On the Ramanujan AGM fraction.
Part II: The complex-parameter case

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Abstract. The Ramanujan continued fraction

$$\mathcal{R}_\eta(a,b) = \cfrac{a}{\eta + \cfrac{b^2}{\eta + \cfrac{4a^2}{\eta + \cfrac{9b^2}{\eta + \ddots}}}}$$

is interesting in many ways; e. g. for certain complex parameters (η, a, b) one has an attractive AGM relation $\mathcal{R}_\eta(a,b) + \mathcal{R}_\eta(b,a) = 2\mathcal{R}_\eta\left((a+b)/2, \sqrt{ab}\right)$. Alas, for some parameters the continued fraction $\mathcal{R}_\eta$ does not converge; moreover, there are converging instances where the AGM relation itself does not hold. To unravel these dilemmas we herein establish convergence theorems, the central result being that $\mathcal{R}_1$ converges whenever $|a| \neq |b|$. We also conjecture on the domain of divergence, expecting such divergence to occur whenever $a = be^{i\phi} \neq 0$ with $\cos^2\phi \neq 1$. We further conjecture that for a/b lying in a certain—and rather picturesque—complex domain, we have both convergence and the truth of the AGM relation.

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1 Background

In a companion treatment [1] we focused on evaluation of the continued fraction

$$\mathcal{R}_1(a,b) = \cfrac{a}{1+\cfrac{b^2}{1+\cfrac{4a^2}{1+\cfrac{9b^2}{1+\ddots}}}} \tag{1.1}$$

for real parameters a, b. Note that, formally, $\mathcal{R}_\eta(a,b) = \mathcal{R}_1(a/\eta, b/\eta)$ so that with impunity we may focus upon the fraction displayed in our Abstract, with $\eta := 1$; thus we have a two-complex-parameter problem. For complex parameters (a, b), convergence of $\mathcal{R}_1$ turns out to be—both historically and currently—problematic.

A formal AGM relation—known to be true at least for positive real a, b [1]—reads

$$\mathcal{R}_1\left(\frac{a+b}{2}, \sqrt{ab}\right) = \frac{\mathcal{R}_1(a,b) + \mathcal{R}_1(b,a)}{2}. \tag{1.2}$$

Yet, one wishes the three relevant fractions to converge prior to any resolution of the truth of such an AGM relation. So we are primarily concerned with a precise determination of the convergence domain

$$\mathcal{D}_0 := \{(a,b) \in \mathcal{C} \times \mathcal{C} :\ \mathcal{R}_1(a,b) \text{ converges on } \hat{\mathcal{C}}\},$$

where $\hat{\mathcal{C}} := \mathcal{C} \cup \{\infty\}$ denotes the extended complex field. It is important to note what is meant by "convergence" on $\hat{\mathcal{C}}$ in the modern complex-continued-fraction context. If p_n/q_n is the n-th convergent to $\mathcal{R}_1$ (we remind ourselves in Section 3 as to the definition of such convergents), we say that $\mathcal{R}_1$ converges if p_n/q_n has a limit in $\hat{\mathcal{C}}$. Thus, divergence (non-convergence) must be oscillatory—say bifurcated, or chaotic (later, we exhibit examples of such divergence scenarios). This modern definition of convergence conveniently handles situations such as the instance that $b^2/(1 + 4a^2/ + \dots)$ converges to a value $(-1) \in \mathcal{C}$, whence $\mathcal{R}_1 = \infty$ still converges on $\hat{\mathcal{C}}$.

Some preliminary nomenclature is relevant. We shall often refer to real cuts, that is sets (α, β) for reals $\alpha < \beta$; when we say a complex number z belongs to (α, β) we mean z must be real with $z \in (a, b)$ in the usual sense of real-interval membership. For example, z is pure-imaginary—i. e. $z = 0 + iy$ with real $y \neq 0$—iff $z^2 \in (-\infty, 0)$. Also the cut $(-\infty, -1/4)$ (and its closure $(-\infty, -1/4]$) will loom important in our convergence analysis.

We are eventually motivated to consider a special set $\mathcal{H}$ that turns out to be the open exterior of a cardioid-knot (the picturesque character of $\mathcal{H}$ is exhibited in the companion treatment [1]) as

$$\mathcal{H} := \{z \in \mathcal{C} : |\sqrt{z}/(1+z)| < 1/2\},$$

where we note for the moment that the classical AGM inequality $(a+b)/2 > \sqrt{ab}$ for positive real $a \neq b$ is true in the sense of *magnitude*—i. e. $|(a+b)/2| > \left|\sqrt{ab}\right|$—when $a/b \in \mathcal{H}$.

