarkably, we find that the leading terms for large $a$ involve the Catalan constant $G := \eta(2)$ as

$$\mathcal{R}(a) = \frac{\pi}{2} - \frac{2G}{a} + \frac{\pi^3}{16a^2} - \dots,$$

a development certainly difficult to infer by casual inspection of the Ramanujan fraction. (Even the asymptote $\mathcal{R}(\infty) = \pi/2$ is difficult to so infer, although such is clear from various of the previous representations for $\mathcal{R}(a)$.)

Using recurrence relations together with various expansions we have derived yields certain results pertaining to the derivatives of $\mathcal{R}$, notably

$$\mathcal{R}'(1) = 8(1-G),$$

$$\mathcal{R}'(1/2) = \pi^2/24.$$

To close this section we note that a peculiar property of the digamma $\psi$ leads to a general, exact evaluation of the imaginary part of $\mathcal{R}(a)$ when $a$ lies on the circle $C_{1/2} := \{z : |z - 1/2| = 1/2\}$ in the complex plane. Because imaginary parts of certain digamma evaluations can be expressed in closed form [**?**, **?**], we have, for $a \in C_{1/2}$ and $y := i(1-1/a)$ (which $y$ is therefore real)

$$\mathrm{Im}(\mathcal{R}(a)) = -\frac{1}{y} + \frac{\pi}{2}\mathrm{cosech}\left(\frac{\pi y}{2}\right).$$

Thus we have an elementary form for $\mathrm{Im}(\mathcal{R})$ on a certain continuum set. Admittedly we have not yet discussed convergence for complex parameters in depth; we do that later in Section 9.

# 5 The $\mathcal{R}$ function at rational arguments

From relations in Section 4 we have, for positive integers $p, q$, and recalling again that $\mathcal{R}(a) := \mathcal{R}_1(a, a)$,

$$\mathcal{R}\left(\frac{p}{q}\right) = 2p\left(\frac{1}{q+p} - \frac{1}{q+3p} + \frac{1}{q+5p} - \dots\right),$$

which is essentially in the form of a particular $L$-function. One way to evaluate $L$ functions in finite form is to apply Fourier-transform techniques to pick out the correct terms from a general logarithmic series. (We note that an equivalent, elementary form for the digamma at rational arguments is a celebrated result of Gauss.) In our case

$$\mathcal{R}\left(\frac{p}{q}\right) = \sum_{0\ <\ \mathrm{odd}\ k\ < 4p} e^{-2\pi 1k(q+p)/(4p)}\left(-\log(1 - e^{2\pi ik/(4p)}) - \sum_{n1}^{q+p-1} e^{2\pi ikn/(4p)}/n)\right).$$

After various simplifications, especially forcing everything to be real-valued, we arrive at a finite series in fundamental numbers, namely

$$\mathcal{R}\left(\frac{p}{q}\right) = -2p\sum_{n1}^{p+q-1}\frac{1}{n}\left(\delta_{n\equiv p+q \bmod 4p} - \delta_{n\equiv 3p+q \bmod 4p}\right) \tag{5.1}$$

$$-2\sum_{0\ <\mathrm{odd}\ k<2p}\left(\cos\left(\frac{(p+q)k\pi}{2p}\right)\log\left(2\sin\left(\frac{\pi k}{4p}\right)\right) - \pi\left(\frac{1}{2} - \frac{k}{4p}\right)\sin\left(\frac{(p+q)k\pi}{2p}\right)\right).$$

Note that when $q = 1$, that is we seek $\mathcal{R}(p)$ for some integer $p$, the first, rational sum vanishes. The manifestly finite series (5.1) (of $O(p+q)$ total terms) leads quickly to exact evaluations such as

$$\mathcal{R}(1/4) = \frac{\pi}{2} - \frac{4}{3}, \qquad \mathcal{R}(1/3) = 1 - \log 2,$$

$$\mathcal{R}(1/2) = 2 - \pi/2, \qquad \mathcal{R}(2/3) = 4 - \frac{\pi}{\sqrt{2}} - \sqrt{2}\log(1+\sqrt{2}),$$
$$\mathcal{R}(1) = \log 2, \qquad \mathcal{R}(3/2) = \pi + \sqrt{3}\log\left(2-\sqrt{3}\right),$$
$$\mathcal{R}(2) = \sqrt{2}\left\{\frac{\pi}{2} - \log(1+\sqrt{2})\right\}, \qquad \mathcal{R}(3) = \frac{\pi}{\sqrt{3}} - \log 2,$$

and, of course, many other attractive forms. It is not hard to establish from the finite series (5.1) that for positive integer $Q$ one has

$$\mathcal{R}(1/q) = \text{rational } + (-1)^{(q-1)/2}\log 2 \quad ; q \text{ odd},$$
$$\mathcal{R}(1/q) = \text{rational } + (-1)^{q/2}\,\pi/2 \quad ; q \text{ even}.$$

These facts can also be derived on knowledge of $\mathcal{R}(1) = \log 2$, $\mathcal{R}(1/2) = 2 - \pi/2$ and the recurrence

$$\mathcal{R}\left(\frac{1}{q}\right) = \frac{2}{q-1} - \mathcal{R}\left(\frac{1}{q-2}\right).$$

On the other hand, one of the more alluring integer-argument evaluations involves the golden mean $\tau = (1+\sqrt{5})/2$, as

$$\mathcal{R}(5) = \frac{\pi}{\sqrt{\tau\sqrt{5}}} + \log 2 - \sqrt{5}\log\tau,$$

although such evaluations—stemming from (5.1)—can involve quite delicate symbolic manipulations. We have not analyzed the possibility of evaluating $\mathcal{R}(a)$ for *irrational* $a$ via the expedient of approximating $a$ first via high-resolution rationals, and then using (5.1), although such a development would be of both computational and theoretical interest.

Incidentally, armed with exact knowledge of $\mathcal{R}(p/q)$ we find some interesting Gauss-fraction results, in the form of rational multiples of $F(\gamma, 1; 1+\gamma; -1) = [\alpha_1, \alpha_2, \ldots]$, for example on the basis of (4.2) we have

$$\mathcal{R}(1) = \log 2 = \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{1 + \ddots}}}},$$

but alas the beginnings of this fraction are misleading; subsequent elements $a_n$ go according to

$$\log 2 = [1, 2, 3, 1, 5, \frac{2}{3}, 7, \frac{1}{2}, 9, \frac{2}{5}, \ldots],$$

being as $\alpha_n = n, 4/n$ respectively as $n$ is odd, even. Similarly, one can derive

$$2 - \log 4 = [1^3, r_2, 2^3, r_4, 3^3, r_6, 4^3, \ldots],$$

where the even-indexed fraction elements $r_{2n}$ are certain rationals. Though these RCFs are not SCFs (integer elements), still the growths of the $\alpha_n$ provide a clue to the convergence rate, which rate we study in a subsequent section.

