1

Introduction and Basic Properties

Overview

The most basic and important theorem concerning polynomials is the Fundamental Theorem of Algebra. This theorem, which tells us that every polynomial factors completely over the complex numbers, is the starting point for this book. Some of the intricate relationships between the location of the zeros of a polynomial and its coefficients are explored in Section 2. The equally intricate relationships between the zeros of a polynomial and the zeros of its derivative or integral are the subject of Section 1.3. This chapter serves as a general introduction to the body of theory known as the geometry of polynomials. Highlights of this chapter include the Fundamental Theorem of Algebra, the Eneström-Kakeya theorem, Lucas' theorem, and Walsh's two-circle theorem.

1.1 Polynomials and Rational Functions

The focus for this book is the polynomial of a single variable. This is an extended notion of the polynomial, as we will see later, but the most important examples are the algebraic and trigonometric polynomials, which we now define. The complex $(n+1)$-dimensional vector space of algebraic polynomials of degree at most n with complex coefficients is denoted by $\mathcal{P}_n^c$.

If $\mathbb{C}$ denotes the set of complex numbers, then

$$\mathcal{P}_n^c := \left\{ p : p(z) = \sum_{k=0}^{n} a_k z^k, \quad a_k \in \mathbb{C} \right\} . \tag{1.1.1}$$

When we restrict our attention to polynomials with real coefficients we will use the notation

$$\mathcal{P}_n := \left\{ p : p(z) = \sum_{k=0}^{n} a_k z^k, \quad a_k \in \mathbb{R} \right\} , \tag{1.1.2}$$

where $\mathbb{R}$ is the set of real numbers. Rational functions of type (m, n) with complex coefficients are then defined by

$$\mathcal{R}_{m,n}^c := \left\{ \frac{p}{q} : p \in \mathcal{P}_m^c, q \in \mathcal{P}_n^c \right\} , \tag{1.1.3}$$

while their real cousins are denoted by

$$\mathcal{R}_{m,n} := \left\{ \frac{p}{q} : p \in \mathcal{P}_m, q \in \mathcal{P}_n \right\} . \tag{1.1.4}$$

The distinction between the real and complex cases is particularly important for rational functions (see E.4).

The set of trigonometric polynomials $\mathcal{T}_n^c$ is defined by

$$\mathcal{T}_n^c := \left\{ t : t(\theta) := \sum_{k=-n}^{n} a_k e^{ik\theta}, \quad a_k \in \mathbb{C} \right\} . \tag{1.1.5}$$

A real trigonometric polynomial of degree at most n is an element of $\mathcal{T}_n^c$ taking only real values on the real line. We denote by $\mathcal{T}_n$ the set of all real trigonometric polynomials of degree at most n. Other characterizations of $\mathcal{T}_n$ are given in E.9. Note that if $z := e^{i\theta}$, then an arbitrary element of $\mathcal{T}_n^c$ is of the form

$$z^{-n} \sum_{k=0}^{2n} b_k z^k, \qquad b_k \in \mathbb{C} \tag{1.1.6}$$

and so many properties of trigonometric polynomials reduce to the study of algebraic polynomials of twice the degree on the unit circle in $\mathbb{C}$.

The most basic theorem of this book, and arguably the most basic nonelementary theorem of mathematics, is the Fundamental Theorem of Algebra. It says that a polynomial of exact degree n (that is, an element

of $\mathcal{P}_n^c \backslash \mathcal{P}_{n-1}^c$) has exactly n complex zeros counted according to their multiplicities.

Theorem 1.1.1 (Fundamental Theorem of Algebra). *If*

$$p(z) := \sum_{i=0}^{n} a_i z^i, \qquad a_i \in \mathbb{C}, \;\; a_n \neq 0\,,$$

then there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$ *such that*

$$p(z) = a_n \prod_{i=1}^{n} (z - \alpha_i)\,.$$

Here the multiplicity of the zero at α_i is the number of times it is repeated. So, for example,

$$(z-1)^3(z+i)^2$$

is a polynomial of degree 5 with a zero of multiplicity 3 at 1 and with a zero of multiplicity 2 at $-i$. The polynomial

$$p(z) := \sum_{i=0}^{n} a_i z^i\,, \qquad a_i \in \mathbb{C}, \;\; a_n \neq 0$$

is called *monic* if its *leading coefficient* a_n equals 1. There are many proofs of the Fundamental Theorem of Algebra based on elementary properties of complex functions (see Theorem 1.2.1 and E.4 of Section 1.2). We will explore this theorem more substantially in the next section of this chapter.

Comments, Exercises, and Examples.