We next establish two-complex-parameter domain definitions

$$\mathcal{D}_2 := \{(a,b) \in \mathcal{C} \times \mathcal{C} : |a| \neq |b|\},$$

$$\mathcal{D}_3 := \{(a,b) \in \mathcal{C} \times \mathcal{C} : a^2 = b^2 \notin (-\infty, 0)\},$$

$$\mathcal{D}_1 := \mathcal{D}_2 \cup \mathcal{D}_3,$$

Our central result will be that

$$\mathcal{D}_1 \subseteq \mathcal{D}_0,$$

and we are eventually led to conjecture that in fact $\mathcal{D}_1 = \mathcal{D}_0$, which conjecture would establish the precise convergence domain for $\mathcal{R}_1(a,b)$.

It is a tribute to the profundity of the Ramanujan construction that in the following treatment we need to rely upon some of the deepest theorems in complex-continued-fraction theory, including Stieltjes-fraction theorems, convergence-set results such as the "parabola-sequence" and "oval" theorems, and yet other results from the finest of the complex-fraction literature.

2 The instance $a^2 = b^2$

Assume $a^2 = b^2$. Clearly, if $\mathcal{R}_1(a,b)$ converges then by the very definition of the $\mathcal{R}_1$ fraction each of the four constructs $\mathcal{R}_1(\pm a, \pm b)$ converges (to $\pm\mathcal{R}_1(a,b)$). So it suffices to analyze just

$$\frac{a}{\mathcal{R}_1(a,a)} = 1 + \cfrac{\alpha_1 z}{1 + \cfrac{\alpha_2 z}{1 + \cfrac{\alpha_3 z}{1 + \cfrac{\alpha_4 z}{1 + \ddots}}}} = 1 + S(z),$$

where $\alpha_n := n^2,\ z := a^2$ and S is a classical Stieltjes fraction (as all α_n are positive real). We are led immediately to:

Theorem 2.1: $a/\mathcal{R}_1(a,a)$ converges to a holomorphic function of a^2 on either half-plane $\mathrm{Re}(a) > 0, \mathrm{Re}(a) < 0$, and so, for $a^2 = b^2$, $\mathcal{R}_1(a,b)$ converges for all $(a,b) \in \mathcal{D}_3$.

Remark: Thus $\mathcal{R}_1(a,a)$ converges on $\hat{\mathcal{C}}$ for all a not pure-imaginary, i. e. $a^2 \notin (-\infty, 0)$.

Proof: This all follows from the Stieltjes theorem [4, Theorem 22, p. 138]. □

We can go further, to establish convergence bounds in the form:

Theorem 2.2: For $(a, b) \in \mathcal{D}_3$, so that $b = \pm a$ and a is not pure-imaginary, the convergents to S satisfy

$$\left| S(a^2) - \frac{p_n}{q_n} \right| < \frac{2|a|^2 \sec\theta}{n^{4/\left(1+\sqrt{1+16|a|^2\sec^2\theta}\right)}}.$$

where $\theta = \min(|\arg(a)|, |\arg(-a)|)$.

Proof: This follows directly from the Gragg–Warner bounds [4, p. 140] for Stieltjes fractions. □

This result can be compared to similar convergence bounds for $\mathcal{R}_1(a, a)$, for real a, in the companion treatment [1]. The situation is, when $a^2 = b^2$, and a is not pure-imaginary, we do have convergence, but said convergence is "poor," i. e. not geometric (by geometric we mean the error relevant to the p_n/q_n approximant would be $O\left(\theta^{-n}\right)$ for some real $\theta > 1$).

3 Even/odd fractions

For a continued fraction

$$x := \cfrac{a_1}{1 + \cfrac{a_2}{1 + \cfrac{a_3}{1 + \cfrac{a_4}{1 + \ddots}}}}$$

a convenient formula with which one may ignite convergence analyses is the classical relation (for $n \geq 1$)

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = (-1)^{n-1} \frac{\prod_{j=1}^{n} a_j}{q_n q_{n-1}}, \tag{3.1}$$

with the standard assignments $(p_0, q_0) := (0, 1)$, $(p_1, q_1) := (a_1, 1)$ and recurrences (for $n \geq 2$) in the form $(p_n, q_n) = (p_{n-1}, q_{n-1}) + a_n(p_{n-2}, q_{n-2})$. We shall say that any continued fraction converges *absolutely* if

$$\sum_{n=1}^{\infty} \left| \frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} \right| < \infty.$$

As pointed out in [4, p. 128], if a fraction converges absolutely then it converges to a finite limit. Similarly, if x has a finite value and $\sum |x - p_n/q_n| < \infty$ then x is absolutely convergent, since $|p_n/q_n - p_{n-1}/q_{n-1}| \leq |x - p_n/q_n| + |x - p_{n-1}/q_{n-1}|$.

