Abstract

Number theorists have been interested in the characteristics of numerical constants like π, e *and* $\log(2)$ *for centuries. These numbers, real irrationals, are composed of an unending string of digits in a specific but seemingly random order. As statistical methods and traditional analysis have revealed very little, it has been proposed that the natural visual capacities of human perception be employed to search for complex correlations in the numerical distributions.*

Keywords irrationality, continued fractions, zero/one polynomials, visualization

1 Introduction

> "I see a confused mass."
>
> Jacques Hadamard (1865-1963)

These are the words the great French mathematician used to describe his initial thoughts when he proved that there is a prime number greater than 11 [10]. His final mental image he described as "... a place somehere between the confused mass and the first point". In commenting on this in his fascinating but quirky monograph, he asks "What may be the use of such a strange and cloudy imagery?".

Hadamard was of the opinion that mathematical thought is visual and that words only interfered. And when he inquired into the thought processes of his most distinguished mid-century colleaugues, he discovered that most of them, in some measure, agreed (A notable exception being George Pólya).

For the non-professional, the idea that mathematicians "see" their ideas may be surprising. However the history of mathematics is marked by many notable developments grounded in the visual. Descartes' introduction of "cartesian" co–ordinates, for example, is arguably the most important advance in mathematics this millenium. It fundamentally reshaped the way mathematicians thought about mathematics. And precisely because it allowed them to "see" better mathematically.

Indeed, mathematicians have long been aware of the significance of visualization and made great effort to exploit it. Carl Friedrich Gauss lamented, in a letter to Heinrich Christian Schumacher, how hard it was to draw the pictures required for making accurate conjectures. Gauss, whom many consider the greatest mathematician of all time, wrote

> "It still remains true that, with negative theorems such as this, transforming personal convictions into objective ones requires deterringly detailed work. To visualize the whole variety of cases, one would have to display a large number of equations by curves; each curve would have to be drawn by its points, and determining a single point alone requires lengthy computations. You do not see from Fig. 4 in my first paper of 1799 , how much work was required for a proper drawing of that curve."
>
> Carl Friedrich Gauss (1777-1855)

The kind of pictures Gauss was looking for would now take seconds to generate on a computer screen.

Newer computational environments have greatly increased the scope for visualizing mathematics. Computer graphics offers magnitudes of improvement in resolution and speed over hand–drawn or mentally conceived images and provides increased utility through color, animation, image processing and user interactivity. And, to some degree, mathematics has evolved to exploit these new tools and techniques. We wish to explore some of the more subtle uses of interactive graphical tools which help us "see" the mathematics more clearly. In particular, we wish to focus on cases where the right picture suggests the "right theorem", or where it indicates structure where none was expected, or which offer the possibility of "visual proof".

For all the various examples we consider we have developed Internet accessible interfaces. They allow the reader interact and explore the mathematics and possibly even discover new results of their own[1].

2 In Pursuit of Patterns

> "Computers make it easier to do a lot of things, but most of the things they make it easier to do don't need to be done."
>
> Andy Rooney

Mathematics can be described as the "science of patterns", pursuing patterns, relationships, generalized descriptions and recognizable structure in space, numbers, and other abstracted entities. Lynn Steen has observed [17],

[1]Readers are encouraged to try out the various interfaces available at www.cecm.sfu.ca/projects/numbers/

> Mathematical theories explain the relations among patterns; functions and maps, operators and morphisms bind one type of pattern to another to yield lasting mathematical structures. Application of mathematics use these patterns to "explain" and predict natural phenomena that fit the patterns. Patterns suggest other patterns, often yielding patterns of patterns.
>
> *Science v240, 1988*

This description conjures up images of cycloids, Sierpinski gaskets, "cowboy hat" surfaces, and multicolored graphs. However it isn't immediately apparent that this patently visual reference to patterns applies to all of mathematics. Many of the higher order relationships in fields like number theory defy pictorial representation or, at least, they don't immediately lend themselves intuitively to a graphic treatment. Much of what is "pattern" in the knowledge of mathematics is instead encoded in a textual, sentential format born out of the logical formalist practices which currently dominate mathematics.

