Interpolation (Part I)

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Overview

- Polynomial interpolation
  - The basic theorem
  - Lagrange form, Neville's method
  - Divided-difference method
  - Hermite interpolation
- Piecewise polynomial interpolation
  - Cubic spline interpolation
- Planar parametric curves

Interpolation: Problem Specification

Input: a set $\mathcal{S}$ of data

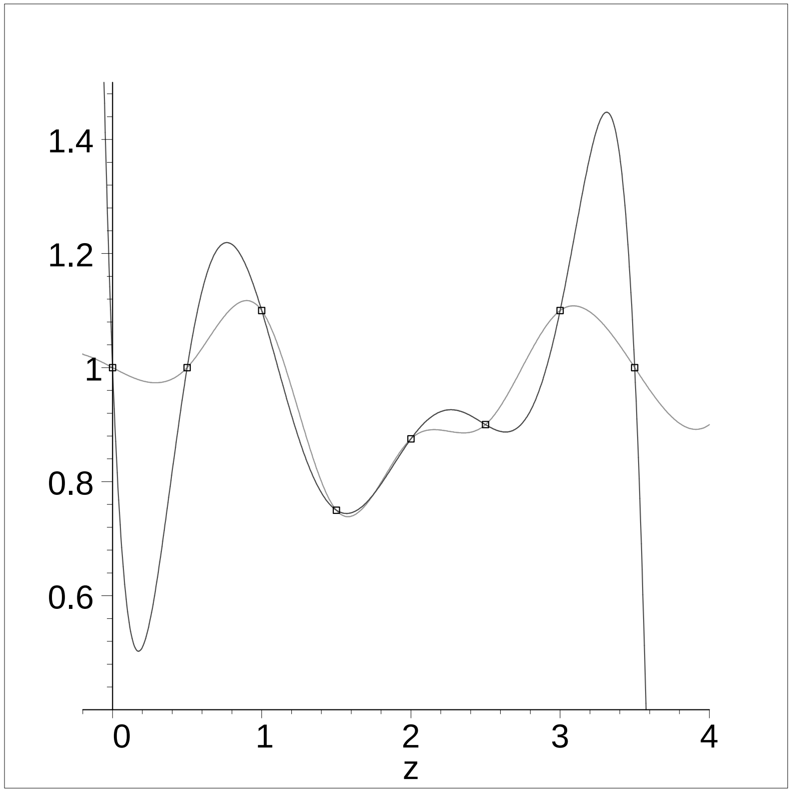
$$\{(x_0, y_0),\ (x_1, y_1), \ldots, (x_n, y_n)\},\ \ (x_i,\ y_i) \in \mathbb{R}^2.$$

Output: a function $P(x)$ which interpolates the given set of data, i.e., $P(x_i) = y_i$, for $0 \le i \le n$.

Note. In general, the interpolating function $P(x)$ is NOT unique.

Interpolation: an Example

$$\mathcal{S} = \left\{(0,1), \left(\frac{1}{2}, 1\right), \left(1, \frac{11}{10}\right), \left(\frac{3}{2}, \frac{3}{4}\right), \left(2, \frac{7}{8}\right), \left(\frac{5}{2}, \frac{9}{10}\right), \left(3, \frac{11}{10}\right), \left(\frac{7}{2}, 1\right)\right\}$$



Polynomial Interpolation

The interpolating function is a polynomial, i.e., $P(x) \in \mathbb{R}[x]$.

Polynomial interpolation is used

- for applications involving small sets of data, i.e., typically n not bigger than 5 or 6;
- as a component of a larger computation including integration, solving differential equations, piecewise polynomial interpolation.

Polynomial Interpolation: the Basic Theorem

Given $(n+1)$ data pairs $(x_i, y_i) \in \mathbb{R}^2, 0 \le i \le n$, with $x_i \neq x_j$ if $i \neq j$, there is *a unique polynomial* $P(x)$ of degree not exceeding n that interpolates this data.