## 6 Transformation of $\mathcal{R}_1(a,b)$

There is one remaining avenue that must be traversed in order to provide a uniformly rapid evaluation scheme for $\mathcal{R}_1(a,b)$ with $a,b$ positive real. We have mentioned that the sech-elliptic series (3.1) (also (3.2)) will converge slowly when $b \approx a$, yet in Sections 4,5 we successfully addressed the case $b = a$. So, we now proceed to establish a series representation for the case that $b < a$ but $b$ is very near to $a$. We employ the wonderful fact that sech is its own Fourier transform, in that

$$\int_{-\infty}^{\infty} e^{i\gamma x}\mathrm{sech}(\lambda x)\,dx = \frac{\pi}{\lambda}\mathrm{sech}\frac{\pi\gamma}{2\lambda}.$$

Using this relation, one can perform a Poisson transform of the sech-elliptic series (3.1). The success of the transform depends on knowing

$$I(\lambda,\gamma) = \int_{-\infty}^{\infty} \frac{\mathrm{sech}\lambda x}{1+x^2} e^{i\gamma x}dx.$$

One may write down a differential equation with source:

$$-\frac{\partial^2 I}{\partial\gamma^2} + I = \frac{\pi}{\lambda}\mathrm{sech}\frac{\pi\gamma}{2\lambda}$$

and solve this—after some delicate machinations—to yield

$$I(\lambda,\gamma) = \frac{\pi}{\cos\lambda}e^{-\gamma} + \frac{2\pi}{\lambda}\sum_{d\in D^+}\frac{(-1)^{(d-1)/2}e^{-\pi d\gamma/(2\lambda)}}{1-\pi^2 d^2/(4\lambda^2)}.$$

where $D^+$ denotes the positive odd integers. In the event that $\lambda = \pi D/2$ for some odd $D$, the $1/\cos$ pole conveniently cancels a corresponding pole in the summation, and the result can be inferred either by avoiding $d = D$ in the sum and inserting a precise residual term

$$\Delta I = \pi(-1)^{(D-1)/2}e^{-\gamma}(\gamma + 1/2)/\lambda,$$

or more simply by taking a numerical limit as $\lambda \to \pi D/2$. When $\gamma \to 0$ we can recover from the sum, via analytic relations for $\psi(z)$, the $\psi$-function form of the integral of $(\mathrm{sech}\lambda x)/(1+x^2)$. Via Poisson transformation of (3.1) we thus obtain, for $0 < b < a$,

$$\mathcal{R}_1(a,b) = \mathcal{R}\left(\frac{\pi a}{2\mathrm{K}'}\right) + \frac{\pi}{\cos\frac{\mathrm{K}'}{a}}\frac{1}{e^{2\mathrm{K}/a}-1} + 8\pi a\mathrm{K}'\sum_{d\in D^+}\frac{(-1)^{(d-1)/2}}{4\mathrm{K}'^2 - \pi^2 d^2 a^2}\frac{1}{e^{\pi d\mathrm{K}/\mathrm{K}'}-1}, \tag{6.1}$$

where $k := b/a, \mathrm{K} := \mathrm{K}(k), \mathrm{K}' := \mathrm{K}(k')$, and $D^+$ again denotes the positive odd integers. A similar Poisson transform obtains from (3.2) in the case $b > a$. Such transformations appear recondite, but we have achieved what we desired: Convergence is rapid for $b \approx a$.

## 7 Convergence results for real parameters

For an RCF of the form $x = [a_0, a_1, \ldots]$ (so that each $a_i$ is nonnegative real but not necessarily integer) one has the usual recurrence relations for convergents

$$p_n = a_n p_{n-1} + p_{n-2},$$

$$q_n = a_n q_{n-1} + q_{n-2},$$

with $(p_0, p_{-1}, q_0, q_{-1}) := (a_0, 1, 1, 0)$. We also have the approximation rule

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}},$$

so that convergence rates can be bounded by virtue of the growth of the $q_n$. One may iterate the recurrence in various ways, obtaining for example

$$q_n = (1 + a_n a_{n-1} + a_n/a_{n-2}) q_{n-2} - (a_n/a_{n-2}) q_{n-4}$$

which relation involving all even or all odd indices on the $q$. An immediate application is

**Theorem 7.1:** For the RCF form of the Gauss fraction, $F(\gamma, 1; 1+\gamma; -1) = [\alpha_1, \alpha_2, \ldots]$, and for $\gamma > 1/2$ we have

$$\left| F - \frac{p_n}{q_n} \right| < \frac{c}{8^{n/2}},$$

where $c$ is an absolute constant.

**Remark:** One can likely obtain sharper bounds, or better $\gamma$-dependent bounds. We intend here just to show geometric convergence; i.e. that the number of good digits grows at least linearly in the number of iterates. Also note that for the $\mathcal{R}(a)$ evaluation of interest, $\gamma = 1/2 + 1/(2a)$ so that the condition on $\gamma$ is natural.

**Proof:** From the element assignments following (4.2) we have

$$\alpha_n \alpha_{n-1} = \frac{4}{(n-1)^2}(n - 1 + \gamma)(n - 2 + \gamma) \quad ; n \text{ odd } > 1,$$

$$\alpha_n \alpha_{n-1} = \frac{1}{(n/2 - 1 + \gamma)^2}(n - 1 + \gamma)(n - 2 + \gamma) \quad ; n \text{ even } .$$

We also have $q_1 1$, $q_2 = 1 + 1/\gamma > 2$ so that for sufficiently large $n$ we have

$$\alpha_n \alpha_{n-1} + 1 > 4, 2$$

respectively as $n$ is odd, even. From $q_n > (\alpha_n \alpha_{n-1} + 1) q_{n-2}$ the desired bound follows.