Now, a typical scenario for divergence of x is that the even convergents p_{2n}/q_{2n} (to the "even part" of x) and the odd convergents p_{2n+1}/q_{2n+1} (to the "odd" part) approach *distinct* limits. If, however, the even/odd parts converge absolutely, that is we have both

$$\sum_{n=0}^{\infty} \left| \frac{p_{2n+2}}{q_{2n+2}} - \frac{p_{2n}}{q_{2n}} \right| < \infty, \quad \sum_{n=0}^{\infty} \left| \frac{p_{2n+3}}{q_{2n+3}} - \frac{p_{2n+1}}{q_{2n+1}} \right| < \infty, \tag{3.2}$$

then much can be gleaned in regard to convergence of the original fraction x, especially if one also knows the Stern–Stolz construct

$$\sum_{n=1}^{\infty}\prod_{k=1}^{n}|a_k|^{(-1)^{n-k+1}}. \tag{3.3}$$

A powerful result in this regard is [4, Lemma 19, p. 127], [3]:

Lemma 3.1 (Jones–Thron): If the even/odd parts of x converge absolutely in the sense of (3.2), then x converges if and only if the Stern–Stolz series (3.3) diverges to infinity.

To employ the Jones–Thron result for the Ramanujan fraction, we first write for positive odd integer M

$$\mathcal{R}_1(a,b) = \cfrac{a}{1+\cfrac{b^2}{1+\cfrac{4a^2}{1+\cdots+\mathcal{S}_M(a,b)}}} \tag{3.4}$$

where

$$\mathcal{S}_M(a,b) := \cfrac{M^2b^2}{1+\cfrac{(M+1)^2a^2}{1+\cfrac{(M+2)^2b^2}{1+\cfrac{(M+3)^2a^2}{1+\ddots}}}}.$$

We shall focus upon these "tail fractions" $\mathcal{S}_M(a,b)$, first dispensing with the Stern–Stolz series issue. Happily, for these tails $\mathcal{S}_M$ we always have divergence to infinity of (3.3), as in

Theorem 3.2: For any positive odd M, the Stern–Stolz series (3.3) for $\mathcal{S}_M(a,b)$ diverges to infinity.

Remark: The companion treatment [1] gives precise, equivalent asymptotics for $M=1$.

Proof: The n-th summand of the Stern–Stolz series (3.3) is, for n even

$$\left|\left(\frac{\Gamma(M/2+n/2)\Gamma(M/2+1/2)}{\Gamma(M/2)\Gamma(M/2+n/2+1/2)}\right)^2 (b/a)^n\right|,$$

while for odd index n the summand is

$$\left|\frac{1}{Mb^2}\left(\frac{\Gamma(M/2+n/2)\Gamma(M/2+1)}{\Gamma(M/2+1/2)\Gamma(M/2+n/2+1/2)}\right)^2 (a/b)^{n-1}\right|.$$

Now, by the standard Stirling formula, each of the squared-gamma factors is asymptotic to (constant)$\times 1/n$, so that the sum (3.3) is divergent to infinity. □

We now establish exact expressions for the even/odd parts of $\mathcal{S}_M(a,b)$ for positive odd M. Using standard even/odd decompositions [4, pp. 83-85], we have

$$\mathcal{S}_M^{\text{even}}(a,b) = \frac{M^2b^2}{1+(M+1)^2a^2+(1+(M+1)^2a^2+M^2b^2)F_M},$$

$$\mathcal{S}_M^{\text{odd}}(a,b) = M^2b^2 + (1+(M-1)^2a^2+M^2b^2)G_M,$$

where we define

$$F_M := \cfrac{c_M(1)}{1+\cfrac{c_M(2)}{1+\cfrac{c_M(3)}{1+\cfrac{c_M(4)}{1+\ddots}}}}$$

and

$$G_M := \cfrac{d_M(1)}{1+\cfrac{d_M(2)}{1+\cfrac{d_M(3)}{1+\cfrac{d_M(4)}{1+\ddots}}}},$$

with the definitions

$$c_M(n) := -\frac{a^2b^2(M+2n-1)^2(M+2n)^2}{(1+(M+2n)^2b^2+(M+2n+1)^2a^2)(1+(M+2n-2)^2b^2+(M+2n-1)^2a^2)},$$

$$d_M(n) := c_M(-M-n+1).$$

With a view to Lemma 3.1, our aim is to show that for certain parameter pairs (a,b), both $\mathcal{S}_M^{\text{even}}$, $\mathcal{S}_M^{\text{odd}}$ converge absolutely (and hence to finite values in $\mathcal{C}$). In such cases we have $\mathcal{S}_M^{\text{even}} = \mathcal{S}_M^{\text{odd}}$ as well.