Within number theory, many problems offer large amounts of "data" which the human mind has difficulty assimilating directly. These include classes of numbers which satisfy certain criteria (eg. primes), distributions of digits in expansions, finite and infinite series and summations, solutions to variable expressions (eg. zeroes of polynomials) and other relatively unmanageable masses of raw information. Typically real insight into such problems has come directly from the mind of mathematician who ferrets out their essence from formalized representations rather than from the data. Now computers are making it possible to "enhance" the human perceptual/cognitive systems through visualization. Consequently, patterns of a new sort are beginning to appear in the morass of numbers.

However the epistemological role of computational visualization in mathematics is still not quite clear, certainly not any clearer than the role of intuition where mental visualization takes place. It can be seen to be fulfilling a number of particular functions in current day practice. These include inspiration and discovery, informal communication and demonstration, and teaching and learning. Lately though, forces in the area of experimental mathematics have been expanding its role to include exploration and experimentation and, prehaps more controversially, formal exposition and proof. Some carefully crafted questions have been posed about how experiment might contribute to mathematics [4]. Yet answers have been slow to come. This is in part due to the general resistance and, some cases, alarm [11] within the mathematical community and finds only conditional support from those who address the issue formally [6, 8].

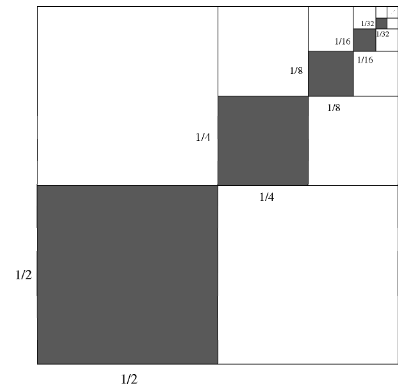

Figure 1: A simple "visual proof" of $\sum_{n=1}^{\infty}(\frac{1}{2})^{2n} = \frac{1}{3}$

The value of visualization hardly seems to be in question. The real issue seems to be what it can be used for. Can it contribute directly to the body of mathematical knowledge? Can an image act as a form of "visual proof"? There are a number of fine examples [6] (including in number theory), most of which are in the form of simplified, heuristic diagrams like Figure 1. They call into question what the epistemological criteria of an acceptable proof are. The full breadth of that issue is outside the scope of this paper. Rather it is suggested that three necessary, but prehaps not sufficient, conditions may be:

- *reliability* ; that the underlying means of arriving at the proof are reliable and that the result is unvarying with each inspection
- *consistency* ; that the means and end of the proof are consistent with other known facts, beliefs and proofs
- *repeatibility* ; that the proof may be confirmed by or demonstrated to others

Each requirement is difficult to satisfy in a single, static visual representation. Most criticisms of images as mathematical knowledge or tools make this clear[7, 13].

It is clear that the nature of traditional exposition differs significantly from that of the visual. In the logical formal mode, proof is provided in linearly connected sentences composed of words that are carefully selected to infer unambiguous meaning. Each sentence follows the previous, specifying an unalterable path through the sequence of statements. Although error and misconception are still possible, the tolerances are extremely demanding and follow the strict conventions of deductivist presentation [12].

In graphical representations, the same facts and relationships are often presented in multiple modes and dimensions. For example, the path through the information is usually indeterminate, leaving the viewer to establish what is important (and what is not) and in what order the dependencies should be assessed. Further, unintended information and relationships may be perceived, either due to the unanticipated interaction of the complex array of details or due to the viewer's own perceptual and cognitive processes.

As a consequence, successful visual representations tend to be spartan in their detail. And the few examples of visual proof which withstand close inspection are limited in their scope and generalizability. In the effort to bring images closer to conformity with the prevailing logical modes of proof, they have subsequently lost the richness which is intrinsic to the visual.

3 In Support of Proof

> "Computers are useless. They can only give you answers."
>
> Pablo Picasso (1881-1973)

In order to offer the reliability, consistency and repeatability of the written word and still provide the potential inherent in the medium, visualization needs to offer more than just the static image. It too must guide, define and relate the information presented. The logical formalist conventions for mathematics have evolved over many decades, resulting in a mode of discourse that is precise in its delivery. To wit, the order of presentation of ideas is critical with definitions preceding their usage, proofs separated from the general flow of the argument for modularity, and references to foundational material listed at the end.