It is appropriate here to mention a clever computational acceleration for Gauss fractions, as described in [**?**, **?**, **?**]. Consider the previously displayed Gauss fraction $\log 2 =$

$[1, 2, 3, 1, 5, 2/3, \ldots]$. Generally a "tail" $t_N$ of this construct, meaning a subfraction starting from the $N$-th element, runs like so:

$$t_N := \cfrac{1}{\frac{4}{N} + \cfrac{1}{N+1+\cfrac{1}{\frac{4}{N+2} + \cfrac{1}{N+3+\ddots}}}}.$$

But—and here is the clever idea from the literature—this tail $t_N$ should be near to the *periodic* fraction $[4/N, N, 4/N, N, \ldots] = N(\sqrt{2}-1)/2$. This suggest that if are evaluating the Gauss fraction and we stop at element $4/N$, this one element should be replaced by $2(1+\sqrt{2})/N$. Indeed, in our own numerical experiments, this expedient always adds a few digits of precision. What is more, as suggested in [**?**], there are higher-order manifestations of this idea; e.g., the use of longer periods for the tail subfraction. As the reference shows via experimentation, the acceleration can become significant.

Now to the convergence of the RCF arising from Ramanujan's original construct, namely

$$\frac{a}{\mathcal{R}_1(a,b)} = [A_0; A_1, A_2, A_3, \ldots],$$

with the $A_i$ defined subsequent to (2.1). It turns out that the $q_n$ convergents consist of linear combinations of terms $a^i b^j$ where $i, j$ are even integers, and certain terms with explicit coefficients can be isolated, leading to

$$q_n \geq 1 + \frac{b^{n-2}}{a^n} \prod_{m \text{ even}}^{n} (1 - 1/m)^2 > 1 + \frac{1}{2n}\frac{b^{n-2}}{a^n} \qquad ; n \text{ even},$$

$$q_n \geq 1/b^2 + \frac{a^{n-1}}{b^{n+1}} \prod_{m \text{ even}}^{n-1} (m/(m+1))^2 > 1/b^2 + \frac{1}{n}\frac{a^{n-1}}{b^{n+1}} \qquad ; n \text{ odd}.$$

Such observations lead to a convergence theorem for the original Ramanujan construct, as

**Theorem 7.2:** For the Ramanujan RCF, $a/\mathcal{R}_1(a,b) = [A_0; A_1, A_2, A_3, \ldots]$, we have for positive reals $b > a$

$$\left|\frac{a}{\mathcal{R}_1(a,b)} - \frac{p_n}{q_n}\right| < \frac{2nb^4}{(b/a)^n},$$

while for positive reals $a > b$ we have

$$\left|\frac{a}{\mathcal{R}_1(a,b)} - \frac{p_n}{q_n}\right| < \frac{nb/a}{(a/b)^n}.$$

**Remark:** Again it should be possible to prove sharper bounds, our motive here being merely to establish essential geometric convergence of the literal fraction when $a, b$ are not near each other.

**Proof:** The given approximation bounds follow directly upon inspection of the products $q_n q_{n+1}$. □

As we previously have intimated, convergence for $a = b$ is slow. What we can prove is

**Theorem 7.3:** For real $a > 0$ the Ramanujan RCF, $a/\mathcal{R}(a)$, has

$$\left| \frac{a}{\mathcal{R}(a)} - \frac{p_n}{q_n} \right| < \frac{c(a)}{n^{h(a)}},$$

where $c(a), h(a)$ are $n$-independent, positive constants. The exponent $h(a)$ can be taken to be $c_0 \min(1, 4\pi^2/a^2)$ where the constant $c_0$ is absolute.

**Remark:** Note the convergence bound is computationally poor; still, as we have noted, convergence does occur. The relevant exponent $h(a)$ could be sharpened—or made more explicit—with more work; we only exhibit the theorem for theoretical completeness. Indeed, for $a = b$ or even $a \approx b$ we have many other, rapidly convergent options.

**Proof:** With a view to induction, assume that for some constants ($n$-independent) $d(a), g(a)$ and for $n \in [1, N-1]$ we have $q_n < dn^g$. Note that the element asymptotics following (2.1) mean that $A_n > f(a)/n$ for an $n$-independent $f$. Then we have a bound for the next $q_N$:

$$q_N > \frac{f}{N} d(N-1)^g + d(N-2)^g.$$

Using the fact that for $g < 1$, $0 < x \le 1/2$ we have $(1-x)^g > 1 - gx - gx^2$, the constants $d, g$ can evidently be arranged such that $q_N > dN^g$ and the induction goes through. □

To clarify the import of the above theorems: The Gauss fraction for $\mathcal{R}(a)$ exhibits (at least) geometric/linear convergence, as does the original Ramanujan form $\mathcal{R}(a, b)$ when $a/b$ or $b/a$ is significantly greater than unity. When $a = b$ we do have convergence, although as suggested by Theorem 7.3 the convergence is far below geometric/linear.

# 8 A uniformly convergent algorithm

We are now in a position to establish a complete algorithm for evaluating the original Ramanujan AGM fraction $\mathcal{R}_\eta(a, b)$ for positive real parameters. The convergence is uniform, in that for any positive real triple $\eta, a, b$ we expect rapid convergence in the sense of $D$ good digits in $< cD$ computational iterations, where $c$ is an absolute constant independent of the magnitudes of $\eta, a, b$. (By iterations here we mean either continued-fraction recurrence steps, or series-summand additions.)

**Algorithm 8.1:** Algorithm for evaluation of $\mathcal{R}_\eta(a,b)$ with real $\eta, a, b > 0$:

0) Observe that $\mathcal{R}_\eta(a,b) = \mathcal{R}_1(a/\eta, b/\eta)$ so that with impunity we may assume $\eta = 1$ and subsequently evaluate only $\mathcal{R}_1$;

1) If ($a/b > 2$ or $b/a > 2$) return the original fraction (1.1), or equivalently (2.1);

2) If ($a = b$) {
  if ($a = p/q$ rational) return finite form (5.1);
    else return the Gauss RCF (4.2) or rational-zeta form (4.1) or (4.3)
    or some other scheme such as rapid $\psi$ computations, etc.;
  }

3) if ($b < a$) {
  if ($b$ is not too close to $a$), return sech-elliptic result (3.1);
    else return Poisson-transform result (6.1);
  }

4) (Here, we must have $b > a$) Return, as in (1.2), $2\mathcal{R}_1\left((a+b)/2, \sqrt{ab}\right) - \mathcal{R}_1(b,a)$. $\square$

It is an implicit tribute to the ingenuity of Ramanujan that the final algorithm step allows the entire procedure to go through for any positive real parameters. One could avoid step (4) by invoking a Poisson transformation of (3.2), but the Ramanujan AGM identity simplifies the procedure.

