Computations with Mahler's Measure

— or —

Mahler's Symphony in C++

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June 2002

Definitions.

- Write $f(x) = \sum_{k=0}^{d} a_k\, x^k = a_d \prod_{k=1}^{d}(x - \beta_k)$.

- **Mahler's measure**

$$\begin{aligned} M(f) &= |a_d| \prod_{k=1}^{d} \max\{1, |\beta_k|\} \\ &= \exp\left(\int_0^1 \log \left| f\left(e^{2\pi i t}\right) \right| \, dt \right) \\ &= \lim_{p \to 0^+} \|f\|_p\,. \end{aligned}$$

- **Height**: $H(f) = \max_{0 \le k \le d} |a_k|$.

- **Length**: $L(f) = \sum_{k=0}^{d} |a_k|$.

Properties.

- $M(f(x)) = M(\pm f(\pm x)) = M(f(x^n)) = M(f^*(x))$, where $f^*(x) = x^d f(1/x)$ is the **reciprocal** of $f(x)$.

- $M(fg) = M(f)M(g)$.

- $M(f) \le \|f\|_2 \le \|f\|_\infty \le L(f) \le 2^d M(f)$.

- $|a_k| \le \binom{d}{k} M(f)$, so only finitely many $f(x) \in \mathbf{Z}[x]$ with $\deg(f) \le d$ and $M(f) \le M$.

- Kronecker: $f(x) \in \mathbf{Z}[x]$, irreducible, $M(f) = 1$, and $f(x) \ne x$ implies $f(x)$ is cyclotomic.

History.

3. Mahler 1960.

   - Introduced term to give simple proofs of some inequalities on polynomials.

2. Lehmer 1933.

   - Method for finding large prime numbers.

   - Small measures (> 1) yield more effective method.

   - Found $\ell(x) = x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1$, $M(\ell) = 1.1762808182599\ldots$.

   - Question: Given $\epsilon > 0$, does there exist $f \in \mathbf{Z}[x]$ with $1 < M(f) < 1 + \epsilon$?

   - Conjecture: No.

1. Reidemeister 1932.

   - $\ell(-x)$ appears as Alexander polynomial of knot.

Special Lower Bounds.

1. Nonreciprocal Polynomials.

   Smyth (1971): $f \in \mathbf{Z}[x]$ nonreciprocal, $f(0)f(1) \neq 0$ then

   $$M(f) \geq M(x^3 - x - 1) = 1.3247179572\ldots.$$

2. Totally Real Polynomials.

   Schinzel (1973): If $f \in \mathbf{Z}[x]$ is monic, totally real, $f(\pm 1) \neq 0$, and $|f(0)| = 1$ then

   $$M(f) \geq \left(\frac{1+\sqrt{5}}{2}\right)^{d/2}.$$

3. Certain Alexander Polynomials.

   Hironaka (2000): $M(\ell)$ is the minimal measure among Alexander polynomials of "pretzel links".

General Lower Bounds.

$$M(f) > 1 + c\left(\frac{\log\log d}{\log d}\right)^3$$

- Dobrowolski (1979): $c = 1 - \epsilon$, $d > d_0(\epsilon)$.
- Cantor and Straus (1982), Rausch (1985): $c = 2 - \epsilon$.
- Louboutin (1983): $c = 9/4 - \epsilon$.
- Voutier (1996): $c = 1/4$, $d \geq 2$.

Searching for Small Measures.

1. Select a family of reciprocal polynomials.
   - Exhaustive.
   - Height 1.
   - Perturbed cyclotomic products.
   - Fewnomials.
   - Littlewood polynomials.
2. Algorithm to process polynomials quickly.
   - Screen.
   - Normalize.
   - Compute measure.

Polynomial Screening Algorithm.

1. Graeffe Root-Squaring.

- Given f, write $f(x) = e(x^2) + xd(x^2)$.
  $G_2(f) := e(x)^2 - xd(x)^2$.

