# Interpolation (Part II)

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# Piecewise Polynomial Interpolation

- Interpolation of data of the form $(x_i, y_i)$, $i = 0, \ldots, n$, for (much) larger $n$ than that for polynomial interpolation.
- The $x$ values are required to be *ordered*, i.e., $x_i < x_{i+1}$. They define a *partition* of $n$ subintervals of the total interval $x_0 < x < x_n$.
- Use of a different (low degree) polynomial for each subinterval. These polynomials are patched together to give a piecewise polynomial approximation.
- $x_k$: a *node*, a *breakpoint*, or a *knot* of the piecewise polynomial.

# Cubic Spine Interpolants

Given a function $f$ defined on $[a, b]$, and a set of $n + 1$ nodes $a = x_0 < x_1 < \cdots < x_n = b$, a *cubic spline interpolant* $S$ for $f$ is a function that satisfies the following conditions:

a. $S(x)$ is a cubic polynomial, denoted $S_j(x)$, on the subinterval $[x_j, x_{j+1}]$ for each $0 \leq j \leq n - 1$;

b. $S(x)$ is an interpolant, i.e., $S(x_j) = f(x_j)$ for each $0 \leq j \leq n$;

c. $S(x)$ is continuous, i.e., $S_{j+1}(x_{j+1}) = S_j(x_{j+1})$ for each $0 \leq j \leq n - 2$;

d. $S(x)$ is continuously differentiable, i.e., $S'_{j+1}(x_{j+1}) = S'_j(x_{j+1})$ for each $0 \leq j \leq n-2$;

e. $S(x)$ is twice continuously differentiable, i.e., $S''_{j+1}(x_{j+1}) = S''_j(x_{j+1})$ for each $0 \leq j \leq n-2$;

f. One of the following sets of boundary conditions is satisfied:

(i) $S''(x_0) = S''(x_n) = 0$ (free or natural boundary);

(ii) $S'(x_0) = f'(x_0)$ and $S'(x_n) = f'(x_n)$ (clamped boundary).

## Boundary Conditions: a Justification

Consider the cubic polynomial $S_j(x)$ defined on $[x_j, x_{j+1}]$:

$$S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3.$$

- $n$ intervals and 4 unknowns ($a_j$, $b_j$, $c_j$ and $d_j$) to determine a cubic polynomial on each subinterval $\Longrightarrow$ $4n$ unknowns.
- $S_j(x_j) = f(x_j), S_j(x_{j+1}) = f(x_{j+1})$, $0 \le j \le n-1 \Longrightarrow 2n$ eqns.

$S'_{j+1}(x_{j+1}) = S'_j(x_{j+1})$ and $S''_{j+1}(x_{j+1}) = S''_j(x_{j+1})$, $0 \le j \le n-2$
$\Longrightarrow$ $2(n-1)$ eqns. Hence, $4n-2$ eqns in total.

- Need 2 more equations $\Longrightarrow$ boundary conditions.

# Cubic Spline Interpolants: a Construction

$$S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3, \ 0 \le j \le n - 1. \quad (1)$$

Define $h_j = x_{j+1} - x_j$, $0 \le j \le n - 1$. By (1),

$$S_j(x_j) = a_j = f(x_j), \quad 0 \le j \le n - 1. \quad (2)$$

- Define $a_n = f(x_n)$. By condition (c),

$$a_{j+1} = a_j + b_j h_j + c_j h_j^2 + d_j h_j^3, \quad 0 \le j \le n - 1. \quad (3)$$

- $S_j'(x) = b_j + 2c_j(x - x_j) + 3d_j(x - x_j)^2$. Define $b_n = S'(x_n)$. By condition (d),

$$b_{j+1} = b_j + 2c_j h_j + 3d_j h_j^2, \quad 0 \le j \le n - 1. \quad (4)$$

- $S_j''(x) = 2c_j + 6d_j(x - x_j)$. Define $c_n = S''(x_n)/2$. By condition (e),

$$c_{j+1} = c_j + 3d_j h_j, \quad 0 \le j \le n-1. \tag{5}$$

By (5), (3), (4), for each $0 \le j \le n-1$,

$$a_{j+1} = a_j + b_j h_j + \frac{h_j^2}{3}(2c_j + c_{j+1}), \tag{6}$$

$$b_{j+1} = b_j + h_j(c_j + c_{j+1}). \tag{7}$$

By (6),

$$b_j = \frac{1}{h_j}(a_{j+1} - a_j) - \frac{h_j}{3}(2c_j + c_{j+1}). \tag{8}$$

By (7) and (8), for $1 \le j \le n-1$,

$$h_{j-1}c_{j-1} + 2(h_{j-1} + h_j)c_j + h_j c_{j+1} = \frac{3}{h_j}(a_{j+1} - a_j) - \frac{3}{h_{j-1}}(a_j - a_{j-1}). \tag{9}$$