Example. For $\mathcal{S} = \{(x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3)\}$, consider $P(x) = c_0 + c_1\, x + c_2\, x^2 + c_3\, x^3$, which interpolates the points in $\mathcal{S}$:

$$\begin{aligned} P(x_0) &= c_0 + c_1 x_0 + c_2 x_0^2 + c_3 x_0^3 = y_0 \\ P(x_1) &= c_0 + c_1 x_1 + c_2 x_1^2 + c_3 x_1^3 = y_1 \\ P(x_2) &= c_0 + c_1 x_2 + c_2 x_2^2 + c_3 x_2^3 = y_2 \\ P(x_3) &= c_0 + c_1 x_3 + c_2 x_3^2 + c_3 x_3^3 = y_3. \end{aligned} \tag{1}$$

In matrix/vector notation:

$$\underbrace{\begin{bmatrix} 1 & x_0 & {x_0}^2 & {x_0}^3 \\ 1 & x_1 & {x_1}^2 & {x_1}^3 \\ 1 & x_2 & {x_2}^2 & {x_2}^3 \\ 1 & x_3 & {x_3}^2 & {x_3}^3 \end{bmatrix}}_{\text{Vandermonde matrix: } V} \cdot \underbrace{\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}}_{c} = \underbrace{\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{y}$$

$\det V = \prod_{i<j}(x_i - x_j) =$

$$(x_2 - x_3)(x_1 - x_3)(x_1 - x_2)(x_0 - x_3)(x_0 - x_2)(x_0 - x_1) \neq 0 \quad (2)$$

since $x_i \neq x_j$ for $i \neq j$.

A basic fact in linear algebra. If V is an $n \times n$ matrix, the following two conditions are equivalent:

1. V is nonsingular;
2. the linear system of equations $V \cdot c = y$ has a unique solution c for any $n \times 1$ matrix y.

It follows from (2) and "a basic fact in linear algebra" that the linear system of equations (1) has a *unique* solution c, i.e., there is a unique polynomial $P(x)$ which interpolates the data set $\mathcal{S}$.

The same argument applies to the general case where $|\mathcal{S}| = n$.

Lagrange Form

Given $\mathcal{S} = \{(x_0, y_0),\ (x_1, y_1), \ldots, (x_n, y_n)\}$, $x_i \neq x_j$ for $i \neq j$, consider the set of $(n+1)$ *Lagrange basis functions* $L_k(x)$, $k = 0, \ldots, n$, defined as

$$L_k(x) = \frac{(x - x_0) \cdots (x - x_{k-1})(x - x_{k+1}) \cdots (x - x_n)}{(x_k - x_0) \cdots (x_k - x_{k-1})(x_k - x_{k+1}) \cdots (x_k - x_n)}.$$

It is easy to show that

$$L_k(x_j) = \begin{cases} 0 & \text{if } j \neq k, \\ 1 & \text{if } j = k. \end{cases} \tag{3}$$

Define $P(x) = y_0\, L_0(x) + y_1\, L_1(x) + \cdots + y_n\, L_n(x)$. By (3), $P(x_i) = y_i$ for $i = 0, \ldots, n$. Hence, $P(x)$ is the interpolating polynomial for the data set $\mathcal{S}$.

Lagrange Form: an Example

i	0	1	2	3	4
x_i	3.2	2.7	1.0	4.8	5.6
$P(x_i)$	22.0	17.8	14.2	38.3	51.7

Use x_0, x_1, and x_2 to construct the second Lagrange interpolating polynomial.

$$\begin{aligned} P_{0,1,2}(x) &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}P(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}P(x_1) + \\ & \quad \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}P(x_2) \\ &\approx 20.0\,(x-2.7)\,(x-1.0) - 20.9\,(x-3.2)\,(x-1.0) + \\ & \quad 3.80\,(x-3.2)\,(x-2.7)\,. \end{aligned}$$

Use x_0, x_1, x_2, and x_3 to construct the third Lagrange interpolating polynomial.

$P_{0,1,2,3}(x)$ equals

$$\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}P(x_0)+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}P(x_1)+$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}P(x_2)+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}P(x_3),$$

which approximates

$$-12.5\,(x-2.7)\,(x-1.0)\,(x-4.8)+10.0\,(x-3.2)\,(x-1.0)\,(x-4.8)-$$
$$1.0\,(x-3.2)\,(x-2.7)\,(x-4.8)+2.99\,(x-3.2)\,(x-2.7)\,(x-1.0)\,.$$

Lagrange Form: Error Estimate

Suppose $x_0, x_1, \ldots, x_n$ are distinct numbers in the interval $[a, b]$ and $f \in C^{n+1}[a, b]$. Then for each x in $[a, b]$, a number $\xi(x)$ in (a, b) exists with

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n),$$

where $P(x)$ is the n-th Lagrange polynomial.

