APPENDIX II: A COMPUTATIONAL CHRONOLOGY FOR PI

The next two tables provide a reasonably complete accounting of conputations from 2000 BCE to 1996 AD.[1]

[1] Tables taken from: David H. Bailey, Jonathan M. Borwein, Peter B. Borwein, and Simon Plouffe, "The Quest for Pi," (June, 1996) to appear in *The Mathematical Intelligencer*.

Babylonians	2000? BCE	1	3.125 ($3\frac{1}{8}$)
Egyptians	2000? BCE	1	3.16045 ($4(\frac{8}{9})^2$)
China	1200? BCE	1	3
Bible (1 Kings 7:23)	550? BCE	1	3
Archimedes	250? BCE	3	3.1418 (ave.)
Hon Han Shu	130 AD	1	3.1622 ($=\sqrt{10}$?)
Ptolemy	150	3	3.14166
Chung Hing	250?	1	3.16227 ($\sqrt{10}$)
Wang Fau	250?	1	3.15555 ($\frac{142}{45}$)
Liu Hui	263	5	3.14159
Siddhanta	380	3	3.1416
Tsu Ch'ung Chi	480?	7	3.1415926
Aryabhata	499	4	3.14156
Brahmagupta	640?	1	3.162277 ($=\sqrt{10}$)
Al-Khowarizmi	800	4	3.1416
Fibonacci	1220	3	3.141818
Al-Kashi	1429	14	
Otho	1573	6	3.1415929
Viete	1593	9	3.1415926536 (ave.)
Romanus	1593	15	
Van Ceulen	1596	20	
Van Ceulen	1615	35	
Newton	1665	16	
Sharp	1699	71	
Seki	1700?	10	
Kamata	1730?	25	
Machin	1706	100	
De Lagny	1719	127	(112 correct)
Takebe	1723	41	
Matsunaga	1739	50	
Vega	1794	140	
Rutherford	1824	208	(152 correct)
Strassnitzky and Dase	1844	200	
Clausen	1847	248	
Lehmann	1853	261	
Rutherford	1853	440	
Shanks	1874	707	(527 correct)

Table 1: History of π Calculations (Pre 20th Century)

Ferguson	1946	620
Ferguson	Jan. 1947	710
Ferguson and Wrench	Sep. 1947	808
Smith and Wrench	1949	1,120
Reitwiesner et al. (ENIAC)	1949	2,037
Nicholson and Jeenel	1954	3,092
Felton	1957	7,480
Genuys	Jan. 1958	10,000
Felton	May 1958	10,021
Guilloud	1959	16,167
Shanks and Wrench	1961	100,265
Guilloud and Filliatre	1966	250,000
Guilloud and Dichampt	1967	500,000
Guilloud and Bouyer	1973	1,001,250
Miyoshi and Kanada	1981	2,000,036
Guilloud	1982	2,000,050
Tamura	1982	2,097,144
Tamura and Kanada	1982	8,388,576
Kanada, Yoshino and Tamura	1982	16,777,206
Ushiro and Kanada	Oct. 1983	10,013,395
Gosper	1985	17,526,200
Bailey	Jan. 1986	29,360,111
Kanada and Tamura	Sep. 1986	33,554,414
Kanada and Tamura	Oct. 1986	67,108,839
Kanada, Tamura, Kubo, et. al	Jan. 1987	134,217,700
Kanada and Tamura	Jan. 1988	201,326,551
Chudnovskys	May 1989	480,000,000
Chudnovskys	Jun. 1989	525,229,270
Kanada and Tamura	Jul. 1989	536,870,898
Kanada and Tamura	Nov. 1989	1,073,741,799
Chudnovskys	Aug. 1989	1,011,196,691
Chudnovskys	Aug. 1991	2,260,000,000
Chudnovskys	May 1994	4,044,000,000
Takahashi and Kanada	Jun. 1995	3,221,225,466
Kanada	Aug. 1995	4,294,967,286
Kanada	Oct. 1995	6,442,450,938

Table 2: History of π Calculations (20th Century)

APPENDIX III - SELECTED FORMULAE FOR PI

Fifty decimal digits of π. One hundred thousand digits may be found in [38].

$$\pi = 3.14159265358979323846264338327950288419716939937510...$$

Archimedes [3,4]

Let $a_0 := 2\sqrt{3}$, $b_0 := 3$ and

$$a_{n+1} := \frac{2a_n b_n}{a_n + b_n} \quad \text{and} \quad b_{n+1} := \sqrt{a_{n+1} b_n}.$$

Then a_n and b_n converge linearly to π (with an error $O(4^{-n})$.)