Example. With $n = 5$, by (9),

$$\begin{aligned} h_0c_0 + 2(h_0 + h_1)c_1 + h_1c_2 &= \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0), \\ h_1c_1 + 2(h_1 + h_2)c_2 + h_2c_3 &= \frac{3}{h_2}(a_3 - a_2) - \frac{3}{h_1}(a_2 - a_1), \\ h_2c_2 + 2(h_2 + h_3)c_3 + h_3c_4 &= \frac{3}{h_3}(a_4 - a_3) - \frac{3}{h_2}(a_3 - a_2), \\ h_3c_3 + 2(h_3 + h_4)c_4 + h_4c_5 &= \frac{3}{h_4}(a_5 - a_4) - \frac{3}{h_3}(a_4 - a_3). \end{aligned} \tag{10}$$

Remark. (10) is a linear system of 4 equations in 6 unknowns $c_i$, $0 \leq i \leq 5$.

## Strictly Diagonally Dominant Matrix

Definition. An $n \times n$ matrix $A$ is strictly diagonally dominant if

$$|a_{ii}| > \sum_{j=1, j \neq i}^{n} |a_{ij}|$$

holds for each $i = 1, 2, \ldots, n$.

Theorem. A strictly diagonally dominant matrix is invertible.

# Natural Boundary Conditions

Recall.

$$S''(x_0) = S''(x_n) = 0. \tag{11}$$

By (11),

$$S_0''(x) = 2c_0 + 6d_0(x - x_0) \quad \Longrightarrow \quad c_0 = \frac{1}{2}\underbrace{S''(x_0)}_{0} = 0. \tag{12}$$

$$c_n = \frac{1}{2}\underbrace{S''(x_n)}_{0} = 0 \tag{13}$$

The combination of (9), (12) and (13) results in a linear system of equations $Ax = b$ to be solved for $x = [c_0, c_1, \ldots, c_n]^t$.

Example. With $n = 5$,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ h_0 & 2(h_0 + h_1) & h_1 & 0 & 0 & 0 \\ 0 & h_1 & 2(h_1 + h_2) & h_2 & 0 & 0 \\ 0 & 0 & h_2 & 2(h_2 + h_3) & h_3 & 0 \\ 0 & 0 & 0 & h_3 & 2(h_3 + h_4) & h_4 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$x = \begin{bmatrix} c_0 & c_1 & c_2 & c_3 & c_4 & c_5 \end{bmatrix}^t,$$

$$\begin{aligned} b \quad = \quad & \Big[0, \quad 3\,\frac{a_2 - a_1}{h_1} - 3\,\frac{a_1 - a_0}{h_0}, \quad 3\,\frac{a_3 - a_2}{h_2} - 3\,\frac{a_2 - a_1}{h_1}, \\ & 3\,\frac{a_4 - a_3}{h_3} - 3\,\frac{a_3 - a_2}{h_2}, \quad 3\,\frac{a_5 - a_4}{h_4} - 3\,\frac{a_4 - a_3}{h_3}, \quad 0\Big]^t. \end{aligned}$$

- The matrix $A$ in the linear system $Ax = b$ is strictly diagonally dominant. Hence, $A$ is invertible, and a unique solution $x = [c_0, c_1, \ldots, c_n]^t$ exists.

- The matrix $A$ is *tridiagonal*: all the entries are zero except for a band which is three-entry wide centered on the main diagonal. Solutions to tridiagonal linear systems can be found very efficiently: only $O(n)$ operations are needed using methods we shall discuss later.

## A Note on Clamped Boundary Conditions

With the clamped boundary conditions: $S'(x_0) = f'(x_0)$, $S'(x_n) = f'(x_n)$, one can derive the following two equations:

$$2h_0c_0 + h_0c_1 = \frac{3}{h_0}(a_1 - a_0) - 3f'(x_0), \quad (14)$$

$$h_{n-1}c_{n-1} + 2h_{n-1}c_n = 3f'(x_n) - \frac{3}{h_{n-1}}(a_n - a_{n-1}). \quad (15)$$

With the combination of (9), (14) and (15), as in the case of natural boundary conditions, the problem of computing the coefficients $x = [c_0, c_1, \ldots, c_n]^t$ is reduced to solving a linear system of equations $Ax = b$. The matrix $A$ is strictly diagonally dominant, and also tridiagonal. Hence, a unique solution exists, and can be efficiently computed.