  Roots of $G_2(f)$ are squares of roots of f.

- Given f_0, define $\{f_m\}$ via $f_{m+1} = G_2(f_m)$.

- $\lim_{m\to\infty} \sqrt[2^m]{L(f_m)} = M(f)$.

- Let $a_{k,m}$ = coefficient of x^k in f_m.

- Boyd: If $M(f) \le M$ then
  $$|a_{k,m}| \le \binom{d}{k} + \binom{d-2}{k-1}(M^{2^m} + M^{-2^m} - 2).$$

- Reject candidate f if this is violated for f_m.

- Compute up to $m = 10$ or 12.

- Use exact arithmetic (NTL).

```
void Polynomial::Graeffe() {
    register int s=1, t, j, k;
    for (k=0; k<=d; k++) w[k] = coef[k];
    for (k=0; k<=d; k++) {
        coef[k] *= w[k];
        if (s < 0) negate(coef[k], coef[k]);
        t = -s;
        for (j=1; j<=imin(k,d-k); j++) {
            u = w[k-j];
            u *= w[k+j];
            u <<= 1;
            if (t < 0) negate(u, u);
            coef[k] += u;
            t = -t;
        }
        s = -s;
    }
}
```

Example: $M = 1.3$.

- Let $f(x) = x^8 - x^7 + x^6 + x^5 + x^3 + x^2 - x + 1$ ($M(f) \approx 1.771$).

- $f_0(x) = (x^8 + x^6 + x^2 + 1) + x(-x^6 + x^4 + x^2 - 1)$, $e(x) = x^4 + x^3 + x + 1, \quad d(x) = -x^3 + x^2 + x - 1$.

- $f_1(x) = (x^4 + x^3 + x + 1)^2 - x(-x^3 + x^2 + x - 1)^2$ $= x^8 + x^7 + 3x^6 + 3x^5 + 3x^3 + 3x^2 + x + 1$.

- $f_2(x) = x^8 + 5x^7 + 3x^6 - 9x^5 - 9x^3 + 3x^2 + 5x + 1$.

- $f_3(x) = x^8 - 19x^7 + 99x^6 + 15x^5 - 192x^4 + 15x^3 + 99x^2 - 19x + 1$.

- $a_{1,3} = -19$, but $|a_{1,3}| \leq 14$ ($M = 1.3$). Reject f.

Remark: Root-cubing (Graefffe Algorithm?).

$$f(x) = a(x^3) + xb(x^3) + x^2c(x^3),$$

$$G_3(f) = a(x)^3 + xb(x)^3 + x^2c(x)^3 - 3xa(x)b(x)c(x).$$

2. Normalize.

- Remove cyclotomic factors.
  - Graeffe detects cyclotomics, clumps cyclotomic factors.
- Avoid $f(x^k)$, $f(-x)$, $f^*(x)$ if $f(x)$ known.
- Store in binary search tree.

3. Compute Measure.

- Bairstow's method.
- PARI, Maple.

Families of Polynomials.

1. Exhaustive.

- Finitely many integer polynomials with bounded degree and measure.
- $O\left(c^{d^2}\right)$.
- Boyd (1989): $d \leq 20$.
- M. (1995): $d \leq 24$ (10^{10} polynomials tested).
- Qiang & Rhin: $d \leq 40$ (?)

2. Height 1.

- If $f(x) \in \mathbf{Z}[x]$ is irreducible and $M(f) < 2$ then there exists $g(x) \in \mathbf{Z}[x]$ so that $H(fg) = 1$.
  - *Excursions* Ch. 3, ex. E8 (box principle).
  - Bombieri & Vaaler: $\deg(g) \ll d \log d / \log(2/M(f))$.
- Hope $g(x)$ exists with $M(g) = 1$ and $\deg(g)$ small.
- $O(c^d)$.
- Boyd (1989), M. (1995): $d \leq 40$ ($5 \cdot 10^9$ polys).
- Finds everything from exhaustive search.