To do the same, visualization must include additional mechanisms or conventions beyond the base image. It isn't appropriate to simply ape the logical conventions and find some visual metaphor or mapping that works similarly (this approach is what limits existing successful visual proofs to very simple diagrams). Instead, an effective visualization needs to offer several key features

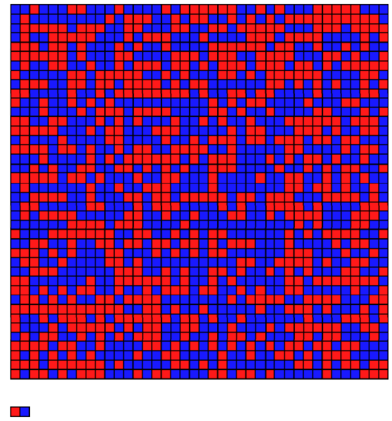

Figure 2: The first 1600 decimal digits of π mod 2.

- *dynamic* ; the representation should vary through some parameter(s) to demonstrate a range of behaviours (instead of the single instance of the static case)

- *guidance* ; to lead the viewer through the appropriate steps in the correct order, the representation should offer a "path" through the information which builds the case for the proof

- *flexibility* ; it should support the viewer's own exploration of the ideas presented, including the search for counter examples or incompleteness

- *openness* ; the underlying algorithms, libraries and details of the programming languages and hardware should be available for inspection and confirmation

With these capabilities available in an interactive representation, the viewer could then follow the argument being made visually, explore all the ramifications, check for counter examples, special cases and incompleteness, and even confirm the correctness of the implementation. In fact, the viewer will be able to perform all of the same inspections on the visual representation as he would be able to on a traditional logical formal proof.

While this does not yet offer any conclusion as to how images and computational tools might impact on mathematical methodologies or the underlying epistemology, it does indicate the direction that subsequent work may take. Examples from recent work done at the CECM offer some insight into how emerging technologies may eventually provide an unambiguous role for visualization in mathematics.

4 The Structure of Numbers

Numbers may be generated by a myriad of means and techniques. Each offers a very small piece of an infinitely large puzzle. Number theory looks for the relationship in and between numbers, tracing the invisible patterns which hint at an underlying fundamental structure. The concreteness of those features belies the seeming abstractness of their existence.

4.1 Binary Expansions

In the 17th century, Gottfried Wilhelm Leibniz[2] asked in a letter to one of the Bernoulli brothers if there might be a pattern in the binary expansion of π. Three hundred years later, the question remains. The numbers in the expansion appear to be completely random. In fact, the most that can be said of any of the classical mathematical constants is that they are largely non-periodic.

With traditional analysis revealing no patterns of interest, generating images from the expansions offers intriguing alternatives. Figures 2 and 3 show 1600 decimal digits of π and 22/7 respectively, both taken mod 2. The light pixels are the even digits and the dark one the odd. The digits read from left to right, top to bottom, like words in a book.

What does one see? The even and odd digits of π in Figure 2 seem to be distributed randomly, as one would expect. And the fact that 22/7 (the widely used approximation for π) is rational appears clearly in Figure 3. Visually representing randomness is not a new idea; Pickover [16] and Voelcker [18] have previously examined the possibility of "seeing randomness". Rather the intention here is to identify patterns where none has so far been seen, in this case in the expansions of irrational numbers.

These are only simple examples but many numbers have structures which are hidden both from simple inspection of the digits and even from standard statistical analysis. Figure 4 shows another rational number 1/65537, this time as a binary expansion, with a period of 65536. Unless graphically represented and with sufficient resolution, the presence of the period might otherwise be missed in then unending string of 0's and 1's.

Figures 5 a) and b) are based on similar calculations using 1600 terms of the simple continued fractions of π and e respectively. Continued fractions take the form

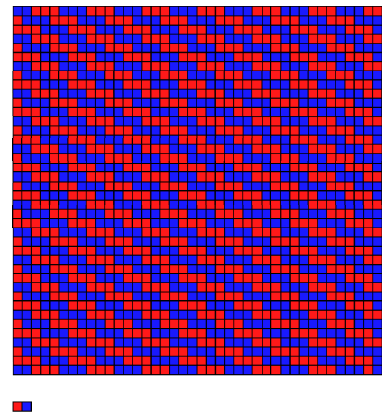

Figure 3: The first 1600 decimal digits of 22/7 mod 2.