Note. The error form for the Lagrange polynomial is quite similar to that for the Taylor polynomial.

Neville's Method

- The error formula is not applied because we do not know the derivative values of f. However, we can get an estimate for the error by examining polynomials of different degrees and by using different nodes.

- Derivation of the interpolating polynomials in a manner that uses the previous calculations.

Definition. Let f be a function defined at $x_0, x_1, x_2, \ldots, x_n$, and suppose that $m_1, m_2, \ldots, m_k$ are k distinct integers, with $0 \leq m_i \leq n$ for each i. The Lagrange polynomial that agrees with $f(x)$ at the k points $x_{m_1}, x_{m_2}, \ldots, x_{m_k}$ is denoted $P_{m_1,m_2,\ldots,m_k}(x)$.

Example. $P_{0,1,2}$ and $P_{0,1,2,3}$ in the last example.

Theorem. Let f be defined at $x_0, x_1, \ldots, x_k$, and let x_i and x_j be two distinct numbers in this set. Then

$$P_{0,\ldots,k}(x) = \frac{(x - x_j)\overbrace{P_{0,\ldots,j-1,j+1,\ldots,k}(x)}^{\hat{Q}(x)} - (x - x_i)\overbrace{P_{0,\ldots,i-1,i+1,\ldots,k}(x)}^{Q(x)}}{x_i - x_j}.$$

Proof.

- $r \neq i$ and $r \neq j$: $Q(x_r) = \hat{Q}(x_r) = f(x_r)$ for $0 \leq r \leq k$. Hence, $P_{0,\ldots,k}(x_r) = f(x_r)$.
- $r = i$: $P_{0,\ldots,k}(x_i) = \hat{Q}(x_i) = f(x_i)$.
- $r = j$: $P_{0,\ldots,k}(x_j) = Q(x_j) = f(x_j)$. ■

$$
\begin{array}{ccccccccc}
P_0 & & & & & & & & \\
& & P_{0,1} & & & & & & \\
P_1 & & & & P_{0,1,2} & & & & \\
& & P_{1,2} & & & & P_{0,1,2,3} & & \\
P_2 & & & & P_{1,2,3} & & & & P_{0,1,2,3,4} \\
& & P_{2,3} & & & & P_{1,2,3,4} & & \\
P_3 & & & & P_{2,3,4} & & & & \\
& & P_{3,4} & & & & & & \\
P_4 & & & & & & & &
\end{array}
$$

Example. $P_{0,1}(x) = \dfrac{(x - x_1)P_0 - (x - x_0)P_1}{x_0 - x_1}$ is derived from P_0 and P_1. Similarly, $P_{0,1,2,3}$ can be derived from $P_{0,1,2}$ and $P_{1,2,3}$.

Neville's Method: an Example

- Problem. Let $x_i = i$ for $i = 0, 1, 2, 3$. It is known that

$$P_{0,1}(x) = 2x + 1, \quad P_{0,2}(x) = x + 1, \quad P_{1,2,3}(2.5) = 3.$$

Find $P_{0,1,2,3}(2.5)$.

- Solution.

$$P_{0,1,2,3}(2.5) = \left. \frac{(x - x_3)P_{0,1,2}(x) - (x - x_0)P_{1,2,3}(x)}{x_0 - x_3} \right|_{x=2.5}. \qquad (4)$$

$$P_{0,1,2}(2.5) = \left. \frac{(x - x_2)P_{0,1}(x) - (x - x_1)P_{0,2}(x)}{x_1 - x_2} \right|_{x=2.5} \approx 2.25.$$

By (4), $P_{0,1,2,3}(2.5) \approx 2.8750$.

Divided-Difference Method

Let $\mathcal{S} = \{(x_0, f(x_0)),\ (x_1, f(x_1)), \ldots, (x_n, f(x_n))\}$, $x_i \neq x_j$ for $i \neq j$. Consider

$$\begin{aligned} P_n(x) &= a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + \\ &\quad a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}). \end{aligned}$$

Note that $P_n(x) \in \mathbb{R}[x]$, and $\deg P_n(x) \leq n$.