The importance of the solution of polynomial equations in the history of mathematics is hard to overestimate. The Greeks of the classical period understood quadratic equations (at least when both roots were positive) but could not solve cubics. The explicit solutions of the cubic and quartic equations in the sixteenth century were due to Niccolo Tartaglia (ca 1500–1557), Ludovico Ferrari (1522–1565), and Scipione del Ferro (ca 1465–1526) and were popularized by the publication in 1545 of the "Ars Magna" of Girolamo Cardano (1501–1576). The exact priorities are not entirely clear, but del Ferro probably has the strongest claim on the solution of the cubic. These discoveries gave western mathematics an enormous boost in part because they represented one of the first really major improvements on Greek mathematics. The impossibility of finding the zeros of a polynomial of degree at least 5, in general, by a formula containing additions, subtractions, multiplications, divisions, and radicals would await Niels Henrik Abel (1802–1829) and his 1824 publication of "On the Algebraic Resolution of Equations." Indeed, so much algebra, including Galois theory, analysis, and particularly complex analysis, is born out of these ideas that it is hard to imagine how the flow of mathematics might have proceeded without these issues being raised. For further history, see Boyer [68].

E.1 Explicit Solutions.

a] Quadratic Equations. Verify that the quadratic polynomial x^2+bx+c has zeros at

$$\frac{-b-\sqrt{b^2-4c}}{2}\,, \qquad \frac{-b+\sqrt{b^2-4c}}{2}\,.$$

b] Cubic Equations. Verify that the cubic polynomial x^3+bx+c has zeros at

$$\alpha+\beta\,, \quad -\left(\frac{\alpha+\beta}{2}\right)+i\sqrt{3}\left(\frac{\alpha-\beta}{2}\right), \quad -\left(\frac{\alpha+\beta}{2}\right)-i\sqrt{3}\left(\frac{\alpha-\beta}{2}\right),$$

where

$$\alpha=\sqrt[3]{\frac{-c}{2}+\sqrt{\frac{c^2}{4}+\frac{b^3}{27}}}$$

and

$$\beta=\sqrt[3]{\frac{-c}{2}-\sqrt{\frac{c^2}{4}+\frac{b^3}{27}}}\,.$$

c] Show that an arbitrary cubic polynomial, x^3+ax^2+bx+c, can be transformed into a cubic polynomial as in part b] by a transformation $x\mapsto ex+f$.

d] Observe that if the polynomial x^3+bx+c has three distinct real zeros, then α and β are necessarily nonreal and hence $4b^3+27c^2$ is negative. So, in this simplest of cases one is forced to deal with complex numbers (which was a serious technical problem in the sixteenth century).

e] Quartic Equations. The quartic polynomial $x^4+ax^3+bx^2+cx+d$ has zeros at

$$-\frac{a}{4}+\frac{R}{2}\pm\frac{\alpha}{2}\,, \qquad -\frac{a}{4}+\frac{R}{2}\pm\frac{\beta}{2}\,,$$

where

$$R=\sqrt{\frac{a^2}{4}-b+y}\,,$$

y is any root of the resolvent cubic

$$y^3-by^2+(ac+4d)y-a^2d+4bd-c^2\,,$$

and

$$\alpha,\beta=\sqrt{\frac{3a^2}{4}-R^2-2b\pm\frac{4ab-8c-a^3}{4R}}\,, \qquad R\neq 0\,,$$

while

$$\alpha,\beta=\sqrt{\frac{3a^2}{4}-2b\pm 2\sqrt{y^2-4d}}\,, \qquad R=0\,.$$

These unwieldy equations are quite useful in conjunction with any symbolic manipulation package.

E.2 Newton's Identities. Write

$$(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n) = x^n - c_1 x^{n-1} + c_2 x^{n-2} - \cdots + (-1)^n c_n\,.$$

The coefficients c_k are, by definition, the *elementary symmetric functions* in the variables $\alpha_1, \ldots, \alpha_n$.

a] For positive integers k, let

$$s_k := \alpha_1^k + \alpha_2^k + \cdots + \alpha_n^k\,.$$

Prove that

$$s_k = (-1)^{k+1} k c_k + (-1)^k \sum_{j=1}^{k-1} (-1)^j c_{k-j} s_j\,, \qquad k \le n$$

and

$$s_k = (-1)^{k+1} \sum_{j=k-n}^{k-1} (-1)^j c_{k-j} s_j\,, \qquad k > n\,.$$

Here, and in what follows, an empty sum is understood to be 0.

A *polynomial of n variables* is a function that is a polynomial in each of its variables. A *symmetric polynomial of n variables* is a polynomial of n variables that is invariant under any permutation of the variables.

b] Show by induction that any symmetric polynomial in n variables (with integer coefficients) may be written uniquely as a polynomial (with integer coefficients) in the elementary symmetric functions $f_1, f_2, \ldots, f_n$.