## 9 Collected results for complex parameters

The issue of *complex* parameters $a, b, \eta$ is profound, as we have discovered via theoretical forays and extensive numerical experimentation. As just one example of the attendant difficulty, we found during this research that certain literature claims are false in regard to convergence of the Ramanujan AGM fraction, so the domain of validity for the AGM relation (1.2) becomes an issue.

To make some sense of this stultifying scenario, let us set $\eta = 1$ and consider various domains of complex parameter pairs $(a, b) \in \mathcal{C} \times \mathcal{C}$. In the modern theory of complex continued fractions [**?**, **?**, **?**], we think of convergents $p_n/q_n$ (see Section 7) as belonging to the extended complexes $\hat{\mathcal{C}} := \mathcal{C} \cup \infty$, so that a fraction *converges* if $p_n/q_n$ approaches a limit—including possibly $\infty$—in $\hat{\mathcal{C}}$ [**?**, p. 4]. One may then ask, what does "diverge" then mean for a continued fraction? A continued fraction diverges if for example $p_n/q_n$ is bounded and yet without a definite limit.

Typically, in our own examples to follow for the Ramanujan fraction, divergence is manifest by the so-called even convergents $p_{2n}/q_{2n}$ approaching one limit (of a certain fraction called the "even part" of $\mathcal{R}_1$), but with the odd convergents $p_{2n+1}/q_{2n+1}$ approaching a *distinct*, alternative limit (of an "odd part"). At any rate, to establish convergence of $\mathcal{R}_1$ it will suffice to study the denominator RCF $[A_0, A_1, \dots]$ in (2.1), or some appropriate transformation of same under which transformation the limit properties of convergents are invariant.

The convergence problem for the Ramanujan AGM fraction thus comes down to determining the precise convergence domain

$$\mathcal{D}_0 := \{(a, b) \in \mathcal{C} \times \mathcal{C} : \mathcal{R}_1(a, b) \text{ converges}\},$$

and beyond such determination is the problem of when the AGM relation (1.2) holds (with all three relevant fractions converging). There is a literature supposition [**?**] that

$$\mathcal{D}_1 := \{(a, b) \in \mathcal{C} \times \mathcal{C} : \mathrm{Re}(a), \mathrm{Re}(b) > 0\} \subseteq \mathcal{D}_0,$$

namely that the assumption of positive real parts for $a, b$ implies convergence. Such a claim is then used to imply the truth of the AGM relation (1.2) via continuation, on the same domain $\mathcal{D}_1$. Our research into the problem of complex parameters began when we discovered that the supposition about $\mathcal{D}_1$ is *false*: There are parameter pairs $(a, b) \in \mathcal{D}_1$ for which the fraction $\mathcal{R}_1(a, b)$ does *not* converge.

Incidentally this discovered restriction on the convergence domain is "algorithmically unfortunate," if you will, because one might look longingly at the *formal* relation possibility (note the "$\stackrel{?}{=}$" signaling suspicion):

$$\frac{\mathcal{R}_1(\sqrt{ab} + i\frac{a-b}{2}, \sqrt{ab} - i\frac{a-b}{2}) + \mathcal{R}_1(\sqrt{ab} - i\frac{a-b}{2}, \sqrt{ab} + i\frac{a-b}{2})}{2} \stackrel{?}{=} \mathcal{R}_1\left(\sqrt{ab}, \frac{a+b}{2}\right),$$

by which one hopes perhaps to forge a "left-handed" AGM relation, possibly giving rise to new iterative algorithms. Alas, this reversed AGM relation is generally false; it appears to hold only for trivial pairs (such as $a = b$), and this is because of the peculiar nature of the true convergence domain $\mathcal{D}_0$.

We shall also make use of lower-dimensional complex domains:

$$\mathcal{C}' := \{z \in \mathcal{C} : |z| = 1, z^2 \neq 1\},$$

$$\mathcal{I} := \{z \in \mathcal{C} : z = it, \text{ with } t \text{ nonzero real}\},$$
$$\mathcal{J} := \{z \in \mathcal{C} : |\arg(z)| < \pi\},$$
$$\mathcal{H} := \{z \in \mathcal{C} : \left|\sqrt{z}/(1+z)\right| < 1/2\},$$
$$\mathcal{K} := \{z \in \mathcal{C} : \left|z/(1+z^2)\right| < 1/2\}.$$

Here, $\mathcal{C}'$ is a "depleted-unit-circle"—i. e. the unit circle with $z = \pm 1$ removed, $\mathcal{I}$ is the imaginary axis but with $0 + 0i$ removed (and so $\mathcal{I}$ could be called the set of "pure imaginaries"), $\mathcal{J}$ is the whole complex plane except for the negative-definite real cut, and $\mathcal{H}$ is a geometrically interesting region about which we later shall have much to say. Note that there exist simple inter-domain connections, such as: $\mathcal{J} = \mathcal{C}\backslash\mathcal{I}^2$, and $\mathcal{K} = \mathcal{H}^2$.

In what follows we shall exhibit a chain of theorems, we then place proofs of the self same results after some additional theoretical development. Our first result pertains to the function $\mathcal{R}(a)$ on which we concentrated earlier in the paper.

**Theorem 9.1:** $\mathcal{R}(a) := \mathcal{R}_1(a, a)$ converges if and only if $a \notin \mathcal{I}$. That is, the fraction diverges if and only if $a$ is pure imaginary. Moreover, for $a \in \mathcal{C}\backslash\mathcal{I}$ the fraction converges to a holomorphic function of $a$ on whichever open half-plane $a$ belongs to.

Theorem 9.1 means that the various representations of Section 4 are valid on the right open half-plane ($\mathrm{Re}(a) > 0$). For the left open half-plane we have $\mathcal{R}(a) = -\mathcal{R}(-a)$ for the (also converging) fractions, and suitable modification of the Section 4 representations is feasible. Note, though, that one cannot generally continue a representation through the "forbidden barrier" of the imaginary axis, where the fraction does not converge (consider the digamma continuation of $\mathcal{R}$ which is holomorphic with poles at the reciprocals of odd negative integers).

**Theorem 9.2:** $\mathcal{R}_1(a, b)$ converges for any real-parameter pair; that is to say whenever $\mathrm{Im(a)} = \mathrm{Im(b)} = 0$.