3. Perturbed cyclotomic products.

- Pinner: $\ell(x) = \Phi_1^2(x)\Phi_2^2(x)\Phi_3^2(x)\Phi_6(x) - x^5$. Similar rep. for many other small measures.

- Given d, form all products of cyclotomic polynomials of degree d. Perturb middle coefficient(s) of each.

- $O(c^{\sqrt{d}})$, $c = \sqrt{70\zeta(3)}/\pi \approx 2.92$.

- M., Pinner, Vaaler (1998): $d \leq 64$ ($7 \cdot 10^8$ polys).

- Finds all known polynomials with $M(f) < 1.23$, and 80% with $M(f) < 1.3$.

- 241 different representations of $\ell(x^k)$ found.

Measure	Polynomial
1.176280 ...	$\Phi_1^2(x)\Phi_2^2(x)\Phi_3^2(x)\Phi_6(x) - x^5$
1.188368 ...	$\Phi_1^2(x)\Phi_2^2(x)\Phi_3^2(x)\Phi_4(x)\Phi_6(x)\Phi_9(x) + x^9$
1.200026 ...	$\Phi_1^2(x)\Phi_2^2(x)\Phi_4(x)\Phi_6(x)\Phi_7(x) + x^7$
1.201396 ...	$(\Phi_1^2(x)\Phi_5^2(x)\Phi_7(x)\Phi_{10}(x) + x^{10})/\Phi_6(x)$
1.202616 ...	$\Phi_1^2(x)\Phi_2^2(x)\Phi_3(x)\Phi_4(x)\Phi_6(x)\Phi_{12}(x) + x^7$
1.205019 ...	$(\Phi_2^2(x)\Phi_{10}(x)\Phi_{16}(x)\Phi_{26}(x) - x^{13})/\Phi_{12}(x)$
1.207950 ...	$\Phi_1^2(x)\Phi_2^2(x)\Phi_3(x)\Phi_4(x)\Phi_6(x)\Phi_7(x)\Phi_9(x)\Phi_{18}(x) + x^{14}$
1.212824 ...	$(\Phi_1^2(x)\Phi_3(x)\Phi_8(x)\Phi_9(x)\Phi_{13}(x) + x^{13})/\Phi_{14}(x)$
1.214995 ...	$\Phi_1^2(x)\Phi_2^2(x)\Phi_3(x)\Phi_5^2(x)\Phi_6(x)\Phi_{10}(x) + x^{10}$
1.216391 ...	$\Phi_1^2(x)\Phi_2^2(x)\Phi_3(x)\Phi_4(x)\Phi_6(x) + x^5$
1.218396 ...	$(\Phi_1^2(x)\Phi_3^2(x)\Phi_4(x)\Phi_6(x)\Phi_7(x)\Phi_{12}^2(x) + x^{12})/\Phi_{10}(x)$
1.218855 ...	$\Phi_1^2(x)\Phi_2^2(x)\Phi_3(x)\Phi_4(x)\Phi_6(x)\Phi_7(x)\Phi_{10}(x)\Phi_{12}(x) + x^{12}$