Hint: For a symmetric polynomial f in n variables, let

$$\sigma(f) := (\nu_1, \nu_2, \ldots, \nu_n)\,, \qquad \nu_1 \ge \nu_2 \ge \cdots \ge \nu_n \ge 0$$

if

$$f(x_1, x_2, \ldots, x_n) = \sum_{\alpha_1=0}^{\nu_1} \sum_{\alpha_2=0}^{\nu_2} \cdots \sum_{\alpha_n=0}^{\nu_n} c_{\alpha_1,\alpha_2,\ldots,\alpha_n} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

and $c_{\nu_1,\nu_2,\ldots,\nu_n} \neq 0$. If

$$\sigma(f) = (\nu_1, \nu_2, \ldots, \nu_n) \quad \text{and} \quad \sigma(g) = (\widetilde{\nu}_1, \widetilde{\nu}_2, \ldots, \widetilde{\nu}_n)\,,$$

then let $\sigma(f) < \sigma(g)$ if $\nu_j \le \widetilde{\nu}_j$ for each j with a strict inequality for at least one index. This gives a (partial) well ordering of symmetric polynomials in n variables, that is, every set of symmetric polynomials in n variables has a minimal element. Now use induction on $\sigma(f)$. □

c] Show that

$$\left(\frac{1+\sqrt{5}}{2}\right)^k \to 0 \pmod 1 .$$

(By convergence to zero (mod 1) we mean that the quantity approaches integral values.)

Hint: Consider the integers

$$s_k := \alpha_1^k + \alpha_2^k ,$$

where $\alpha_1 := \frac{1}{2}(1+\sqrt{5})$ and $\alpha_2 := \frac{1}{2}(1-\sqrt{5})$. □

d] Find another algebraic integer α with the property that

$$\alpha^k \to 0 \pmod 1 .$$

Such numbers are called *Salem numbers* (see Salem [63]). It is an open problem whether any nonalgebraic numbers $\alpha > 1$ satisfy $\alpha^k \to 0 \pmod 1$.

E.3 Norms on $\mathcal{P}_n$. $\mathcal{P}_n$ is a vector space of dimension $n+1$ over $\mathbb{R}$. Hence $\mathcal{P}_n$ equipped with any norm is isomorphic to the Euclidean vector space $\mathbb{R}^{n+1}$, and these norms are equivalent to each other. Similarly, $\mathcal{P}_n^c$ is a vector space of dimension $n+1$ over $\mathbb{C}$. Hence $\mathcal{P}_n^c$ equipped with any norm is isomorphic to the Euclidean vector space $\mathbb{C}^{n+1}$, so these norms are also equivalent to each other. Let

$$p_n(x) := \sum_{k=0}^{n} a_k x^k, \qquad a_k \in \mathbb{R} .$$

Some common norms on $\mathcal{P}_n$ and $\mathcal{P}_n^c$ are

$$\begin{aligned}
\|p\|_A &:= \sup_{x\in A} |p(x)| && \textit{supremum norm}\\
&:= \|p\|_{L_\infty(A)} && L_\infty \textit{ norm}\\
\|p\|_{L_p(A)} &:= \left(\int_A |p(t)|^p\, dt\right)^{1/p} && L_p \textit{ norm, } p \ge 1\\
\|p\|_{l_\infty} &:= \max_k\{|a_k|\} && l_\infty \textit{ norm}\\
\|p\|_{l_p} &:= \left(\sum_{k=0}^{n} |a_k|^p\right)^{1/p} && l_p \textit{ norm, } p \ge 1 .
\end{aligned}$$

In the first case A must contain $n+1$ distinct points. In the second case A must have positive measure.

a] Conclude that there exist constants C_1, C_2, and C_3 depending only on n so that

$$\|p'\|_{[-1,1]} \le C_1 \|p\|_{[-1,1]} ,$$

$$\sum_{i=0}^{n} |a_i| \le C_2 \|p\|_{[-1,1]} ,$$

$$\|p\|_{[-1,1]} \le C_3 \|p\|_{L_2[-1,1]}$$

for every $p \in \mathcal{P}_n^c$, and, in particular, for every $p \in \mathcal{P}_n$.

These inequalities will be revisited in detail in later chapters, where precise estimates are given in terms of n.

b] Show that there exist extremal polynomials for each of the above inequalities. That is, for example,

$$\sup_{0 \not\equiv p \in \mathcal{P}_n} \frac{\|p'\|_{[-1,1]}}{\|p\|_{[-1,1]}}$$

is achieved.