A key function of which we shall make both computational and theoretical use is

$$c(a,b) := -\frac{a^2b^2}{(a^2+b^2)^2}, \tag{3.5}$$

for this is the asymptotic large-n limit of either $c_M(n),\ d_M(n)$ when $a^2+b^2 \neq 0$. In fact, for $a^2+b^2 \neq 0$ we have

$$c_M(n), d_M(n) \sim c(a,b) + O(1/(M+n)) \qquad (3.6).$$

A useful collection of straightforward results is

Lemma 3.3: We have $c(a,b) \notin (-\infty, -1/4]$ if and only if $|a| \neq |b|$. In particular, if $a/b = e^{i\phi}$ then $c(a,b) = -(1/4)\sec^2\phi$. Finally, if $c(a,b) \notin (\infty, -1/4]$ then the two roots of $\omega^2 - \omega - c(a,b) = 0$ are unequal in magnitude.

Proof: If a real ρ has $-\rho \in (\infty, -1/4]$ the supposition

$$a^2b^2/(a^2+b^2)^2 = \rho$$

means, with $\rho \geq 1/4$,

$$a/b = \left(\frac{1-2\rho \pm i\sqrt{4\rho-1}}{2\rho}\right)^{1/2},$$

so that $|a/b| = 1$. For the converse, with $a = be^{i\phi}$ (and so the sec-identity is immediate from definition (3.5)) and in case ϕ is real, we have $c(a,b) = -\frac{1}{4}\sec^2\phi \in (-\infty, -1/4]$. Finally, the quadratic roots are $\omega = (1/2)\left(1 \pm \sqrt{1+4c(a,b)}\right)$. It is a simple geometric observation in the complex plane that $|1-z| = |1+z|$ if and only if $Re(z) = 0$. Thus the roots can only be equal in magnitude if $c(a,b)$ is real and $\leq -1/4$. □

4 γ-fractions

With a view to the even/odd decompositions F_M, G_M of the previous section, we introduce the concept of a γ-fraction, as

$$x := \cfrac{\gamma_1}{1+\cfrac{\gamma_2}{1+\cfrac{\gamma_3}{1+\cdots}}} \tag{4.1}$$

where the γ elements approach a finite complex limit, say $\gamma_n \to c \in \mathcal{C}$. For our analysis, it is a welcome property of the Ramanujan fraction $\mathcal{R}_1$ that both F_M, G_M of the previous section are, for $a^2+b^2 \neq 0$, gamma-fractions, with $\gamma_n \to c(a,b)$.

It is instructive to consider first the canonical case in which the gamma-fraction x has $\gamma_n = c$ for all $n \in \mathcal{Z}^+$, whence we have the classical result (see, e. g. [5]):

Theorem 4.1: Assume that every $\gamma_n = c$ with $c \notin (-\infty, -1/4)$ (note here the real cut is open). Then x given by (4.1) converges absolutely to the value $r-1$, where r is the larger (in magnitude—see Lemma 3.3) of the roots $\left(1 \pm \sqrt{1+4c}\right)/2$ of $\omega^2 - \omega - c = 0$. In particular, the convergents of x satisfy

$$\left|\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}\right| = \left|r(1-s/r)^2\right| \left|\frac{s}{r}\right|^n,$$

where $s \neq r$ is the other quadratic root (and by Lemma 3.3, $|s| < |r|$).

Remark: For $c = -1/4$ exactly, the fraction x does converge (to the value $x = -1/2$), but not absolutely. In fact, $|1/2 + p_n/q_n| = 1/(2n+2)$ for all $n \geq 0$, and this slow convergence is a hint as to how nonabsolute convergence might occur for some continued fractions (4.1) with $\gamma_n \to c$ more intricately.

Proof: All follows from a closed form for the convergents p_n/q_n to x, namely

$$p_n = c(r^n - s^n)/(r-s),$$

$$q_n = (r^{n+1} - s^{n+1})/(r-s),$$

and that $|r| > |s|$. $\square$

It turns out that for any $c \in (-\infty, -1/4)$ we have divergence [5]. For example, with $c := -1/2$ one has

$$\frac{p_n}{q_n} = -\frac{1}{\sqrt{2}} \frac{\sin(n\pi/4)}{\sin((n+1)\pi/4)},$$

which values oscillate endlessly though $(0, -1/2, -1, \infty)$. Such observations and Theorem 4.1 completely settle the convergence problem for γ-fractions with all $\gamma_n = c$.