1.219057 ...	$(\Phi_1^2(x)\Phi_3(x)\Phi_4(x)\Phi_5(x)\Phi_{44}(x) + x^{15})/\Phi_{14}(x)$
1.219446 ...	$\Phi_1^2(x)\Phi_2^2(x)\Phi_3^2(x)\Phi_4^2(x)\Phi_6(x)\Phi_{12}(x) + x^9$
1.219720 ...	$\Phi_1^2(x)\Phi_2^2(x)\Phi_3(x)\Phi_4(x)\Phi_8(x)\Phi_9(x) + x^9$
1.220287 ...	$(\Phi_1^2(x)\Phi_2^2(x)\Phi_7(x)\Phi_{10}(x)\Phi_{14}(x)\Phi_{19}(x) + x^{19})/\Phi_{12}(x)$
1.223447 ...	$\Phi_1^2(x)\Phi_2^2(x)\Phi_3(x)\Phi_4(x)\Phi_5(x)\Phi_6(x)\Phi_{10}(x)\Phi_{20}(x)\Phi_{42}(x) - x^{19}$
1.223777 ...	$(\Phi_1^2(x)\Phi_2^2(x)\Phi_4(x)\Phi_6(x)\Phi_8(x)\Phi_{17}(x)\Phi_{18}(x) + x^{17})/\Phi_{15}(x)$
1.224278 ...	$\Phi_2^2(x)\Phi_6(x)\Phi_{18}^2(x) + x^8$
1.225503 ...	$(\Phi_5(x)\Phi_{48}(x) + x^{10})/\Phi_4(x)$
1.225619 ...	$\Phi_2^2(x)\Phi_4(x)\Phi_6(x)\Phi_{10}(x)\Phi_{26}(x)\Phi_{30}(x) - x^{15}$
1.225810 ...	$(\Phi_1^2(x)\Phi_2^2(x)\Phi_5(x)\Phi_{10}(x)\Phi_{14}(x)\Phi_{17}(x) + x^{17})/\Phi_{12}(x)$
1.226092 ...	$\Phi_2^2(x)\Phi_4(x)\Phi_6(x)\Phi_{20}(x)\Phi_{26}(x) - x^{13}$
1.226493 ...	$\Phi_1^2(x)\Phi_2^2(x)\Phi_3(x)\Phi_6(x)\Phi_9(x)\Phi_{17}(x)\Phi_{18}(x) + x^{18}$
1.226993 ...	$\Phi_1^2(x)\Phi_2^2(x)\Phi_3^2(x)\Phi_4(x)\Phi_6(x)\Phi_{12}^2(x) + x^{10}$
1.227785 ...	$\Phi_1^2(x)\Phi_2^2(x)\Phi_3^2(x)\Phi_4(x)\Phi_6(x) + x^6$
1.228140 ...	$(\Phi_1^2(x)\Phi_4(x)\Phi_5(x)\Phi_{12}(x)\Phi_{13}(x)\Phi_{36}(x) - x^{18})/\Phi_{14}(x)$
1.229482 ...	$(\Phi_1^2(x)\Phi_2^2(x)\Phi_6(x)\Phi_{11}(x)\Phi_{13}(x)\Phi_{18}(x)\Phi_{22}(x) + x^{22})/\Phi_{15}(x)$
1.229566 ...	$(\Phi_1^2(x)\Phi_5(x)\Phi_7(x)\Phi_{36}(x) - x^{12})/\Phi_6(x)$
1.229999 ...	$\Phi_1^2(x)\Phi_2^2(x)\Phi_3(x)\Phi_6(x)\Phi_{17}(x)\Phi_{22}(x) + x^{17}$