**Theorem 9.3:** There exist parameter pairs $(a, b)$ such that the fraction $\mathcal{R}_1(a, b)$ is divergent. In particular, the even/odd parts of $\mathcal{R}_1(1, i)$ converge but to distinct limits. Moreover, $\mathcal{D}_1 \not\subset \mathcal{D}_0$; that is, there exist $(a, b)$ with $\mathrm{Re}(a), \mathrm{Re}(b) > 0$ but $\mathcal{R}_1(a, b)$ diverges.

**Theorem 9.4:** If $a/b \in \mathcal{K}$ then $\mathcal{R}_1(a, b)$ and $\mathcal{R}_1(b, a)$ converge.

**Theorem 9.5:** $\mathcal{H} \subset \mathcal{K}$ (the containment being proper).

The claims so far can be used finally to give a region of validity for the AGM relation, in the form of:

**Theorem 9.6:** If $a/b \in \mathcal{H}$ then $\mathcal{R}_1(a,b)$ and $\mathcal{R}_1(b,a)$ both converge, and the arithmetic mean $(a+b)/2$ dominates the geometric mean $\sqrt{ab}$ in absolute value.

A primary observation we shall require involves the growth of the fraction elements themselves. Modern continued fraction theory for a fraction of the form

$$\cfrac{a_1}{1+\cfrac{a_2}{1+\cfrac{a_3}{1+\cfrac{a_4}{1+\ddots}}}} \tag{9.7}$$

often makes use of the classical Stern–Stoltz summation

$$\sum_{n=1}^{\infty}\prod_{k=1}^{n}|a_k|^{(-1)^{n-k+1}}. \tag{9.8}$$

In many theorems, (absolute) divergence of a given fraction relies on the boundedness of this sum, or conversely. It turns out that the summation is equivalent to the summation for the RCF elements in our proof of Theorem 2.1, except with summands $|A_i|$, and again from the asymptotic relations for the elements $A_i$, we easily obtain unboundedness of the Stern–Stoltz construct (9.8), for *any* complex parameters $a, b$.

A second key observation is that the even and odd parts of $\mathcal{R}_1(a,b)$ can be derived explicitly and elegantly, as follows.

Define, for $n \in \mathcal{Z}$, and parameters $a, b$ fixed,

$$c(n) := -\frac{a^2b^2(2n)^2(2n+1)^2}{(1+(2n)^2a^2+(2n-1)^2b^2)\,(1+(2n+2)^2a^2+(2n+1)^2b^2)}, \tag{9.9}$$

with the understanding that for certain $a, b, n$ there may be poles in $c(n)$, which poles can be handled on a case-by-case basis in the subsequent analysis. Now define a fraction $\mathcal{S}$ by

$$\mathcal{S}(a,b) := \cfrac{b^2}{1+\cfrac{4a^2}{1+\cfrac{9b^2}{1+\cfrac{16a^2}{1+\ddots}}}}.$$

It is not hard to derive from the standard even/odd recurrence relations [?, pp. 83-85] the even/odd parts of $\mathcal{S}$ respectively as

$$\mathcal{S}^{(e)} = \frac{b^2}{1+4a^2+(1+4a^2+b^2)\mathcal{F}^{(+)}},$$

$$\mathcal{S}^{(o)} = b^2 + (1+b^2)\mathcal{F}^{(-)},$$

where

$$\mathcal{F}^{(\pm)} = \cfrac{c(\pm 1)}{1 + \cfrac{c(\pm 2)}{1 + \cfrac{c(\pm 3)}{1 + \cfrac{c(\pm 4)}{1 + \ddots}}}}.$$

Remarkably, this procedure yields the even/odd parts of the original fraction $\mathcal{R}_1$ respectively in the explicit form

$$\mathcal{R}_1^{(e)}(a,b) = \frac{a}{1+\mathcal{S}^{(e)}},$$
$$\mathcal{R}_1^{(o)}(a,b) = \frac{a}{1+\mathcal{S}^{(o)}}.$$

Now, if $\mathcal{R}_1$ is to converge, we *must* have $\mathcal{S}^{(e)}, \mathcal{S}^{(o)}$ both converging and *agreeing* in their limiting values.

In the event $a^2 + b^2 \neq 0$, we obtain a (converging) expansion of $c(n)$ for large $n$ in the form

$$c(n) = -\frac{a^2 b^2}{(a^2+b^2)^2}\left(1 + \frac{1}{n}\frac{b^2-a^2}{a^2+b^2} + \frac{1}{n^2}\frac{5a^4 - 2a^2(1+2b^2) + b^2(-2+3b^2)}{4(a^2+b^2)^2} + \cdots\right). \tag{9.10}$$

This $c(n)$ expansion is a sharp indicator of continued fraction convergence, especially in the leading ($n$-independent) term. Indeed, the even/odd parts of $\mathcal{S}$, and perforce of $\mathcal{R}_1$, are now seen to be, when $a^2 + b^2 \neq 0$, so-called "1-limit-periodic" fractions [?]; i. e. the elements $c(\pm n)$ approach a definite, common limit as $n \to \infty$.

**Proof of Theorem 9.1:** The Stieltjes theorem on $S$-fractions [?, Theorem 22, p. 138] says that the fraction (9.7)—with $a_n$ replaced by $\alpha_n z$, with each $\alpha_n$ positive real—has its even/odd parts both converging locally uniformly to a holomorphic function of $z \in \mathcal{J}$. Furthermore, if the Stern–Stoltz summation (9.8) is unbounded (we know it is for $a \neq 0$), the fraction (9.7) converges to a holomorphic function for $a^2 \in \mathcal{J}$. This establishes that $\mathcal{R}(a) := \mathcal{R}_1(a,a)$ converges for $\mathrm{Re}(a) > 0$. Being as $\mathcal{R}(-a) = -\mathcal{R}(a)$ when the latter converges, we have the holomorphic convergence for $a$ in either the right or left half-plane. (Note that the *parabola theorem* [?, Theorem 20, p. 130] also gives convergence for $a^2 \in \mathcal{J}$, as the $\{n^2 a^2\}$ fraction elements lie on a ray (degenerate parabola), but the Stieltjes approach offers some extra analyticity results.)