A computational digression is relevant here: It is of interest that the function $c(a,b)$ defined in (3.5) can be used to accelerate rather sluggish situations, in the following way (a similar idea is enunciated in our companion treatment [1] for Gauss continued fractions). We use (3.5) as an approximation to $c_M(n)$ for some large n, so that when $a^2 \neq b^2$, the continued fraction $\mathcal{R}_1(a,b)$ can be calculated according to the chain starting with (3.4), $M = 1$, but at a key juncture using the fact that a periodic fraction defined

$$x(a,b) := \cfrac{c(a,b)}{1 + \cfrac{c(a,b)}{1 + \cfrac{c(a,b)}{1 + \cdots}}}$$

is given, via Theorem 4.1, by

$$x(a,b) = -\frac{a^2}{a^2+b^2} \quad \text{or} \quad -\frac{b^2}{a^2+b^2},$$

whichever is larger in magnitude. We may therefore attempt to calculate

$$\mathcal{R}_1(a,b) = \frac{a}{1 + \frac{b^2}{1+4a^2+(1+4a^2+b^2)F_1}},$$

with an approximation presumed accurate for suitably large n, namely we use the *finite* continued fraction development

$$F_1 \approx \cfrac{c_1(1)}{1 + \cfrac{c_1(2)}{1 + \cdots \cfrac{c_1(n-1)}{1 + x(a,b)}}}.$$

That is, in this computational procedure the tail fraction from $c_1(n)$ inclusive is replaced by the *number* $x(a,b)$. This expedient of tail approximation really does improve matters when $|a| \approx |b|$. For example, for $a = b = 1$ and the known evaluation $\mathcal{R}_1(1,1) = \log 2$ (see [1]), we found that p_{1000}/q_{1000} is correct only to about 3 good decimals for the original continued fraction (1.1); yet, the same amount of work using the even convergents p_{2000}/q_{2000} but also doing the tail-substitution with $x(1,1) = -1/2$ yields 10 good decimals. Incidentally, rate-bounding in regard to the "oval" theorems in the literature [4, pp. 141-146] can be used to effect good bounds on the rate of convergence of such approximations.

We now revert to the theoretical avenue by observing that a relevant set of complex numbers not on a certain real cut can be characterized

$$\{c \in \mathcal{C} : c \notin (-\infty, -1/4]\} \;=\; \{c \in \mathcal{C} : |c| < 1/4\} \cup \{c \in \mathcal{C} : |\arg(c)| < \pi\}.$$

There is overlap in this union, but convenient theorems are possible for each component of said union. We start with

Theorem 4.2: Assume $|c| < 1/4$ and set $\varepsilon := 1/4 - |c|$. If in the γ-fraction (4.1) we have

$$|\gamma_n - c| < \varepsilon/2$$

then x is absolutely convergent, with

$$\left|\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}\right| < \frac{2}{(1+2\varepsilon)^{2n}}.$$

Proof: Employing the Śleszyński–Pringsheim expedient [4, p. 35] for such bounded elements γ_n, we write the equivalent form

$$x := \cfrac{2\gamma_1}{2 + \cfrac{4\gamma_2}{2 + \cfrac{4\gamma_3}{2 + \cdots}}}$$

and observe for this continued fraction that

$$|q_n| > 2|q_{n-1}| - (1 - 2\varepsilon)|q_{n-2}|.$$

Thus $|q_n| > (1 + 2\varepsilon)|q_{n-1}|$ and so by (3.1),

$$\left|\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}\right| < \frac{4^n}{2}\frac{\prod_{k=1}^{n}\gamma_k}{(1+2\varepsilon)^{2n-3}},$$

and the result follows. □

To complete this foray for the set $\{c \notin (-\infty, -1/4]\}$, we now establish

Theorem 4.3: Assume $\theta := |\arg(c)| < \pi$ and that for the γ-fraction x defined (4.1) we have

$$|\gamma_n - c| < h := \frac{2}{9}\cos^2(\theta/2).$$

Then x is absolutely convergent, with

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{\sqrt{h}} \frac{|c| + h}{(1 + h/(|c| + h))^{n-1}}.$$

Proof: This follows quickly from the parabola-sequence theorem [4, Theorem 21, pp. 136-137], with the multiplier assignment $g_k := 1/3$. □

Now we have the central result of the present treatment, namely

Theorem 4.4: For $|a| \neq |b|$, the Ramanujan fraction $\mathcal{R}_1(a, b)$ converges on $\hat{\mathcal{C}}$.

Proof: By Lemma 3.3, $|a| \neq |b|$ implies $c(a, b) \notin (-\infty, -1/4]$. By Lemma 3.1 and Theorems 4.2, 4.3, and the observation that for sufficiently large odd M the bounds on $|\gamma_n - c(a, b)|$ in the two stated theorems are indeed met either for $\gamma_n := c_M(n)$ or for $\gamma_n := d_M(n)$, we have absolute convergence of the even/odd parts of $\mathcal{S}_M$, and hence convergence of the original fraction $\mathcal{R}_1(a, b)$. □

Corollary 4.5: $\mathcal{D}_1 \subseteq \mathcal{D}_0$, that is, $\mathcal{R}_1(a, b)$ converges on $\hat{\mathcal{C}}$ if $|a| \neq |b|$ or $a^2 = b^2$ with a not pure-imaginary.