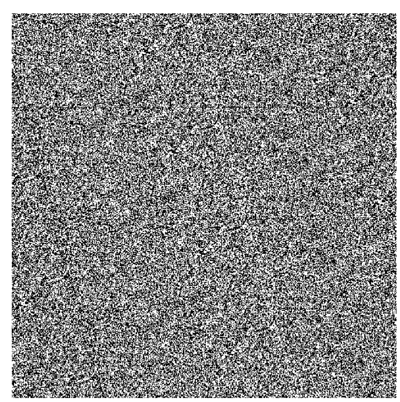

Figure 4: The first million binary digits of 1/65537 reveal the subtle diagonal structure from the periodicity.

[2] G. Leibniz was a German philosopher and mathematician best known for having developed the differential and integral calculus independently of Sir Isaac Newton

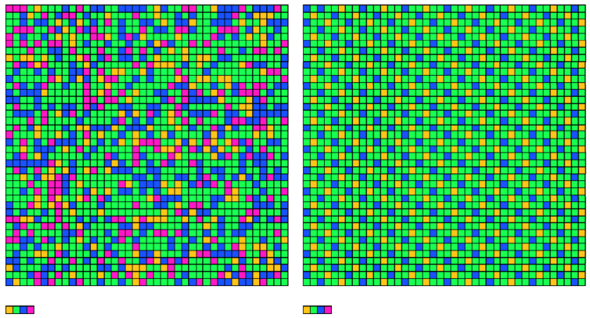

Figure 5: The first 1600 values of the continued fraction for a) π on the left and b) e, both mod 4

of

$$\frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}}$$

In these images, the decimal values have been taken mod 4[3]. Again the distribution of the a_i of π appear random – though now, as one would expect, there are more odds than evens. However for e, the pattern appears highly structured. This is no surprise on closer examination as the continued fraction for e is

$$[2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, \ldots]$$

and is a rational number mod 4. It is apparent from the images that the natures of the various distributions are quite distinct and recognizable. In contrast no such simple pattern exists for expansions like exp(3) mod 4 [4]

Presumably this representation of numbers offers a qualitative handle on their character. It tags them in an instantly distinguishable fashion which would be almost impossible to do otherwise.

4.2 Sequences of Polynomials

> "Few things are harder to put up with than the annoyance of a good example. "
>
> Mark Twain (1835-1910)

In a similar vein, structures are found in the coefficients of sequences of polynomials. The first example in Figure 6 shows the binomial coefficients $\binom{n}{m}$ mod 3, or equivalently Pascal's Triangle mod 3. For the sake of what follows, it is convenient to think of the ith row as the coefficients of the polynomial $(1+x)^i$

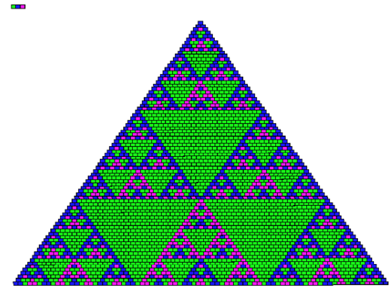

Figure 6: Eighty rows of Pascal's Triangle mod 3

taken modulo three. This apparently fractal pattern [5] has been the object of much careful study [9].

Figure 7 shows the coefficients of the first eighty Cebyshev polynomials mod 3 laid out like the binomial coefficients of Figure 6. Recall that the nth Cebyshev polynomial T_n is defined by $T_n(x) := \cos(n \arccos x)$. They have the explicit representation :

$$T_n(x) = \frac{n}{2} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{(n-k-1)!}{k!(n-2k)!} (2x)^{n-2k},$$

and satisfy the recursion

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)\,, \quad n = 2, 3, \ldots.$$

Note that the expression for $T_n(x)$ resembles the $\binom{n}{m}$ form of the binomial coefficients and its recursion relation is similar to that for the Pascal's Triangle.