In the remaining case, when $\mathrm{Re}(a) = 0$ or equivalently $a \in \mathcal{I}$, the expansion (9.10) for such pure-imaginary $a$ reads

$$c(n) = -\frac{1}{4} + \frac{1}{16n^2}\left(\frac{1}{a^2} - 1\right) + \cdots .$$

Now, the Jacobsen–Masson theory [?, Theorem 32, p. 159] shows that if negative-real fraction elements $c(n)$ are eventually less than $-\frac{1}{4} - \frac{r}{16n^2}$ for some real $r > 1$, then the fraction diverges. This divergence—of both even/odd parts—happens, then, for $a \in \mathcal{I}$. $\square$

**Proof of Theorem 9.2:** We need to extend Theorem 2.1 to all real-parameter pairs $(a, b)$. Observe that $\mathcal{R}_1(0, b) = 0,\ \mathcal{R}_1(a, 0) = a$, and that if any one of the eight fractions $\mathcal{R}_1(\pm a, \pm b), \mathcal{R}_1(\pm a^*, \pm b^*)$ converges to a value $V$, then, by virtue of the arithmetic of convergents $p_n/q_n$, each of the other seven fractions also converges (and furthermore, to one of the values $\pm V,\ \pm V^*$). □

**Proof of Theorem 9.3:** Consider $(a, b) = (1, i)$, for which the form (9.9) becomes

$$c(n) = \frac{n(2n+1)^2}{4(n+1)},$$

which we note is finite, positive for integer $n \neq -1$. For this particular $c$ we have even/odd fractions

$$\mathcal{S}^{(e)} = \cfrac{-1}{5 + \cfrac{4c(1)}{1 + \cfrac{c(2)}{1 + \cfrac{c(3)}{1 + \ddots}}}} \in [-0.15, -0.14],$$

$$\mathcal{S}^{(o)} = -1 + \cfrac{-1}{1 + \cfrac{c(-2)}{1 + \cfrac{c(-3)}{1 + \cfrac{c(-4)}{1 + \ddots}}}} \in [-1.5, -1.4],$$

both of which converge by the parabola theorem mentioned in the proof of Theorem 9.1. In this way we find via simple calculation and easy bounds that the two distinct even/odd convergence limits of $\mathcal{R}_1$ have

$$\mathcal{R}_1^{(e)}(1, i) \in [1.0, 1.1]$$

and

$$\mathcal{R}_1^{(o)}(1, i) \in [-2.4, -2.3].$$

Similar analysis finishes the theorem, via a pair such as $(a, b) = (e^{i\pi/4}, e^{-i\pi/4})$, for which one may show there are again two distinct even/odd limits for $\mathcal{R}_1$, one of which limits lies strictly inside, the other outside the unit circle. □

**Proof of Theorem 9.4:** For $a/b \in \mathcal{K}$ we have

$$c(n) \sim -\frac{a^2 b^2}{(a^2 + b^2)^2},$$

with $|c(n)| < 1/4$ for sufficiently large $n$. But the Worpitsky theorem [**?**, Theorem 3, p. 35] states that any fraction of the form (9.1) with $|a_n| \leq 1/4$ converges (and furthermore, to a value whose absolute value is no greater than $1/2$). Thus, both even/odd parts of $\mathcal{R}_1(a, b)$ converge.

To show the parts converge to a common limit, we first apply the Śleszyński–Pringsheim approach [?, Proof of Theorem 1, p. 31]—from which the Worpitsky result can be derived—thus establishing that the even/odd parts converge *absolutely*, in the sense that $\sum |p_{2n+2}/q_{2n+2} - p_{2n}/q_{2n}|$ and $\sum |p_{2n+1}/q_{2n+1} - p_{2n-1}/q_{2n-1}|$ are both finite. Then we apply a typical lemma on even/odd parts [?, Lemma 19, p. 127] that, under the condition of absolute convergence of said parts, together with divergence of the Stern–Stoltz summation, we have convergence of the $\mathcal{S}$ fraction and thus of $\mathcal{R}_1(a, b)$. □

**Proof of Theorem 9.5:** We proceed to show that if $|\sqrt{z}/(1 + z)| < 1/2$, then also $|z/(1+z^2)| < 1/2$, which will establish containment. The proper character of containment follows from, say, $3i \notin \mathcal{H}$ but $3i \in \mathcal{K}$. So it remains to show that

$$g(z) := \left|2\frac{\sqrt{z}}{1+z}\right| < 1 \Rightarrow \left|2\frac{z}{1+z^2}\right| < 1.$$

Let $z = re^{i\theta}$, $r \geq 0$. What is required is equivalent to showing that

$$4r < 1 + r^2 + 2r\cos(\theta) \tag{1}$$

implies

$$4r^2 < 1 + r^4 + 2r^2\cos(2\theta). \tag{2}$$

Since $\cos(2\theta) = 2\cos^2(\theta) - 1$, proving (2) is equivalent to proving

$$0 < 1 - 6r^2 + r^4 + 4r^2\cos^2(\theta). \tag{3}$$

By (1), $(4r - 1 - r^2)^2 \leq 4r^2\cos^2(\theta)$ and so to prove (3) it suffices to show that

$$0 < 1 - 6r^2 + r^4 + (4r - 1 - r^2)^2. \tag{4}$$

But this is immediate since the right-hand-side of (4) reduces to $2(r - 1)^4$, and $r \neq 1$ as $g(e^{i\theta}) \geq 1$.

An alternative proof follows from applying the Kuhn-Tucker theorem to the problem of minimizing $g^2(z)$ subject to $g^2(z^2) \geq 1$. □

**Proof of Theorem 9.6:** From Theorem 9.5, $a/b \in \mathcal{H}$ implies $a/b \in \mathcal{K}$ and $b/a \in \mathcal{K}$ so both indicated fractions converge. As for $|(a + b)/2| > |\sqrt{ab}|$, this follows from the very definition of $\mathcal{H}$. □

While inferring the above results, using both theory and computation, we also noticed some intriguing phenomena we are not yet able to prove. For example we posit:

**Conjecture 9.11:** The precise domain of convergence for the fraction $\mathcal{R}_1(a, b)$ is

$$\mathcal{D}_0 = \{(a, b) \in \mathcal{C} \times \mathcal{C} : (a/b \notin \mathcal{C}') \text{ or } (a^2 = b^2,\ b \notin \mathcal{I})\}.$$

In particular, for $a/b \in \mathcal{C}'$ we have divergence. Moreover, $\mathcal{R}_1(a, b)$ converges to an analytic function of both $a$ and $b$ on the domain

$$\mathcal{D}_2 := \{(a, b) \in \mathcal{C} \times \mathcal{C} :\ |a/b| \neq 1\} \subset \mathcal{D}_0.$$

One could take Conjecture 9.11 into yet more detail. On the basis of numerical experimentation we expect "bifurcation" divergence—i. e. both even/odd fractions converging but to distinct limits—except in the cases $a^2 = b^2, b \in \mathcal{I}$ (exemplified by the instance $\mathcal{R}_1(i, i)$ where we see "chaos", with both even/odd fractions diverging).