Example. With $n = 5$,

$$A = \begin{bmatrix} 2h_0 & h_0 & 0 & 0 & 0 & 0 \\ h_0 & 2(h_0+h_1) & h_1 & 0 & 0 & 0 \\ 0 & h_1 & 2(h_1+h_2) & h_2 & 0 & 0 \\ 0 & 0 & h_2 & 2(h_2+h_3) & h_3 & 0 \\ 0 & 0 & 0 & h_3 & 2(h_3+h_4) & h_4 \\ 0 & 0 & 0 & 0 & h_4 & 2h_4 \end{bmatrix},$$

$$x = \begin{bmatrix} c_0 & c_1 & c_2 & c_3 & c_4 & c_5 \end{bmatrix}^t,$$

$$\begin{aligned} b \quad = \quad & \Big[ 3\frac{a_1 - a_0}{h_0} - 3f'(x_0),\ 3\,\frac{a_2 - a_1}{h_1} - 3\,\frac{a_1 - a_0}{h_0},\ 3\,\frac{a_3 - a_2}{h_2} - 3\,\frac{a_2 - a_1}{h_1}, \\ & 3\,\frac{a_4 - a_3}{h_3} - 3\,\frac{a_3 - a_2}{h_2},\ 3\,\frac{a_5 - a_4}{h_4} - 3\,\frac{a_4 - a_3}{h_3},\ 3f'(x_5) - 3\frac{a_5 - a_4}{h_4} \Big]^t. \end{aligned}$$

## Cubic Spline Interpolants: Error-Bound Formula

Let $f \in C^4[a,b]$ with $\max_{a \le x \le b} |f^{(4)}(x)| = M$. If $S$ is the unique clamped cubic spline interpolant to $f$ with respect to the nodes $a = x_0 < x_1 < \cdots < x_n = b$, then

$$\max_{a \le x \le b} |f(x) - S(x)| \le \frac{5M}{384} \max_{0 \le j \le n-1} (x_{j+1} - x_j)^4.$$

Remark. A fourth-order error bound also exists for the case of natural boundary conditions.

# Parametric Curves

The interpolating polynomials and splines can only be used to interpolate *functions*.

Given $\mathcal{S} = \{(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)\}$ with

$$x_0 < x_1 < \cdots < x_n :$$

- define a parameter $t$ on the interval $[t_0, t_n]$ with

$$t_0 < t_1 < \cdots < t_n;$$

- construct interpolating functions $x(t)$ and $y(t)$ separately:

$$x(t_i) = x_i, \quad y(t_i) = y_i \quad \text{for } 0 \leq i \leq n$$

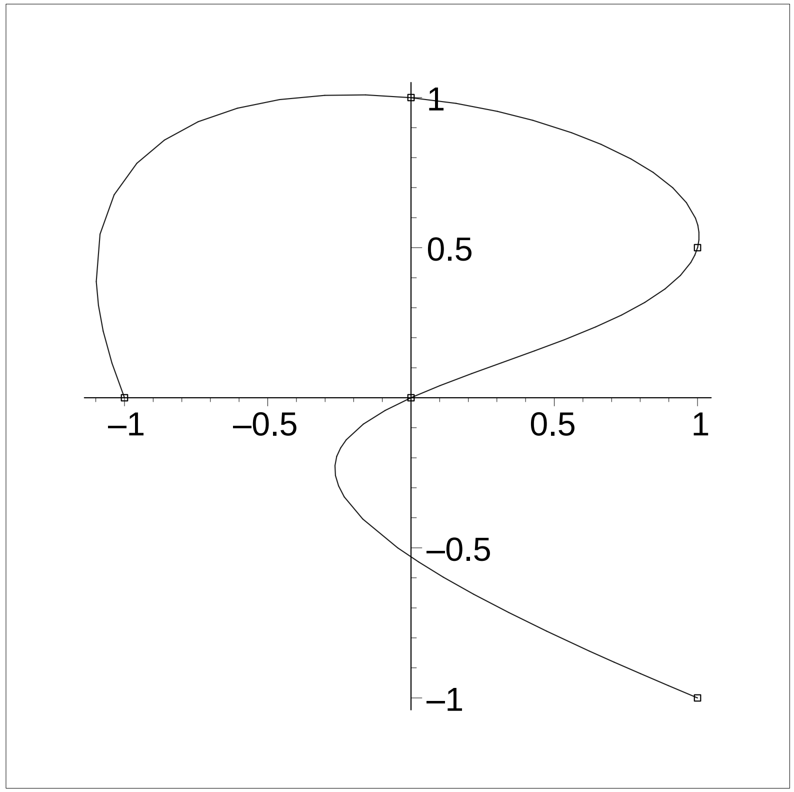
  using interpolating polynomials or splines.