Problem. Find a_i's, $0 \leq i \leq n$ so that $P_n(x) = f(x)$ at $x = x_i$, $0 \leq i \leq n$.

kth Divided Difference

zeroth: $f[x_i] = f(x_i)$.

first: $f[x_i, x_{i+1}] = \dfrac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i} = \dfrac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$

second: $f[x_i, x_{i+1}, x_{i+2}] = \dfrac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$

kth: $f[x_i, x_{i+1}, \ldots, x_{i+k-1}, x_{i+k}] =$

$$\frac{f[x_{i+1}, x_{i+2}, \ldots, x_{i+k}] - f[x_i, x_{i+1}, \ldots, x_{i+k-1}]}{x_{i+k} - x_i}$$

Divided Differences: a Table

x	$f(x)$	First divided differences	Second divided differences
x_0	$f[x_0]$		
		$f[x_0, x_1] = \frac{f[x_1]-f[x_0]}{x_1-x_0}$	
x_1	$f[x_1]$		$f[x_0, x_1, x_2] = \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0}$
		$f[x_1, x_2] = \frac{f[x_2]-f[x_1]}{x_2-x_1}$	
x_2	$f[x_2]$		$f[x_1, x_2, x_3] = \frac{f[x_2,x_3]-f[x_1,x_2]}{x_3-x_1}$
		$f[x_2, x_3] = \frac{f[x_3]-f[x_2]}{x_3-x_2}$	
x_3	$f[x_3]$		

Newton's Divided Difference Formula

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}). \quad (5)$$

$P_n(x_0) = f(x_0) = f[x_0] = a_0$.

$P_n(x_1) = f(x_1) = f[x_1] = f[x_0] + a_1(x - x_0)$. Hence,

$$a_1 = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = f[x_0, x_1].$$

$$a_k = f[x_0, x_1, \ldots, x_k].$$

By (5), $P_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, \ldots, x_k](x - x_0)\cdots(x - x_{k-1}) =$

$f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \cdots +$
$f[x_0, x_1, \ldots, x_n](x - x_0)(x - x_1)\cdots(x - x_{n-1})$.

Newton's Formula: an Example

$$\mathcal{S} = \{(3.2, 22.0),\ \ (2.7, 17.8),\ \ (4.8, 38.3)\}\,.$$

$$\begin{aligned}
P_3(x) &= f[x_0]+f[x_0,x_1](x-x_0)+f[x_0,x_1,x_2](x-x_0)(x-x_1)\\
f[x_0] &= f(x_0) = 3.2\\
f[x_0,x_1] &= \frac{f[x_1]-f[x_0]}{x_1-x_0} = 8.4000\\
f[x_1,x_2] &= \frac{f[x_2]-f[x_1]}{x_2-x_1} = 9.7619\\
f[x_0,x_1,x_2] &= \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0} = 0.85119
\end{aligned}$$

Hence,

$P_3(x) = 3.2000+8.4000(x-3.2000)+0.85119(x-3.2000)(x-2.7000).$

Newton Forward-Difference Formula

Equal spacing: $x_{i+1} - x_i = h$, $0 \le i \le n-1$. Then $x_i = x_0 + ih$.
Let $x = x_0 + sh$. We have

$$\begin{aligned}\prod_{i=0}^{k-1}(x-x_i) &= (x_0+sh-x_0)(x_0+sh-x_0-h)\cdots(x_0+sh-x_0-h(k-1)) \\ &= s(s-1)\cdots(s-k+1)h^k = \binom{s}{k} k!\, h^k.\end{aligned}$$

Hence,

$$\begin{aligned}P_n(x) &= f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, \ldots, x_k](x-x_0)\cdots(x-x_{k-1}) \\ &= f[x_0] + \sum_{k=1}^{n} \binom{s}{k} k!\, h^k\, f[x_0, x_1, \ldots, x_k].\end{aligned}$$

Foward-difference operator.

$\Delta f(x_i) = f(x_{i+1}) - f(x_i)$, $\Delta^k f(x_i) = \Delta^{k-1}(\Delta f(x_i))$ for $k = 2, 3, \ldots$.