One way to bring a certain closure to this research would be to establish Conjecture 9.11 and therefore to have solved the AGM issue in the sense that the cardioid-knot exterior would be the essential domain of AGM validity, as in

**Theorem 9.12:** On Conjecture 9.11, $a/b \in \mathcal{H}$ implies the truth of the AGM relation (1.2) with all three fractions converging.

**Proof:** If $a/b \in H$ then none of $|a/b|, |b/a|, |(a+b)/(2\sqrt{ab})|$ equals 1. Thus $(a, b), (b, a), ((a+b)/2, \sqrt{ab})$ are all members of $\mathcal{D}_2$, so all three fractions in the AGM relation (1.2) converge. Because the sech series (3.1), (3.2) are analytic on their respective unit disks $|k| < 1$ (see technical parenthetical below), and by the analyticity claim of Conjecture 9.11, the AGM result of Theorem 3.1 can be continued over the entire domain $\mathcal{H}$. This is Berndt's argument [**?**, p. 165], but with the redefined analyticity domain $\mathcal{D}_2$ in force, instead of $\mathcal{D}_1$. (As a technical matter, we take the elliptic integral $\mathrm{K}(k)$ for complex $k$ to be defined by $\mathrm{K}(k) = \frac{\pi}{2}\,\mathrm{F}(1/2, 1/2; 1; k^2)$ with the hypergeometric F converging absolutely for all $k$ within the unit complex disk ($|k| < 1$).) $\square$

**Remarks 9.13.**

**1.** Sophisticated continued-fraction theory would no doubt be required to establish Conjecture 9.1. One can make headway via the explicit even/odd fraction elements $c(n)$, the problematic aspect being to ascertain when the even/odd limits agree. This was possible for the Worpitsky scenario $|\lim c(n)| < 1/4$ as in Theorem 9.4, but the domain in Conjecture 9.11 has limiting $c(n)$ values outside the Worpitsky disk.

**2.** An observation that led us to some of the above results is that in the real-parameter scenario one implicitly uses, for positive real $a \neq b$, and perforce for Jacobi parameter $q := \min(a, b)/\max(a, b)$ in $[0, 1)$, the fact that $\theta_2/\theta_3 < 1$. However, if one plots the *complex* $q$ such that this $\theta$-ratio has absolute value $< 1$, one sees a frightfully complicated fractal structure in the complex $q$-plane, as shown in Figure **??**. We also exhibit a related Figure **??**, and all this in turn leads into the theory of modular forms [**?**]. Accordingly, the sech relations of Theorem 2.1 are in generally suspect for complex $q$. Indeed, the identities appear to fail numerically when $|\theta_2(q)/\theta_3(q)|$ exceeds unity. $\square$

It is a classic and elementary observation that for positive real $a, b$ the arithmetic mean strictly dominates the geometric mean. Via Theorem 9.6 we are saying that for the Ramanujan AGM relation to hold true for $\mathcal{R}_1$, one requires this classic arithmetic-geometric inequality to hold in the sense of absolute value. A picturesque interpretation of this idea is effected as follows. $\mathcal{H}$ is actually the (open) exterior of a "cardioid-knot" which in turn is the contour determined by the polar relation

$$r^2 + (2\cos\phi - 4)r + 1 = 0$$

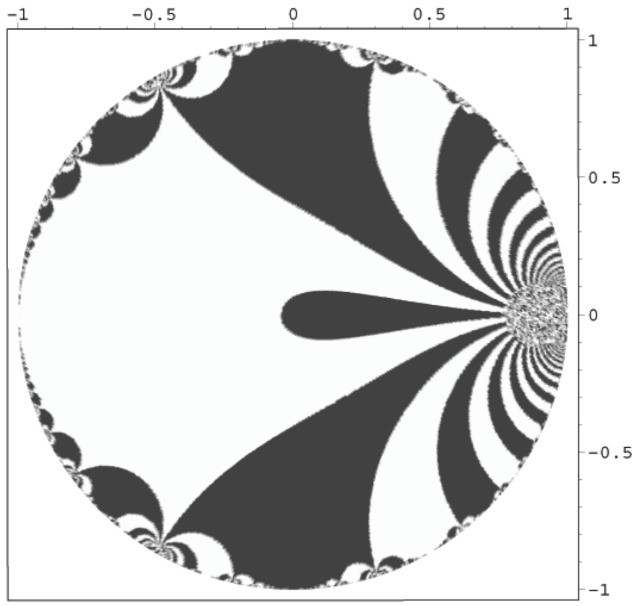


Figure 1: Where $|\theta_2/\theta_3| < 1$ in the complex $q$-plane. Note that the real interval $(-c, 1)$ for some positive real $c$ is monochrome black.

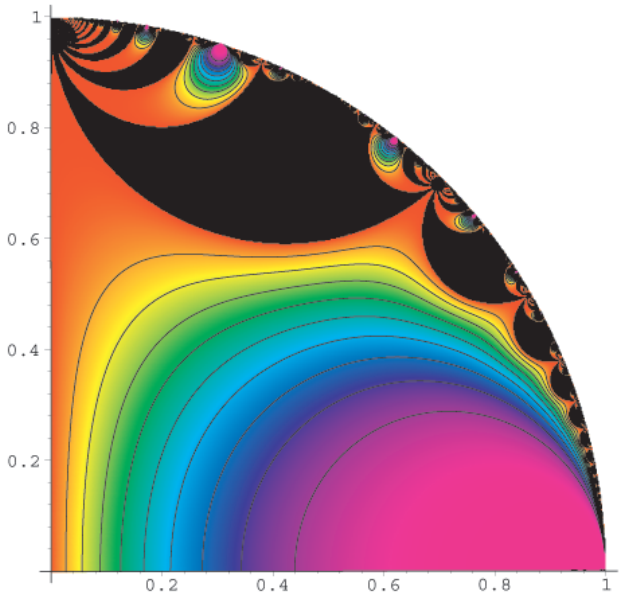


Figure 2: Values of $|\theta_4/\theta_3|$ (in first $q$-quadrant). The colors represent gradations of values between zero and one.