Figure 8 shows the Stirling numbers of the second kind mod 3 organized as a triangle as well. Recall that Stirling numbers of the second kind are defined by

$$S(n,m) := \frac{1}{m!} \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} k^n$$

and give number of ways of partitioning a set of n elements into m non-empty subsets. Once again the form of $\binom{n}{m}$ appears in its expression.

While the forms for each of the polynomials are relatively well-known, it is apparent that they are graphically related to each other (and distinguishable from each other). Each is a variant on the binomial coefficients.

It is possible to find similar sorts of structure in virtually any sequence of polynomials: Legendre polynomials; Euler polynomials; sequences of Padé denominators to the exponential or to $(1-x)^\alpha$ with α rational. Then, selecting any moduli, a distinct pattern

[3] Modulo or "mod" is the remainder after division by the specified value

[4] These images can be generated by "the Colour Calculator" using the function **see_cf**.

[5] The well-known Sierpinski gasket is recovered by taking Pascal's Triangle mod 2.

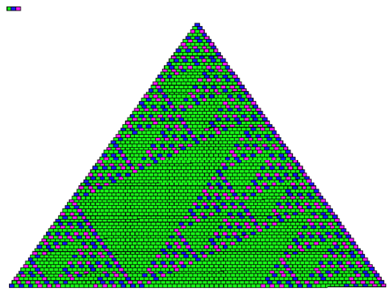

Figure 7: Eighty Cebychev Polynomials mod 3

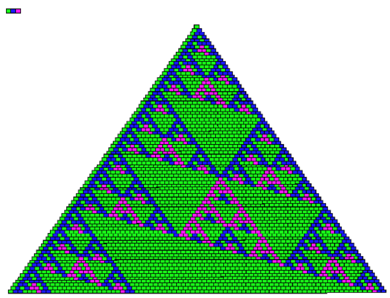

Figure 8: Eighty rows of Stirling Numbers of the second kind mod 3

will emerge. These intriguing images hint at an underlying structure within the polynomials themselves and demand some explanation. While conjectures exist for their origin, an incontrovertible proof for the theorems suggested by these pictures is not yet in hand. And when there finally is a proof, might it be offered in some visual form?

4.3 Quasi-Rationals

> "For every problem, there is one solution which is simple, neat and wrong."
>
> H.L. Mencken (1880-1956)

Having established a visual character for irrationals and their expansions, it is interesting to note the existence of "quasi-rational" numbers. These are certain well-known irrational numbers which generate images appearing suspiciously rational. The sequences pictured in Figures 9 and 10 are $\{i\pi\}_{i=1}^{1600}$ mod 2 and $\{ie\}_{i=1}^{1600}$ mod 2, respectively. One way of thinking about these sequences is as binary expansions of the numbers

$$\sum_{n=1}^{\infty} \frac{[m\alpha] \bmod 2}{2^i}$$

where α is, respectively, π and e.

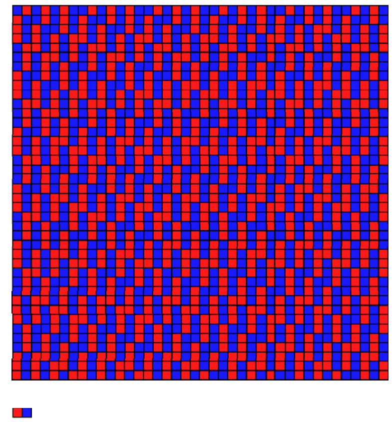

Figure 9: Integer part of $\{i\pi\}_{i=1}^{1600}$ mod 2; note the slight irregularities in the pseudo-periodic pattern.

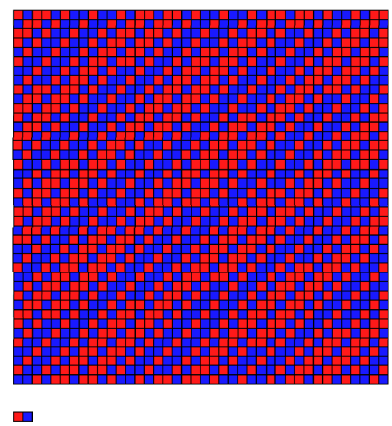

Figure 10: Integer part of $\{ie\}_{i=1}^{1600}$ mod 2; note the slight irregularities in the pseudo-periodic pattern.