Proof: This follows from Theorems 2.1 and 4.4.

5 Divergence

A special case of divergence of $\mathcal{R}_1$ runs as follows:

Theorem 5.1: If a is pure-imaginary, that is $a^2 \in (-\infty, 0)$, the fraction $\mathcal{R}_1(a, a)$ diverges. In particular, $\mathcal{R}_1(i, i)$ diverges.

Proof: We have in this case

$$c_1(n) = -\frac{1}{4} + \frac{1}{16n^2} \left(\frac{1}{a^2} - 1 \right) + \cdots .$$

Now, the Jacobsen–Masson theory (see [4, Theorem 32, p. 159] and references therein) shows that if negative-real fraction elements $c_1(n)$ are eventually less than $-\frac{1}{4} - \frac{r}{16n^2}$ for some real $r > 1$, then the fraction diverges. Thus $\mathcal{S}_1^{even}(a, a)$ diverges and so $\mathcal{R}_1(a, a)$ cannot converge. (Similarly, the odd part $\mathcal{S}_1^{odd}$ diverges.) □

In attempting to establish divergence for other parameter pairs, in particular the cases $a = bi$, we developed means to combine computation and theory, to prove inequality of even/odd parts, even though both parts often themselves converge. The technique starts with the assumption of a fraction (4.1), but not a gamma-fraction, as $\gamma_n \to \infty$), instead:

$$\gamma_n := (n + \delta_n)^2,$$

which assignment —when we know $c_1(n), d_1(n)$ for cases $a = bi$—implicitly defines the pertubations δ_n. An attractive recurrence-transformation results if we define ρ_n implicitly by

$$q_n = \rho_n \prod_{j=1}^{n+1} (j + \delta_j),$$

whence the usual recurrence ($q_0 = q_1 = 1$), $q_n = q_{n-1} + \gamma_n q_{n-2}$, yields

$$\rho_n = \frac{\rho_{n-1} + (n + \delta_n)\rho_{n-2}}{n + 1 + \delta_{n+1}}.$$

In turn, we have an exact formula

$$\Delta_n := \frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^n}{\rho_n \rho_{n-1}} \frac{1}{n + 1 + \delta_{n+1}}.$$

For suitably bounded $|\delta_n|$ and for ρ_n confined to, say, a circle in the proper right half-plane, the series for the fraction $x = \sum_{n\geq 1} \Delta_n$ is convergent; moreover we can establish bounds on the error relevant to the p_n/q_n approximant. Our technique, then, is to calculate even/odd parts to some level n, and bound the error such that we know rigorously the inequality of said even/odd parts.

An exemplary application of this computational-theoretical fusion is

Theorem 5.2: $\mathcal{R}_1(1, i)$ and $\mathcal{R}_1(e^{i\pi/4}, e^{-i\pi/4})$ diverge.

Remark: The second case of the theorem contradicts previous literature claims that convergence occurs for $\mathrm{Re}(a), \mathrm{Re}(b) > 0$; see [1].

Proof: For $(a, b) = (1, i)$ we have

$$S_1^{even}(1, i) = \frac{-1}{5 + 4F_1},$$

where

$$F_1 := \cfrac{c_1(1)}{1 + \cfrac{c_1(2)}{1 + \cfrac{c_1(3)}{1 + \ddots}}}$$

with, here,

$$c_1(n) := \frac{n(2n + 1)^2}{4(n + 1)}$$

(and note relation (3.6) does not apply, as $a^2 + b^2 = 0$). This F_1 does converge to a finite value according to the above analysis involving the ρ_n, or the"parabola" theorem [4, Theorem 20, p. 130]. In this particular case $(a, b) = (1, i)$ the error analysis can be simplified. We have $n^2 < c_1(n) < n^2 + 1/4$, so the recursion $q_n = q_{n-1} + c_1(n)q_{n-2} >$

$q_{n-1} + n^2 q_{n-2}$ tells us that in fact $q_n \geq (n+1)!/2$. Thus we have (the first inequality here is allowed when all fraction elements are positive real)

$$\left|F_1 - \frac{p_n}{q_n}\right| \leq \left|\frac{\prod_{j=1}^n c_1(j)}{q_n q_{n-1}}\right| < \frac{d}{n+1},$$

for a positive constant d. The convergence is "slow", nonabsolute, but one may use this convergence bound together with computation up to appropriate n, to establish

$$\mathcal{S}_1^{even}(1, i) \in [-0.15, -0.14].$$

On the other hand, one may show in similar fashion that

$$S_1^{odd}(1, i) = -1 + \cfrac{-1}{1 + \cfrac{c_1(-2)}{1 + \cfrac{c_1(-3)}{1 + \cfrac{c_(-4)}{1 + \ddots}}}} \in [-1.5, -1.4],$$

so $\mathcal{S}_1(1, i)$ is shown to have distinct even/odd parts. Being as $\mathcal{R}_1(a, b) = a/(1 + S_1(a, b))$ we thus see that the even, odd parts of $\mathcal{R}_1$ are known as

$$\mathcal{R}_1^{even}(1, i) \approx 1.167,$$

$$\mathcal{R}_1^{odd}(1, i) \approx -2.38\ldots,$$

provably correct to the implied precision; thus $\mathcal{R}_1(1, i)$ diverges.