E.4 On $\mathcal{R}_{n,m}$.

a] $\mathcal{R}_{n,m}$ is not a vector space because it is not closed under addition.

b] **Partial Fraction Decomposition.** Let $r_{n,m} \in \mathcal{R}_{n,m}^c$ be of the form

$$\frac{p(x)}{\prod_{k=1}^{m'} (x - \alpha_k)^{m_k}} , \qquad p \in \mathcal{P}_n^c , \quad \alpha_k \text{ distinct} , \quad p(\alpha_k) \neq 0 .$$

Then there is a unique representation of the form

$$r_{n,m}(x) = q(x) + \sum_{k=1}^{m'} \sum_{j=1}^{m_k} \frac{a_{k,j}}{(x - \alpha_k)^j} , \qquad q \in \mathcal{P}_{n-m}^c , \quad a_{k,j} \in \mathbb{C}$$

(if $m > n$, then $\mathcal{P}_{n-m}^c$ is meant to be $\{0\}$).

Hint: Consider the type and dimension of expressions of the above form. □

c] Show that if

$$r_{n,m} \in \mathcal{R}_{n,m}^c ,$$

then

$$\mathrm{Re}(r_{n,m}(\cdot)) \in \mathcal{R}_{n+m,2m} .$$

This is an important observation because in some problems a rational function in $\mathcal{R}_{n,n}^c$ can behave more like an element of $\mathcal{R}_{2n,2n}$ than $\mathcal{R}_{n,n}$.

E.5 Horner's Rule.

a] We have

$$\sum_{i=0}^{n} a_i x^i = (\cdots((a_n x + a_{n-1})x + a_{n-2})x + \cdots + a_1)x + a_0 \,.$$

So every polynomial of degree n can be evaluated by using at most n additions and n multiplications. (The converse is clearly not true; consider x^{2^n}.)

b] Show that every rational function of type $(n-1, n)$ can be put in a form so that it can be evaluated by using n divisions and n additions.

E.6 Lagrange Interpolation. Let z_i and y_i be arbitrary complex numbers except that the z_i must be distinct ($z_i \neq z_j$, for $i \neq j$). Let

$$l_k(z) := \frac{\prod_{i=0, i\neq k}^{n}(z - z_i)}{\prod_{i=0, i\neq k}^{n}(z_k - z_i)}\,, \qquad k = 0, 1, \ldots, n\,.$$

a] Show that there exists a unique $p \in \mathcal{P}_n^c$ that takes $n+1$ specified values at $n+1$ specified points, that is,

$$p(z_i) = y_i\,, \qquad i = 0, 1, \ldots, n\,.$$

This $p \in \mathcal{P}_n^c$ is of the form

$$p(z) = \sum_{k=0}^{n} y_k l_k(z)$$

and is called the *Lagrange interpolation polynomial.*

If all the z_i and y_i are real, then this unique interpolation polynomial is in $\mathcal{P}_n$.

b] Let

$$\omega(z) := \prod_{i=0}^{n} (z - z_i)\,.$$

Show that l_k is of the form

$$l_k(z) = \frac{\omega(z)}{(z - z_k)\omega'(z_k)}$$

and

$$p(z) = \sum_{k=0}^{n} \frac{y_k \omega(z)}{(z - z_k)\omega'(z_k)}\,.$$

c] **An Error Estimate.** Assume that the points $z_i \in [a,b]$, $i = 0,1,\ldots,n$, are distinct and $f \in C^{n+1}[a,b]$ (that is, f is an $n+1$ times continuously differentiable real-valued function on $[a,b]$). Let $p \in \mathcal{P}_n$ be the Lagrange interpolation polynomial satisfying

$$p(z_i) = f(z_i)\,, \qquad i = 0,1,\ldots,n\,.$$

Show that for every $x \in [a,b]$ there is a point $\xi \in (a,b)$ so that

$$f(x) - p(x) = \frac{1}{(n+1)!}\, f^{(n+1)}(\xi)\, \omega(x)\,.$$

Hence

$$\|f - p\|_{[a,b]} \le \frac{1}{(n+1)!}\, \|f^{(n+1)}\|_{[a,b]}\, \|\omega\|_{[a,b]}\,.$$

Hint: Choose λ so that $\varphi := f - p - \lambda w$ vanishes at x, that is,

$$\lambda := (f(x) - p(x))/\omega(x)\,.$$

Then repeated applications of Rolle's theorem yield that

$$\varphi^{(n+1)} = f^{(n+1)} - \lambda(n+1)!$$

has a zero ξ in (a,b). □

E.7 Hermite Interpolation.

a] Let $z_i \in \mathbb{C}$, $i = 1,2,\ldots k$, be distinct. Let m_i, $i = 1,2,\ldots,k$, be positive integers with $n+1 := \sum_{i=1}^{k} m_i$, and let

$$y_{i,j} \in \mathbb{C}, \quad i = 1,2,\ldots,k\,, \quad j = 0,1,\ldots,m_i - 1$$

be fixed. Show that there is a unique $p \in \mathcal{P}_n^c$, called the *Hermite interpolation polynomial*, so that

$$p^{(j)}(z_i) = y_{i,j}\,, \quad i = 1,2,\ldots,k\,, \quad j = 0,1,\ldots,m_i - 1\,.$$

If all the z_i and $y_{i,j}$ are real, then this unique interpolation polynomial is in $\mathcal{P}_{n-1}$.