4. Fewnomials.

- Height 1 with fixed number of terms, n.
- Running time: $O(d^{\lceil n/2 \rceil})$.
- Finds all known polynomials with $M(f) < 1.3$.
- M. (1998). $n \leq 7$: $d \leq 181$; $n \leq 9$: $d \leq 131$, etc. ($8 \cdot 10^8$ polynomials).
- Lisonek (2000): A few more with $174 \leq d \leq 180$.

5. Littlewood polynomials.

- Height 1, with no missing terms.
- Minimal nonreciprocal measure: Golden ratio.
- M. (1999): Exhaustive $d \leq 31$; Reciprocal $d \leq 58$.

Measure	d	ν	Half of Coefficients
1.49671107561	19	1	++++−+−+−−
1.50613567955	11	1	++−−−+
1.50646000575	35	2	++−−−−−−+++−−+−−−+
1.53691794778	23	2	++++−−−−+−+−
1.55107223951	23	2	+++++++−−++−
1.55603019132	6	1	++−−
1.58234718368	7	1	++−+
1.58501169305	35	5	++++++−−−−−−−+−+−+
1.59341317381	19	3	++++−++++−
1.59504631132	53	5	+−+−+−+−+−+−+−+−+−−−−−−−−−−
1.59700500917	17	1	++−++−++−
1.59918220880	41	4	+−+−+−+−+−+−+−+++++++

Multivariable Mahler Measures.

- $\log M(f(x,y)) = \displaystyle\int_0^1 \int_0^1 \log \left| f\left(e^{2\pi i s}, e^{2\pi i t}\right)\right| \, ds\, dt.$

- Boyd, Lawton: $\lim_{n\to\infty} M(f(x,x^n)) = M(f(x,y))$.

- Four $f(x,y)$ known with $M(f) < 1.3247\ldots$.

- Specialization generates most known $f(x)$ with small measure.

- Smallest known limit of two-variable measures: $M(1+x+y) = 1.38135\ldots$.

- Boyd conjecture (1981):
  The set $L = \bigcup_{n\geq 1}\{M(f) : f \in \mathbf{Z}[x_1,\ldots,x_n]\}$ is closed. (Implies Lehmer's conjecture.)

- Four methods for searching for $f(x,y)$ with small $M(f)$.

1. Patterns in Coefficients of One-Variable Measures.

- Most small limit points found this way. Example:

$$x^{28} + x^{20} + x^{17} - x^{16} + x^{15} + x^{8} + 1,$$

$$x^{48} + x^{35} + x^{27} - x^{26} + x^{25} + x^{13} + 1,$$

$$x^{60} + x^{44} + x^{33} - x^{32} + x^{31} + x^{16} + 1.$$

Suggests

$$x^{4n} + x^{3n-1} + x^{2n+3} - x^{2n+2} + x^{2n+1} + x^{n+1} + 1.$$

Substituting y for x^n yields

$$xy^4 + y^3 + x^4y^2 - x^3y^2 + x^2y^3 + x^2y + 1,$$

which has measure $1.309098\ldots$.

2. Sparse Multiples of Sporadic Polynomials.

- $f(x) = x^{44} - x^{42} + x^{35} - x^{33} + x^{31} - x^{29} + x^{26} - x^{24} + x^{22} - x^{20} + x^{18} - x^{15} + x^{13} - x^{11} + x^9 - x^2 + 1$ has $M(f) = 1.291273\ldots$.

- Let $g_0(x) = x^m$, $g_k(x) = x^{m+k} + x^{m-k}$, $1 \leq k \leq m$.

- Use LLL on lattice spanned by (half) coefficients of $f(x)g_k(x)$.