# Parametric Curves: an Example

| $i$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $t_i$ | 0 | 0.25 | 0.5 | 0.75 | 1 |
| $x_i$ | -1 | 0 | 1 | 0 | 1 |
| $y_i$ | 0 | 1 | 0.5 | 0 | -1 |

- Using Lagrange interpolating polynomials for $x(t)$ and $y(t)$:

$$\begin{aligned} x(t) &= -\frac{32}{3}\,(t-1/4)\,(t-1/2)\,(t-3/4)\,(t-1) + 64\,t\,(t-1/4)\times \\ &\quad (t-3/4)\,(t-1) + \frac{32}{3}\,t\,(t-1/4)\,(t-1/2)\,(t-3/4)\,, \\ y(t) &= -\frac{128}{3}\,t\,(t-1/2)\,(t-3/4)\,(t-1) + 32\,t\,(t-1/4)\times \\ &\quad (t-3/4)\,(t-1) - \frac{32}{3}\,t\,(t-1/4)\,(t-1/2)\,(t-3/4)\,. \end{aligned}$$




Remark. Moving a single data point effects the entire curve. It is desirable, e.g., in computer graphics, that changing one portion of a curve should have little or no effect on other portions of the curve.

# Piecewise Cubic Hermite Polynomials

Let $x(t)$ and $y(t)$ be the parametric representation of the curve which interpolates the points $(x_i, y_i)$, $0 \leq i \leq n$. For each portion of the curve, the following 6 conditions hold:

$$x(t_i) = x_i,\ y(t_i) = y_i,\ x(t_{i+1}) = x_{i+1},\ y(t_{i+1}) = y_{i+1},$$
$$dy/dx|_{t=t_i} = y'(x_i),\ dy/dx|_{t=t_{i+1}} = y'(x_{i+1}).$$

However, each cubic polynomial $x(t)$ and $y(t)$ has 4 parameters for a total of 8.

Suppose that the endpoints are at $t = 0$ and $t = 1$. Then the following conditions on the quotients should hold:

$$\left.\frac{dy}{dx}\right|_{t=0} = \frac{y'(0)}{x'(0)}, \quad \left.\frac{dy}{dx}\right|_{t=1} = \frac{y'(1)}{x'(1)}. \qquad (16)$$

The actual values of $x'(0)$ and $y'(0)$ can be scaled by a common factor and still satisfy the first relation in (16). Similarly, the actual values of $x'(1)$ and $y'(1)$ can be scaled by a common factor and still satisfy the second relation in (16).

To simplify the process of specifying the slopes and to obtain a unique curve, commercial software commonly specifies a second point, called a *guidepoint*, which lies on the desired tangent line.

Let $(x_0, y_0)$ and $(x_1, y_1)$ be the endpoints, and $(x_0 + \alpha_0, y_0 + \beta_0)$ and $(x_1 - \alpha_1, y_1 - \beta_1)$ be the guidepoints. The cubic Hermite polynomial $x(t)$ satisfies

$$x(0) = x_0, \;\; x(1) = x_1, \;\; x'(0) = \alpha_0, \;\; x'(1) = \alpha_1. \qquad (17)$$

Similarly, the cubic Hermite polynomial $y(t)$ satisfies

$$y(0) = y_0, \;\; y(1) = y_1, \;\; y'(0) = \beta_0, \;\; y'(1) = \beta_1. \qquad (18)$$

Then the unique cubic polynomials $x(t)$ and $y(t)$ which satisfy the conditions in (17) and (18) respectively are

$$\begin{aligned} x(t) &= (2(x_0 - x_1) + (\alpha_0 + \alpha_1))t^3 + \\ &\quad (3(x_1 - x_0) - (\alpha_1 + 2\alpha_0))t^2 + \alpha_0 t + x_0, \\ y(t) &= (2(y_0 - y_1) + (\beta_0 + \beta_1))t^3 + \\ &\quad (3(y_1 - y_0) - (\beta_1 + 2\beta_0))t^2 + \beta_0 t + x_0. \end{aligned}$$