Hint: Use induction on n. □

b] Assume that the points $z_i \in [a,b]$ are distinct and $f \in C^n[a,b]$. Let $p \in \mathcal{P}_{n-1}$ be the Hermite interpolation polynomial satisfying

$$p(z_i) = f^{(j)}(z_i)\,, \quad i = 1,2,\ldots,k\,, \quad j = 0,1,\ldots,m_i - 1\,.$$

Show that for every $x \in [a,b]$ there is a point $\xi \in (a,b)$ so that

$$f(x) - p(x) = \frac{1}{n!}\, f^{(n)}(\xi)\, \omega(x)$$

with

$$\omega(x) := \prod_{i=1}^{k} (x - x_i)^{m_i - 1}\,.$$

Hence

$$\|f - p\|_{[a,b]} \le \frac{1}{n!}\, \|f^{(n)}\|_{[a,b]}\, \|\omega\|_{[a,b]}\,.$$

Hint: Follow the hint given for E.6 c]. □

Polynomial interpolation and related topics are studied thoroughly in Davis [75]; Lorentz, Jetter, and Riemenschneider [83]; and Szabados and Vértesi [92].

E.8 On the Zeros of a $p \in \mathcal{P}_n$. Show that if $p \in \mathcal{P}_n$, then the nonreal zeros of p form conjugate pairs (that is, if z is a zero of p, then so is $\overline{z}$).

E.9 Factorization of Trigonometric Polynomials.

a] Show that $t \in \mathcal{T}_n$ (or $t \in \mathcal{T}_n^c$) if and only if t is of the form

$$t(z) = a_0 + \sum_{k=1}^{n} (a_k \cos kz + b_k \sin kz)\,, \qquad a_k, b_k \in \mathbb{R} \text{ (or } \mathbb{C})\,.$$

b] Show that if $t \in \mathcal{T}_n \backslash \mathcal{T}_{n-1}$, then there are numbers $z_1, z_2, \ldots, z_{2n}$ and $0 \ne c \in \mathbb{C}$ such that

$$t(z) = c \prod_{j=1}^{2n} \sin \frac{z - z_j}{2}\,.$$

Show also that the nonreal zeros z_j of t form conjugate pairs.

E.10 Newton Interpolation and Integer-Valued Polynomials. Let $\Delta^k f(x)$ be defined inductively by

$$\Delta^0 f(x) := f(x)\,, \quad \Delta f(x) = \Delta^1 f(x) := f(x+1) - f(x)$$

and

$$\Delta^{k+1} f(x) := \Delta(\Delta^k f(x))\,, \qquad k = 1, 2, \ldots\,.$$

Let

$$\binom{x}{k} := \frac{x(x-1)\cdots(x-k+1)}{k!}\,.$$

a] Show that $\binom{x}{k}$ is a polynomial of degree k that takes integer values at all integers.

b] Let f be an m times differentiable function on $[a, a+m]$. Show that there is a $\xi \in (a, a+m)$ such that

$$\Delta^m f(a) = f^{(m)}(\xi)\,.$$

c] Show that if $p \in \mathcal{P}_n^c$, then

$$p(x) = \sum_{k=0}^{n} \Delta^k p(0) \binom{x}{k}\,.$$

d] Suppose $p \in \mathcal{P}_n^c$ is integer-valued at all integers. Show that

$$p(x) = \sum_{k=0}^{n} a_k \binom{x}{k}$$

for some integers $a_0, a_1, \ldots, a_n$. Note that this characterizes such polynomials.

e] Show that if $p \in \mathcal{P}_n^c$ takes integer values at $n+1$ consecutive integers, then p takes integer values at every integer.

f] Suppose $c \in \mathbb{R}$ and n^c is an integer for every $n \in \mathbb{N}$. Use part b] to show that c is a nonnegative integer.

1.2 The Fundamental Theorem of Algebra

The following theorem is a quantitative version of the Fundamental Theorem of Algebra due to Cauchy [1829]. We offer a proof that does not assume the Fundamental Theorem of Algebra, but does require some elementary complex analysis.