- Detects sparse multiple

  $x^{52} + x^{51} + x^{39} + x^{38} + x^{26} + x^{14} + x^{13} + x + 1,$

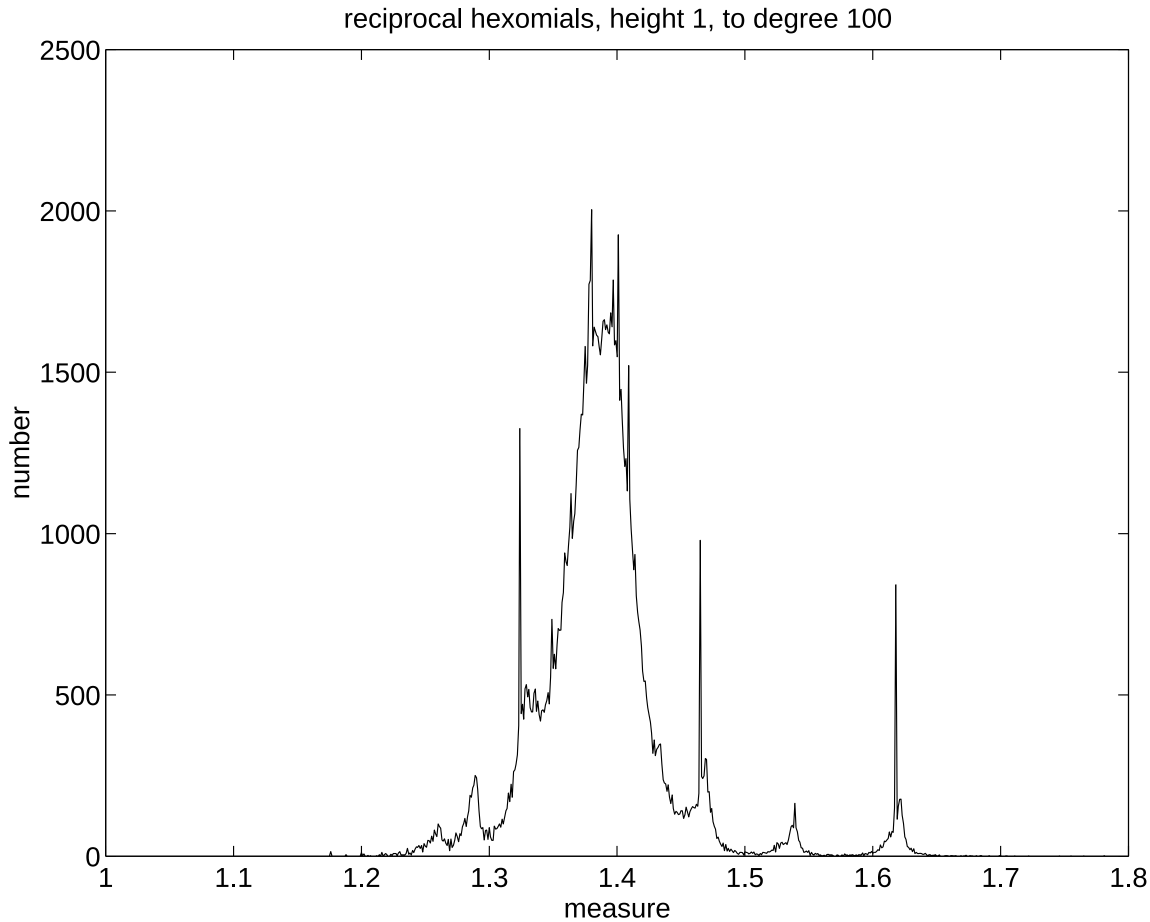
  suggesting

  $xy^4 + y^4 + xy^3 + y^3 + xy^2 + x^2y + xy + x^2 + x,$

  which has measure $1.332051\ldots$.

3. Clustering of Measures of Classes of Fewnomials.

- D. Boyd (1995).



4. Systematic Searches.

- Symmetric, reciprocal $f(x, y)$ with $H(f) = 1$ and $\deg_x(f) \leq 6$.

- Reciprocal $f(x, y)$ with $H(f) = 1$, $\deg_x(f) \leq 9$, and $\deg_y(f) = 2$.

- $f(x, y) = (1 - x^a) + x^r y(1 - x^b) + x^s y^2(1 - x^a)$.

- $f(x, y) = 1 + y(x^{a+b} \pm x^a \pm x^b + 1) + x^{a+b} y^2$.