The resulting images are very regular. And yet these are transcendental numbers; having observed this phenomenon, we were subsequently able to prove this rigourously from the study of

$$\sum_{n=1}^{\infty} \frac{[m\alpha]}{2^i}$$

which is transcendental for all irrational α. This follows from the remarkable continued fraction expansion of Böhmer [3]

$$\sum_{n=1}^{\infty} [m\alpha] z^n = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(1-z^{q_n})(1-z^{q_{n+1}})}$$

Here (q_n) is the sequence of denominators in the simple continued fraction expansion of α.

Careful examination of Figures 9 and 10 show that they are only *pseudo*-periodic; slight irregularities appear in the pattern. This rational–like behaviour follows from the very good rational approximations provided by this expansion. Or put another way, there are very large terms in the continued fraction expansion. For example, the expansion of

$$\sum_{n=1}^{\infty} \frac{[m\pi]mod2}{2^i}$$

is

$$[0, 1, 2, 42, 638816050508714029100700827905, 1, 126, \ldots]$$

with a similar phenomenon for e.

This behaviour makes it clear that there is subtlety in the nature of these numbers. Indeed, while were able to rigourously establish the results shown above, many related phenomena exist whose proofs are not yet in hand. For example, a similar result is not yet available for

$$\sum_{n=1}^{\infty} \frac{[m\pi]mod2}{3^i}$$

A proof for these graphic results might well offer further refinements to their representations, leading to yet another critical graphic characterization [6].

4.4 Complex Zeros

Polynomials with constrained coefficients have been much studied [2, 15, 5]. They relate to the Littlewood conjecture and many other problems. Littlewood notes [14] that "these raise fascinating questions".

Certain of these polynomials demonstrate suprising complexity when their zeros are appropriately plotted. Figure 11 shows the complex zeros of all polynomials

$$P_n(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

of degree $n \leq 18$ where $a_i = \{-1, +1\}$ as they appear on the complex plane. This image, reminiscent of pictures for polynomials with all coefficients in the set $\{0, +1\}$ [15], does raise many questions:

> Is the set fractal and what is its boundary? Are there holes at infinite degree? How do the holes vary with the degree? What is the relationship between these zeros and those of polnomials with real coefficients in the neighbourhood of $\{-1, +1\}$?

[6] The pictures from this section can be generated using the "Colour Calculator" and the function **see_seq**.

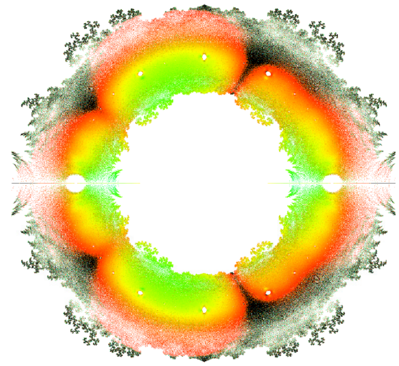

Figure 11: Roots of Littlewood Polynomials of degree at most 18 for coefficients ± 1.

Some, but definitely not all, of these questions have found some analytic answer [15, 5]. Others have been shown to relate subtly to standing problems of some significance in number theory. For example, the nature of the holes involves a old problem known as Lehmer's conjecture [1]. It is not yet clear how these images will contribute to a solution to such problems. However visualization has provided definitive description the polynomials, pointing a remarkable finger at previously unseen depths [7].

5 Conclusion

Visualization extends the natural capacity of the mathematician to intuit his subject, to see the entities and objects which are part of his work with the aid of software and hardware. Since the graphics representations are firmly rooted in verifiable algorithms and machines, the images and interfaces may also provide new forms of exposition and possibly even proof. Most important of all, like spacecraft, diving bells and electron microscopes, it takes the human mind to places it has never been and shows it images from a realm yet unseen.

Acknowledgements

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[7] A tool to make these type of pictures is available at www.cecm.sfu.ca/cgi-bin/organics/polyform.

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