Theorem 1.2.1. *The polynomial*

$$p(z) := a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 \in \mathcal{P}_n^c\,, \qquad a_n \neq 0$$

has exactly n zeros. These all lie in the open disk of radius r centered at the origin, where

$$r := 1 + \max_{0 \le k \le n-1} \frac{|a_k|}{|a_n|}\,.$$

Proof. We may suppose that $a_0 \neq 0$, or we may first divide by z^k for some k. Now observe that

$$g(x) := |a_0| + |a_1|x + \cdots + |a_{n-1}|x^{n-1} - |a_n|x^n$$

satisfies $g(0) > 0$ and $\lim_{x\to\infty} g(x) = -\infty$. So by the intermediate value theorem, g has a zero in $(0,\infty)$ (which is, on considering $(g(x)/x^n)'$, in fact unique). Let s be this zero. Then for $|z| > s$,

$$|p(z) - a_n z^n| \le |a_0| + |a_1 z| + \cdots + |a_{n-1} z^{n-1}| < |a_n z^n|\,. \tag{1.2.1}$$

This, by Rouché's theorem (see E.1), shows that $p(z)$ and $a_n z^n$ have exactly the same number of zeros, namely, n, in any disk of radius greater than s.

It remains to observe that if $x \ge r$, then $g(x) < 0$ so $s < r$. Indeed,

$$\begin{aligned} g(x) &\le |a_n|x^n \left(-1 + \left(\max_{k=0,\dots,n-1} \frac{|a_k|}{|a_n|} \right) \sum_{k=0}^{n-1} x^{k-n} \right) \\ &< |a_n|x^n \left(-1 + \left(\max_{k=0,\dots,n-1} \frac{|a_k|}{|a_n|} \right) \frac{1}{x-1} \right) \\ &\le 0 \end{aligned}$$

for

$$x \ge 1 + \max_{k=0,\dots,n-1} \frac{|a_k|}{|a_n|}\,.$$

□

The exact relationship between the coefficients of a polynomial and the location of its zeros is very complicated. Of course, the more information we have about the coefficients, the better the results we can hope for. The following pretty theorem emphasizes this:

Theorem 1.2.2 (Eneström-Kakeya). *If*

$$p(z) := a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$$

with

$$a_0 \ge a_1 \ge \cdots \ge a_n > 0\,,$$

then all the zeros of p lie outside the open unit disk.

Proof. Consider

$$(1-z)p(z) = a_0 + (a_1 - a_0)z + \cdots + (a_n - a_{n-1})z^n - a_n z^{n+1}\,.$$

Then

$$|(1-z)p(z)| \ge a_0 - [(a_0 - a_1)|z| + \cdots + (a_{n-1} - a_n)|z|^n + a_n|z|^{n+1}]\,.$$

Since $a_k - a_{k+1} \ge 0$, the right-hand expression above decreases as $|z|$ increases. Thus, for $|z| < 1$,

$$|(1-z)p(z)| > a_0 - [(a_0 - a_1) + \cdots + (a_{n-1} - a_n) + a_n] = 0\,,$$

and the result follows. □

Corollary 1.2.3. *Suppose*

$$p(z) := a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$$

with $a_k > 0$ for each k. Then all the zeros of p lie in the annulus

$$r_1 := \min_{k=0,\ldots,n-1} \frac{a_k}{a_{k+1}} \le |z| \le \max_{k=0,\ldots,n-1} \frac{a_k}{a_{k+1}} =: r_2 \,.$$

Proof. Apply Theorem 1.2.2 to $p(r_1 z)$ and $z^n p(r_2/z)$. □

This is a theme with many variations, some of which are explored in the exercises.

Theorem 1.2.4. *Suppose $p > 1$, $q > 1$, and $p^{-1} + q^{-1} = 1$. Then the polynomial $h \in \mathcal{P}_n^c$ of the form*

$$h(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 \,, \qquad a_n \neq 0$$

has all its zeros in the disk $\{z \in \mathbb{C} : |z| \le r\}$, where

$$r := \left\{ 1 + \left(\sum_{j=0}^{n-1} \frac{|a_j|^p}{|a_n|^p} \right)^{q/p} \right\}^{1/q} .$$

Proof. See E.6. □

Comments, Exercises, and Examples.

The Fundamental Theorem of Algebra appears to have been given its name by Gauss, although the result was familiar long before; it resisted rigorous proof by d'Alembert (1740), Euler (1749), and Lagrange (1772). It was more commonly formulated as a real theorem, namely: *every real polynomial factors completely into real linear or quadratic factors.* (This is an essential result for the integration of rational functions.) Girard has a claim to priority of formulation. In his "Invention Nouvelle en L'Algèbra" of 1629 he wrote "every equation of degree n has as many solutions as the exponent of the highest term." Gauss gave the first satisfactory proof in 1799 in his doctoral dissertation, and he gave three more proofs during his lifetime. His first proof, while titled "A new proof that every rational integral function of one variable can be resolved into real factors of the first or second degree," was in fact the first more-or-less satisfactory proof. Gauss' first proof is a geometric argument that the real and imaginary parts of a polynomial, u and v, have the property that the curves $u = 0$ and $v = 0$ intersect, and by modern standards has some topological problems. His third proof of 1816 amounts to showing that

$$\int_{|z|=r} \frac{p'(z)}{p(z)}\,dz$$

must vanish if p has no roots, which leads to a contradiction and is a genuinely analytic proof (see Boyer [68], Burton [85], and Gauss [1866]).