For $(a, b) = (e^{i\pi/4}, e^{-i\pi/4})$ the parabola theorem applies with

$$c_1(n) := \frac{2n^2(2n+1)^2}{-2 - i + (4 - 4i)n + 8n^2}, \quad d_1(n) := c_1(-n),$$

so both F_1, G_1 converge to finite values. This convergence can also be shown via the aformentioned definition $\gamma_n := (n + \delta_n)^2$ with

$$\delta_n = \sqrt{i/8} + O(1/n^2).$$

The computation-bounding technique for say $n = 10^5$ and a suitable error bound (we omit the details on bounding of ρ_n) yields

$$\mathcal{R}_1^{even}(e^{i\pi/4}.\ e^{-i\pi/4}) \approx 0.8185 + 0.867i,$$

$$\mathcal{R}_1^{odd}(e^{i\pi/4}.\ e^{-i\pi/4}) \approx -0.103 + 0.583i,$$

both approximations correct to the implied precision. Thus $\mathcal{R}_1$ does not converge for the given parameter pair. □

Such isolated divergence results, together with extensive computations, have led us to:

Conjecture 5.3: $\mathcal{D}_0 = \mathcal{D}_1$. Equivalently, given Corollary 4.5, $\mathcal{R}_1(a,b)$ diverges if $a/b = e^{i\phi}$ with $\cos^2\phi \neq 1$.

We have been able to refine Conjecture 5.3—which conjecture would completely settle the convergence question for the Ramanujan fraction—down to the following (experimentally motivated) form, amounting to a dynamical equivalent for divergence:

Conjecture 5.4: For complex nonzero a and real ϕ with $\cos^2\phi \neq 1$, or $a \in \mathcal{I}$ and $\cos^2\phi = 1$, and any complex initial values (r_0, r_1), the sequence (r_n) determined by the recurrence $(n > 1)$

$$r_n = \frac{1}{a(n+1/2)} r_{n-1} + \frac{n^2}{n^2 - 1/4} r_{n-2}, \quad n \text{ even},$$

$$r_n = \frac{1}{a(n+1/2)} r_{n-1} + \frac{n^2 e^{2i\phi}}{n^2 - 1/4} r_{n-2}, \quad n \text{ odd},$$

is bounded in $\mathcal{C}$.

Remark: One could also posit that a recurrence

$$\rho_n = \frac{\rho_{n-1} + n\omega_n \rho_{n-2}}{n+1}$$

with $\omega_n = a^2, a^2 e^{2i\phi}$ as n be even/odd respectively, has $\rho_n = O(a^n/\sqrt{n})$, yielding an equivalent analysis. The advantage of the particular recurrence form in Conjecture 5.4 is the simple goal of boundedness of the $|r_n|$, while the advantage of the ρ-recurrence suggested here is that the algebra is less recondite. We note that Conjecture 5.4 has been indirectly settled, via Theorems 5.1, 5.2 (and the analysis in the following Theorem 5.5) for the cases a pure-imaginary and $\phi = 0$, and $(a, \phi) = (1, \pi/2), (\sqrt{i}, -\pi/2)$. Also, though we believe the boundedness of the r_n is independent of initial values, we could if necessary posit a conjecture having $r_0 := 1/\Gamma(3/2)$, $r_1 := 1/(a\Gamma(5/2))$ (or for the alternative ρ sequence, $\rho_0 := 1$, $\rho_1 := 1/2$) for such initial values are consistent with $q_0 = q_1 := 1$ for the original fraction.

The fascinating recurrence in Conjecture 5.4—or its various equivalent recurrences as in the Remark—give rise to

Theorem 5.5: Conjecture 5.4 implies Conjecture 5.3, i.e. that $\mathcal{D}_0 = \mathcal{D}_1$.