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{\Delta f(x_0)}{h}.$$

$$\begin{aligned} f[x_0, x_1, x_2] &= \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{1}{2h}\left(\frac{\Delta f(x_1) - \Delta f(x_0)}{h}\right) \\ &= \frac{1}{2h^2}\left(\Delta(f(x_1) - f(x_0))\right) = \frac{1}{2!\, h^2}\Delta^2 f(x_0). \end{aligned}$$

In general, $f[x_0, x_1, \ldots, x_k] = \frac{1}{k!\, h^k}\Delta^k f(x_0)$. Hence,

$$\begin{aligned} P_n(x) &= f[x_0] + \sum_{k=1}^{n} \binom{s}{k} k!\, h^k\, f[x_0, x_1, \ldots, x_k] \\ &= f[x_0] + \sum_{k=1}^{n} \binom{s}{k} \Delta^k f(x_0). \end{aligned}$$

Hermite Interpolation

The interpolating polynomial $P(x)$ is to have the same "shape" as the function $f(x)$ at the data points in the sense that the *tangent lines* to the polynomial and to the function agree at $(x_i, f(x_i))$, for $0 \le i \le n$, i.e.,

$$\begin{aligned} P(x_i) &= f(x_i), \\ P'(x_i) &= f'(x_i). \end{aligned}$$

Remark. Need to know the values of $f(x_i)$ and $f'(x_i)$ for $0 \le i \le n$.

Hermite Polynomials

Theorem. If $f \in C^1[a,b]$ and $x_0, \ldots, x_n \in [a,b]$ are distinct, the *unique polynomial* of least degree agreeing with f and f' at $x_0, \ldots, x_n$ is the Hermite polynomial of degree at most $2n+1$:

$$H(x) = \sum_{j=0}^{n} f(x_j) H_j(x) + \sum_{j=0}^{n} f'(x_j) \hat{H}_j(x),$$

where

$$H_j(x) = \left(1 - 2(x - x_j) L_j'(x_j)\right) L_j^2(x), \quad \bar{H}_j(x) = (x - x_j) L_j^2(x).$$

Moreover, if $f \in C^{2n+2}[a,b]$, then for some ξ with $a < \xi < b$,

$$f(x) = H(x) + \frac{(x - x_0)^2 \cdots (x - x_n)^2}{(2n+2)!} f^{(2n+2)}(\xi).$$

Hermite Polynomials: an Example

j	x_j	$f(x_j)$	$f'(x_j)$
0	1.3	0.6200860	-0.5220232
1	1.6	0.4554022	-0.5698959
2	1.9	0.2818186	-0.5811571

$$\begin{aligned} H(x) &= \sum_{j=0}^{2} f(x_j)H_j(x) + \sum_{j=0}^{2} f'(x_j)\hat{H}_j(x) \\ &= f(x_0)H_0(x) + f(x_1)H_1(x) + f(x_2)H_2(x) + \\ &\quad f'(x_0)\hat{H}_0(x) + f'(x_1)\hat{H}_1(x) + f'(x_2)\hat{H}_2(x). \end{aligned} \tag{6}$$

$$
\begin{aligned}
H_0(x) &= (1-2(x-x_0)L_0'(x_0))\, L_0^2(x), & \hat{H}_0(x) &= (x-x_0)L_0^2(x)\\
H_1(x) &= (1-2(x-x_1)L_1'(x_1))\, L_1^2(x), & \hat{H}_1(x) &= (x-x_1)L_1^2(x)\\
H_2(x) &= (1-2(x-x_2)L_2'(x_2))\, L_2^2(x), & \hat{H}_2(x) &= (x-x_2)L_2^2(x)
\end{aligned}
$$

$$
\begin{aligned}
L_0(x) &= \tfrac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} &=& \tfrac{50}{9}x^2 - \tfrac{175}{9}x + \tfrac{152}{9},\\
L_1(x) &= \tfrac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} &=& -\tfrac{100}{9}x^2 + \tfrac{320}{9}x - \tfrac{247}{9},\\
L_2(x) &= \tfrac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} &=& \tfrac{50}{9}x^2 - \tfrac{145}{9}x + \tfrac{104}{9}.
\end{aligned}
$$

$$
L_0'(x) = \frac{100}{9}x - \frac{175}{9},\ L_1'(x) = -\frac{200}{9}x + \frac{320}{9},\ L_2'(x) = \frac{100}{9}x - \frac{145}{9}.
$$

Given L_j's, L_j''s, compute H_j's and $\hat{H}_j$'s, then plug in the values of $f(x_j)$, $f'(x_j)$, $H_j(x)$, $\hat{H}_j(x)$, for $0 \leq j \leq 2$, into (6).