An almost purely algebraic proof using Galois theory, but based on ideas of Legendre, may be found in Stewart [73].

The "geometry of polynomials" is extensively studied in Marden [66] and Walsh [50], where most of the results of the section and much more may be accessed. See also Barbeau [89] and Pólya and Szegő [76].

Theorem 1.2.2 is due to Kakeya [12]. It is a special case of Corollary 1.2.3, due to Eneström [1893]. The Eneström-Kakeya theorem and related matters are studied thoroughly in Anderson, Saff, and Varga [79] and [81] and in Varga and Wu Wen-da [85], and a number of interesting properties are explored. For example, it is shown in the first of the above papers that the zeros of all p satisfying the assumption of Corollary 1.2.3 are dense in the annulus $\{z \in \mathbb{C} : r_1 \le |z| \le r_2\}$.

E.1 Basic Theorems in Complex Analysis. We collect a few of the basic theorems of complex analysis that we need. (Proofs may be found in any complex variables text such as Ahlfors [53] or Ash [71].)

a] Cauchy's Integral Formula. Let $D_r := \{z \in \mathbb{C} : |z| < r\}$. Suppose f is analytic on D_r and continuous on the closure $\overline{D}_r$ of D_r. Let ∂D_r denote the boundary of D_r. Then

$$0 = \int_{\partial D_r} f(t)\,dt\ ,$$

$$f(z) = \frac{1}{2\pi i}\int_{\partial D_r} \frac{f(t)}{t-z}\,dt\,, \qquad z \in D_r\,,$$

and

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{\partial D_r} \frac{f(t)}{(t-z)^{n+1}}\,dt\,, \qquad z \in D_r\,.$$

Unless otherwise specified, integration on a simple closed curve is taken anticlockwise. (We may replace ∂D_r and D_r by any simple closed curve and its interior, respectively, though for most of our applications circles suffice.)

b] Rouché's Theorem. *Suppose f and g are analytic inside and on a simple closed path γ (for most purposes we may use γ a circle). If*

$$|f(z) - g(z)| < |f(z)|$$

for every $z \in \gamma$, then f and g have the same number of zeros inside γ (counting multiplicities).

A function analytic on $\mathbb{C}$ is called *entire*.

c] **Liouville's Theorem.** *A bounded entire function is constant.*

d] **Maximum Principle.** *An analytic function on an open set $U \subset \mathbb{C}$ assumes its maximum modulus on the boundary. Moreover, if f is analytic and takes at least two distinct values on an open connected set $U \subset \mathbb{C}$, then*

$$|f(z)| < \sup_{z \in U} |f(z)|\,, \qquad z \in U\,.$$

e] **Unicity Theorem.** *Suppose f and g are analytic on an open connected set U. Suppose f and g agree on S, where S is an infinite compact subset of U, then f and g agree everywhere on U.*

E.2 Division.

a] Suppose p is a polynomial of degree n and $p(\alpha) = 0$. Then there exists a polynomial q of degree $n-1$ such that

$$p(x) = (x - \alpha)q(x)\,.$$

Hint: Consider the usual division algorithm for polynomials. □

b] A polynomial of degree n has at most n roots.

This is the easier part of the Fundamental Theorem of Algebra. The remaining content is that every nonconstant polynomial has at least one complex root.

The next exercise develops the basic complex analysis tools mostly for polynomials on circles. The point of this exercise is to note that the proofs in this case are particularly straightforward.

E.3 Polynomial Complex Analysis.

a] Deduce Cauchy's integral formula for polynomials on circles.

Hint: Integrate z^n on ∂D_r. □

b] If $p(z) = a_n \prod_{i=1}^{n}(z - \alpha_i)$, then the number of indices i for which $|\alpha_i| < r$ is

$$\frac{1}{2\pi i}\int_{\partial D_r} \frac{p'(z)}{p(z)}\,dz\,,$$

provided no α_i lies on ∂D_r.

Hint: We have

$$\frac{p'(z)}{p(z)} = \sum_{i=1}^{n} \frac{1}{z - \alpha_i}$$

and

$$\frac{1}{2\pi i}\int_{\partial D_r} \frac{dz}{z - \alpha_i} = \begin{cases} 1 & \text{if} \quad |\alpha_i| < r \\ 0 & \text{if} \quad |\alpha_i| > r\,. \end{cases}$$

□

c] Deduce Rouché's theorem from part b] for polynomials f and g given by their factorizations, and for circles γ.