Proof: Let p_n/q_n be the convergents to the fraction

$$\mathcal{S}_1(a,b) := \cfrac{b^2}{1 + \cfrac{4a^2}{1 + \cfrac{9b^2}{1 + \cfrac{16a^2}{1 + \ddots}}}},$$

where $a/b = e^{i\phi}$ with real ϕ, $\cos^2\phi \neq 1$. We have $q_0 = q_1 = 1$. Now define

$$r_n := \frac{q_n}{a^n \Gamma(n+3/2)},$$

so that the r_n satisfy the recurrences of Conjecture 5.4. For the $\mathcal{S}_1$ fraction, we have, for n even, via relation (3.1),

$$\Delta_n := \left|\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}\right| = \frac{n!^2}{\Gamma(n+3/2)\Gamma(n+1/2)} \left|\frac{a e^{in\phi}}{r_n r_{n-1}}\right|.$$

Thus, by Conjecture 5.4, Δ_n is thus bounded below, and so $\mathcal{S}_1$ and hence $\mathcal{R}_1$ is divergent. □

Conjecture 5.4—which would completely settle the convergence problem for the Ramanujan fraction—is motivated by extensive numerical experiments: The r_n of said conjecture appear to be bounded (alternatively, the ρ_n/a^n decay like $1/\sqrt{n}$) in every case we have studied. One thing we can say at this juncture: The theory of Gill on Möbius transforms [2] implies that for $a/b = e^{i\phi}$ with $\cos^2\phi \neq 1$, then *both* even and odd parts of $\mathcal{R}_1(a,b)$ do converge. (Indeed, we saw two manifestations of this in Theorem 5.2.) We are saying via our conjectures that such even/odd parts should converge to distinct limits. Thus there is a kind of "bifurcation" for $a/b = e^{i\phi}$ with $\cos^2\phi \neq 1$. For the parameter instances $a^2 = b^2, b \in \mathcal{I}$ it may be that *both* even/odd parts of $\mathcal{R}_1(a,b)$ are always themselves bifurcated, or in some way chaotic. These collective divergence issues beg for further analysis beyond the scope of the present treatment.

6 AGM relation revisited

The remarkable AGM relation (1.2) that motivated both this and the companion [1] treatments can now be put in perspective via

Theorem 6.1: If $a/b \in \mathcal{H}$ then each of the three fractions

$$\mathcal{R}_1(a,b),\ \mathcal{R}_1(b,a),\ \mathcal{R}_1\left((a+b)/2, \sqrt{ab}\right)$$

converges on $\hat{\mathcal{C}}$.

Proof: For $a/b \in \mathcal{H}$ none of the relevant parameter pairs enjoy equal magnitudes, so Theorem 4.4 settles the issue. □

It is fascinating that, in spite of Theorem 6.1–and as suggested in our Abstract—there are parameter pairs (a,b) with all three fractions converging, and yet the AGM relation (1.2) is false. For example.

$$\mathcal{R}_1(2i,1) + \mathcal{R}_1(1,2i) \neq 2\mathcal{R}_1(1/2+i, 1+i),$$

which fact can be gleaned easily via some computation and the relatively strong bounds of Theorem 4.3.

Conjecture 6.2: For $a/b \in \mathcal{H}$ the AGM relation (1.2) holds on $\hat{\mathcal{C}}$ (with, as we know, all fractions converging on $\hat{\mathcal{C}}$).

In regard to Theorem 6.1 and Conjecture 6.2, one must take care to observe certain anomalies. For example, it turns out that $\mathcal{R}_1(a,b)$, converges to infinity when

$$a := i\,\frac{\Gamma^2(1/4)}{4\pi^{3/2}}, \quad b := i\,\frac{\Gamma^2(1/4)}{4\pi^{3/2}\sqrt{2}},$$

even though $a/b \in \mathcal{H}$, and indeed Conjecture 6.2 remains intact, in the sense that the AGM relation for this pair (a,b) then reads $\infty = \infty$. Note that for this peculiar parameter pair (and certain others) the fraction

$$\mathcal{S}_1(a,b) := \cfrac{b^2}{1+\cfrac{4a^2}{1+\cfrac{9b^2}{1+\cfrac{16a^2}{1+\ddots}}}}$$

actually converges to the finite value (-1). Such singularities in the AGM relation can also be inferred from the sech identities (3.1), (3.2) in the treatment [1], which identities revealing the possibility of infinitely many poles in the summation.

We believe it *very* likely that Conjecture 6.2 would follow from careful examination of the analyticity properties (in η, a, b) of the aforementioned sech series and the corresponding properties for the continued fractions with $|a| \neq |b|$.

7 Open issues

- We still do not know an exact evaluation—in the sense, say, of closed forms as in [1] for $\mathcal{R}_1(a,a)$, certain a—for unequal a, b; except, as we state in Section 6, we do know some (a,b) with $\mathcal{R}_1(a,b) = \infty$.

- Conjecture 5.4, as we have intimated, is numerically reasonable. Evidently, we need to show that the terms r_n have a certain phase relationship that allows rigorous bounding. What is the procedure (hypergeometric analysis, asymptotic-series analysis) that might prove Conjecture 5.4, and thus settle completely the convergence issue for $\mathcal{R}_1(a,b)$? This the basic question that begs further analysis of such recurrences as those of Conjecture 5,4.

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