Hermite Polynomials: Newton IDD Formula

- A direct application of the theorem for constructing Hermite polynomials requires evaluation of the Lagrange polynomials and their derivatives. This is tedious even for small values of n.
- We instead use Newton interpolatory divided difference formula.
- Main idea. Define a new sequence $z_0, z_1, \ldots, z_{2n+1}$ by

$$z_{2i} = z_{2i+1} = x_i, \qquad \text{for each } \ i = 0, 1, \ldots, n,$$

construct the divided difference table that uses $z_0, z_1, \ldots, z_{2n+1}$.

Problem. For $0 \le i \le n$, the first divided differences $f[z_{2i}, z_{2i+1}]$ are not defined.

Fix. $f[z_{2i}, z_{2i+1}] \approx f'(z_{2i}) = f'(x_i)$.

A small table:

x	$f(x)$	First divided differences	Second divided differences
$z_0 = x_0$	$f[z_0] = f(x_0)$		
		$f[z_0, z_1] = f'(x_0)$	
$z_1 = x_0$	$f[z_1] = f(x_0)$		$f[z_0, z_1, z_2] = \frac{f[z_1, z_2] - f[z_0, z_1]}{z_2 - z_0}$
		$f[z_1, z_2] = \frac{f[z_2] - f[z_1]}{z_2 - z_1}$	
$z_2 = x_1$	$f[z_2] = f(x_1)$		$f[z_1, z_2, z_3] = \frac{f[z_2, z_3] - f[z_1, z_2]}{z_3 - z_1}$
		$f[z_2, z_3] = f'(x_1)$	
$z_3 = x_1$	$f[z_3] = f(x_1)$		

The Hermite polynomial is then given by

$$H(x) = f[z_0] + \sum_{k=1}^{2n+1} f[z_0, \ldots, z_k](x - z_0)(x - z_1) \cdots (x - z_{2k-1}).$$

Newton IDD Formula: an Example

Use the data in the last example (p26), we have

$$
\begin{aligned}
H(x) &= f[z_0] + \sum_{k=1}^{5} f[z_0, \ldots, z_k](x - z_0)(x - z_1) \cdots (x - z_{k-1}) \\
&= f[z_0] + f[z_0, z_1](x - z_0) + f[z_0, z_1, z_2](x - z_0)(x - z_1) + \\
&\quad f[z_0, z_1, z_2, z_3](x - z_0)(x - z_1)(x - z_2) + \\
&\quad f[z_0, z_1, z_2, z_3, z_4](x - z_0)(x - z_1)(x - z_2)(x - z_3) + \\
&\quad f[z_0, z_1, z_2, z_3, z_4, z_5](x - z_0)(x - z_1)(x - z_2)(x - z_3)(x - z_4).
\end{aligned}
$$

The following table of divided differences should give all the information needed for the construction of the Hermite polynomial $H(x)$ given above.

z	$f(z)$	1-st	2-nd	3-rd	4-th	5-th
1.3	0.6200860					
		$\underbrace{-0.5220232}_{f[z_0,z_1]}$				
1.3	0.6200860		$\underbrace{-0.0897427}_{f[z_0,z_1,z_2]}$			
		$\underbrace{-0.5489460}_{f[z_1,z_2]}$		$\underbrace{0.0663657}_{f[z_0,z_1,z_2,z_3]}$		
1.6	0.4554022		$\underbrace{-0.0698330}_{f[z_1,z_2,z_3]}$		0.0026663	
		$\underbrace{-0.5698959}_{f[z_2,z_3]}$		$\underbrace{0.0679655}_{f[z_1,z_2,z_3,z_4]}$		-0.0027738
1.6	0.4554022		$\underbrace{-0.0290537}_{f[z_2,z_3,z_4]}$		0.0010020	
		$\underbrace{-0.5786120}_{f[z_3,z_4]}$		$\underbrace{0.0685667}_{f[z_2,z_3,z_4,z_5]}$		
1.9	0.2818186		$\underbrace{-0.0084837}_{f[z_3,z_4,z_5]}$			
		$\underbrace{-0.5811571}_{f[z_4,z_5]}$				
1.9	0.2818186					