(ca 250 BC)

Francois Viète [9]

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots$$

(ca 1579)

John Wallis [10,11]

$$\frac{\pi}{2} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdots}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\cdots}$$

(ca 1650)

William Brouncker

$$\pi = \cfrac{4}{1+\cfrac{1}{2+\cfrac{9}{2+\cfrac{25}{2+\cdots}}}}$$

(ca 1650)

Mādhava, James Gregory, Gottfried Wilhelm Leibnitz [7,13,14]

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

(1450–1671)

Isaac Newton [16]

$$\pi \;=\; \frac{3\sqrt{3}}{4} + 24\left(\frac{2}{3\cdot 2^3} - \frac{1}{5\cdot 2^5} - \frac{1}{28\cdot 2^7} - \frac{1}{72\cdot 2^9} - \cdots\right)$$

(ca 1666)

Machin Type Formulae [31,32,37]

$$\frac{\pi}{4} = 4\arctan(\frac{1}{5}) - \arctan(\frac{1}{239})$$

$$\frac{\pi}{4} = \arctan(\frac{1}{2}) + \arctan(\frac{1}{3})$$

$$\frac{\pi}{4} = 2\arctan(\frac{1}{2}) - \arctan(\frac{1}{7})$$

$$\frac{\pi}{4} = 2\arctan(\frac{1}{3}) + \arctan(\frac{1}{7})$$

(1706–1776)

Leonard Euler [17]

$$\begin{aligned} \frac{\pi^2}{6} &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots \\ \frac{\pi^4}{90} &= 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \cdots \end{aligned}$$

$$\frac{\pi^2}{6} = 3\sum_{m=1}^{\infty} \frac{1}{m^2\binom{2m}{m}}$$

(ca 1748)

Srinivasa Ramanujan [29,64]

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \begin{pmatrix} 2n \\ n \end{pmatrix}^3 \frac{42n+5}{2^{12n+4}}.$$

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4} \frac{[1103 + 26390n]}{396^{4n}}.$$

Each additional term of the latter series adds roughly 8 digits.

(1914)

Louis Comtet

$$\frac{\pi^4}{90} = \frac{36}{17}\sum_{m=1}^{\infty} \frac{1}{m^4\binom{2m}{m}}$$

(1974)

Eugene Salamin [46], **Richard Brent** [47]

Set $a_0 = 1, b_0 = 1/\sqrt{2}$ and $s_0 = 1/2$. For $k = 1, 2, 3, \cdots$ compute

$$\begin{aligned} a_k &= \frac{a_{k-1} + b_{k-1}}{2} \\ b_k &= \sqrt{a_{k-1} b_{k-1}} \\ c_k &= a_k^2 - b_k^2 \\ s_k &= s_{k-1} - 2^k c_k \\ p_k &= \frac{2a_k^2}{s_k} \end{aligned}$$

Then p_k converges *quadratically* to π.

(1976)

Jonathan Borwein and Peter Borwein [64]

Set $a_0 = 1/3$ and $s_0 = (\sqrt{3} - 1)/2$. Iterate

$$\begin{aligned} r_{k+1} &= \frac{3}{1 + 2(1 - s_k^3)^{1/3}} \\ s_{k+1} &= \frac{r_{k+1} - 1}{2} \\ a_{k+1} &= r_{k+1}^2 a_k - 3^k (r_{k+1}^2 - 1) \end{aligned}$$

Then $1/a_k$ converges *cubically* to π.

(1991)

Set $a_0 = 6 - 4\sqrt{2}$ and $y_0 = \sqrt{2} - 1$. Iterate

$$\begin{aligned} y_{k+1} &= \frac{1 - (1 - y_k^4)^{1/4}}{1 + (1 - y_k^4)^{1/4}} \\ a_{k+1} &= a_k (1 + y_{k+1})^4 - 2^{2k+3} y_{k+1} (1 + y_{k+1} + y_{k+1}^2) \end{aligned}$$

Then a_k converges *quartically* to $1/\pi$.

(1985)

David Chudnovsky and Gregory Chudnovsky [63]

$$\frac{1}{\pi} = 12 \sum_{n=0}^{\infty} (-1)^n \frac{(6n)!}{(n!)^3 (3n)!} \frac{13591409 + n545140134}{(640320^3)^{n+1/2}}.$$

Each additional term of the series adds roughly 15 digits.

(1989)

Jonathan Borwein and Peter Borwein

$$\frac{1}{\pi} = 12 \sum_{n=0}^{\infty} \frac{(-1)^n (6n)!}{(n!)^3 (3n)!} \frac{(A + nB)}{C^{n+1/2}}$$

where

$$\begin{aligned}
A &:= 212175710912\sqrt{61} + 1657145277365 \\
B &:= 13773980892672\sqrt{61} + 107578229802750 \\
C &:= [5280(236674 + 30303\sqrt{61})]^3.
\end{aligned}$$

Each additional term of the series adds roughly 31 digits.

(1989)

The following is not an identity but is correct to over 42 billion digits

$$\left(\frac{1}{10^5}\sum_{n=-\infty}^{\infty} e^{-\frac{n^2}{10^{10}}}\right)^2 \doteq \pi.$$

(1985)

Roy North [65]
Gregory's series for π, truncated at 500,000 terms gives to forty places

$$4\sum_{k=1}^{500,000} \frac{(-1)^{k-1}}{2k-1}$$

$$= 3.14159\underline{0}6535897932\underline{40}4626433832\underline{6}9502884197.$$

Then only the underlined digits are incorrect.

(1989)

David Bailey, Peter Borwein and Simon Plouffe [70]

$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i}\left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6}\right)$$

(1996)