1.	1.2554338662666087457	(2, 3)
2.	1.2857348642919862749	(1, 3) or (2, 1)
3.	1.3090983806523284595	++000, +0−0+, 000++
4.	1.3156927029866410935	(3, 5)
5.	1.3247179572447460260	++0−, −0++
6.	1.3253724973075860349	(3, 4)
7.	1.3320511054374193142	(2, 5)
8.	1.3323961294587154121	+0000, ++0++, 0000+
9.	1.3381374319388410775	(3, 2)
10.	1.3399999217381835332	(4, 7)
11.	1.3405068829308471079	(3, 1)
12.	1.3497161046696958653	+++0−−, −−0+++
13.	1.3500148321630142650	(3, 7)
14.	1.3503169790598690950	+00000, +−00−+, 00000+
15.	1.3511458956697046903	(4, 5)
16.	1.3524680625188602961	(5, 9)
17.	1.3536976494626355711	++00000, +00000+, 00000++
18.	1.3567481051456008311	(4, 3)
19.	1.3567859884526454967	(5, 8)
20.	1.3581296324044179208	++00000, +0−−−0+, 00000++
21.	1.3585455903960511404	(4, 1)
22.	1.3592080686995589268	(4, 9)
23.	1.3598117752819405021	(6, 11)
24.	1.3598158989877492950	+0000000, ++0000++, 0000000+
25.	1.3599141493821189216	+0−+0−+, +−0+−0+
26.	1.3602208408592842371	(5, 7)
27.	1.3627242816569882815	(5, 6)
28.	1.3636514981864992177	+00000000, +00+0+00+, 00000000+
29.	1.3641995455827723418	+0−00+, +00−0+
30.	1.3644358117806362770	+000, 00++, ++00, 000+
31.	1.364545987	(7, 13)
32.	1.364655729	(5, 11)

Processing Two-Variable Polynomials.

1. Sieving.

   - Select θ and $n_1, \ldots, n_m$.
   - Keep f if mean of $M(f(x, x^{n_i})) > \theta$.
   - Repeat.

2. Computing Measures ($\deg_y(f)$ small).

   - Let $\beta_i(t)$ denote roots in y of $f(e(t), y) = 0$ for $0 \le t \le 1 \quad (e(t) = \exp(2\pi i t))$.
   - Jensen's formula:
     $$\log M(f(x,y)) = 2 \sum_i \int_{\substack{|\beta_i(t)|>1 \\ 0 \le t \le 1/2}} \log |\beta_i(t)| \, dt.$$
   - Automate with Maple.

Example.

- $f(x,y) = (1+x+x^2) + y(1+x+x^2+x^3)$

  $+xy^2(1+x+x^2).$

- $f(e(t),y) = e(t)s(t) + 2ye(3t/2)r(t) + y^2e(2t)s(t)$;

  $r(t) = \cos 3\pi t + \cos \pi t, \quad s(t) = 2\cos 2\pi t + 1.$

- Roots of discriminant: $t_1 \approx .301$, $t_2 \approx .388$.

- $\displaystyle\int_{t_1}^{t_2} \log\left(\sqrt{r(t)^2 - s(t)^2} - r(t)\right) dt \approx -.0106005.$

- $\displaystyle\int_{t_1}^{t_2} \log|s(t)|\ dt = (t_2 - z)\log|s(t_2)| + (z - t_1)\log|s(t_1)| + \int_{t_1}^{t_2} (z-t)\frac{s'(t)}{s(t)}\, dt \approx -.151447. \quad (z = 1/3).$

- $M(f(x,y)) \approx \exp(2(.151447 - .0106005)) = 1.32537.$

Large Measures.

- Maximize $M(f)/\left\|f\right\|_2$.
- Mahler (1962): For $f \in \mathbf{C}[x]$, take $|a_k| = 1$ for all k.
- Integer case: Littlewood polynomials.
- Related to open problems of Erdős, Littlewood regarding locally flat polynomials with ± 1 coefficients.
- Largest known normalized measure.
  - $f(x) = x^{12} + x^{11} + x^{10} + x^9 + x^8 - x^7 - x^6 + x^5 + x^4 - x^3 + x^2 - x + 1.$
  - Barker polynomial.
  - $M(f)/\left\|f\right\|_2 \approx .986.$
- Current research (Granville, M.):
  Large asymptotic normalized measures.