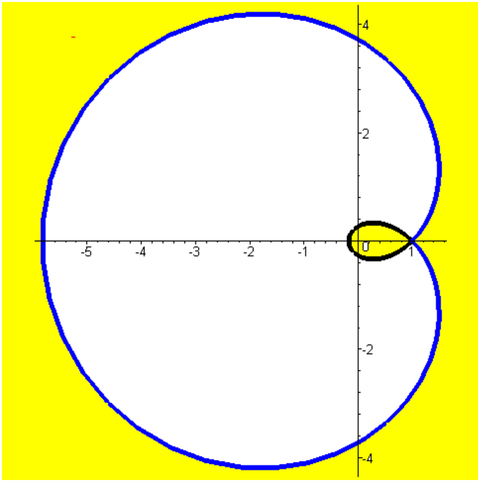


Figure 3: A cardioid-knot, the exterior $\mathcal{H}$ (in yellow) of which domain ensures the truth of the Ramanujan AGM relation (1.2) with the relevant $\mathcal{R}_1$ fractions hypothesized to converge.

in the complex plane. One can think of said contour as two contours:

$$r = 2 - \cos\theta \pm \sqrt{(1-\cos\theta)(3-\cos\theta)},$$

that is, we fuse the orbits of the $\pm$ instances, both for $\theta \in [0, 2\pi]$. One sees a small loop encompassing the origin, with left-intercept $\sqrt{8} - 3 + 0i$, and a wider contour whose left-intercept is $-3 - \sqrt{8} + 0i$. So $\mathcal{H}$ consists of all points outside the cardioid-knot, including the points in the inner lobe. One can call a point within said lobe an exterior point on the basis of the classical Jordan-curve rule: A point is outside a (smooth) contour if a ray to infinity from said point crosses the contour an even number of times. See Figure **??** and Figure **??**. In the latter we have superimposed the cardioid-knot and a "double-knot" contour whose (open) exteriors are the domains $\mathcal{H}, \mathcal{K}$ respectively. One can see that $\mathcal{H} \subset \mathcal{K}$ as in Theorem 9.5.

It is remarkable that the condition $a/b \in \mathcal{H}$ or $\mathcal{K}$ is not really asymmetrical as it might appear: Indeed, if $a/b$ is in one of the stated domains then so is $b/a$ in the same domain. On these rules, the AGM relation is valid when $a/b$ lies on certain exterior rays:

$$\frac{a}{b} \text{ or } \frac{b}{a} \in (\sqrt{8} - 3, \infty) \cup (\infty, -3 - \sqrt{8}),$$

thus including the truth of the AGM relation for all positive real pairs $a, b$ and for a somewhat wider class of pairs. Similarly, the AGM relation would hold for pairs $(a, b) = \{1, i\beta\}$ for $\pm\beta \in [0, 2-\sqrt{3}) \cup (2+\sqrt{3}, \infty)$. Incidentally we have not spoken of the *boundary* $\partial\mathcal{H}$ (which is just the cardioid-knot contour) as such analysis is more intricate than might be expected. However, we do know that the AGM relation is valid for *some* instances of $a/b \in \partial\mathcal{H}$; for example, $(a, b) = (1, 1)$ has $a/b$ on the knot intersection.

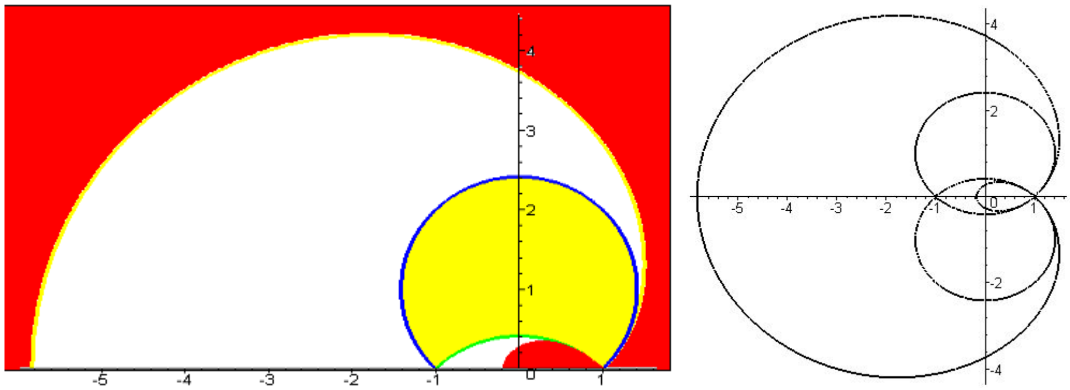


Figure 4: The double-knot superimposed on the cardioid-knot has exterior $\mathcal{K}$ (the open region of *non*yellow), showing where $\mathcal{R}_1$ exists and where the AGM relation (1.2) holds.

We close with some open problems:

- Again on the basis of numerical experiments, we wonder whether some "deeper" AGM identity might hold. There are complex pairs $(a, b)$ for which the average $(\mathcal{R}_1(a, b)+\mathcal{R}_1(b, a))/2$ is not equal to $\mathcal{R}_1((a+b)/2, \sqrt{ab})$ but said average *does* agree numerically with some alternative, call it $\mathcal{S}_1((a+b)/2, \sqrt{ab})$ with such a function $\mathcal{S}_1$ naively calculated as one of the series (3.1), (3.2) (recall that elliptic K can be defined for a wide range of complex $k$). Such coincidences are remarkable, and difficult so far to predict. We maintain hope that there should ultimately be a comprehensive theory under which the wonderful AGM relation—with suitable modification—holds for general complex parameters.

- We know that the fraction $\mathcal{R}(i) := \mathcal{R}_1(i, i)$ does not converge, yet the $\psi$-function representation of Section 4 does have a definite value at $a = i$. Could it be that some limit such as $\lim_{\epsilon \to 0} \mathcal{R}_1(i + \epsilon, i)$ does exist? (Conjecture 9.11 implies that the indicated fraction converges for every $\epsilon > 0$.)

- What is the precise domain of validity of the sech formulae (3.1), (3.2)?

- In spite of a menagerie of closed forms given for $\mathcal{R}(a) := \mathcal{R}_1(a, a)$, we do not know a single, nontrivial closed form for an $\mathcal{R}_1(a, b)$ with $a \neq b$. What can be done about this impasse?

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