Hint: Let $h := 1 + (g - f)/f$. So $fh = g$ and

$$\frac{1}{2\pi i}\int_{\partial D_r} \frac{g'(z)}{g(z)}\,dz = \frac{1}{2\pi i}\int_{\partial D_r} \frac{f'(z)}{f(z)}\,dz + \frac{1}{2\pi i}\int_{\partial D_r} \frac{h'(z)}{h(z)}\,dz\,.$$

Show that the last integral is zero by expanding h^{-1} and applying b]. □

d] Deduce E.1 c] and E.1 d] from E.1 a].

e] Observe that the unicity theorem can be sharpened for polynomials as follows. If $p, q \in \mathcal{P}_n^c$ and $p(z) = q(z)$ for $n+1$ distinct values of $z \in \mathbb{C}$, then p and q are identical, that is, $p(z) = q(z)$ for every $z \in \mathbb{C}$. Equivalently, a polynomial $p \in \mathcal{P}_n^c$ is either identically 0 or has at most n zeros. (This is trivial from the Fundamental Theorem of Algebra, but as in E.2, it does not require it.)

E.4 The Fundamental Theorem of Algebra. *Every nonconstant polynomial has at least one complex zero.*

Prove this directly from Liouville's theorem.

E.5 Pellet's Theorem. *Suppose* $a_p \neq 0$, $|a_{p+1}| + \cdots + |a_n| > 0$, *and*

$$g(x) := |a_0| + |a_1|x + \cdots + |a_{p-1}|x^{p-1} - |a_p|x^p + |a_{p+1}|x^{p+1} + \cdots + |a_n|x^n$$

has exactly two positive zeros $s_1 < s_2$. *Then*

$$f(z) := a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0 \in \mathcal{P}_n^c$$

has exactly p *zeros in the disk* $\{z \in \mathbb{C} : |z| \leq s_1\}$ *and no zeros in the annulus* $\{z \in \mathbb{C} : s_1 < |z| < s_2\}$.

Proof. Let $s_1 < t < s_2$. Then $g(t) < 0$, that is,

$$\sum_{\substack{j=0\\ j\neq p}}^{n} |a_j|t^j < |a_p|t^p\,.$$

Now apply Rouché's theorem to the functions

$$F(z) := a_p z^p \qquad \text{and} \qquad G(z) := \sum_{j=0}^{n} a_j z^j\,.$$

□

E.6 Proof of Theorem 1.2.4.

a] Hölder's inequality (see E.7 of Section 2.2) asserts that

$$\sum_{k=1}^{n} |a_k b_k| \le \left(\sum_{k=1}^{n} |a_k|^p \right)^{1/p} \left(\sum_{k=1}^{n} |b_k|^q \right)^{1/q},$$

where $p^{-1} + q^{-1} = 1$ and $p \ge 1$. So if

$$p(z) := a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 \in \mathcal{P}_n^c,$$

then

$$\sum_{k=1}^{n-1} |a_k||z|^k \le \left(\sum_{k=0}^{n-1} |a_k|^p \right)^{1/p} \left(\sum_{k=0}^{n-1} |z|^{kq} \right)^{1/q}.$$

b] Thus, for $|z| > 1$,

$$\begin{aligned} |p(z)| &\ge |a_n||z|^n - \sum_{k=0}^{n-1} |a_k||z|^k \\ &\ge |a_n||z|^n \left\{ 1 - \left(\sum_{k=0}^{n-1} \left| \frac{a_k}{a_n} \right|^p \right)^{1/p} \left(\sum_{k=0}^{n-1} \frac{|z|^{kq}}{|z|^{nq}} \right)^{1/q} \right\} \\ &\ge |a_n||z|^n \left\{ 1 - \left(\sum_{k=0}^{n-1} \left| \frac{a_k}{a_n} \right|^p \right)^{1/p} \frac{1}{(|z|^q - 1)^{1/q}} \right\}. \end{aligned}$$

c] When is the last expression positive?

E.7 The Number of Positive Zeros of a Polynomial. Suppose

$$p(z) := \sum_{j=0}^{n} a_j z^j$$

has m positive real roots. Then

$$m^2 \le 2n \log \left(\frac{|a_0| + |a_1| + \cdots + |a_n|}{\sqrt{|a_0 a_n|}} \right).$$

This result is due to Schur though the proof more or less follows Erdős and Turán [50]. It requires using Müntz's theorem from Chapter 4.

a] Suppose

$$p(z) = a_n \prod_{k=1}^{n} (z - r_k e^{i\theta_k})$$

and

$$q(z) := \prod_{k=1}^{n} (z - e^{i\theta_k}) .$$

Note that for $|z| = 1$,

$$\frac{|z - re^{i\theta}|^2}{|r|} \geq |z - e^{i\theta}| .$$

Use this to deduce that

$$|q(z)|^2 \leq \frac{|p(z)|^2}{|a_0 a_n|} \leq \left(\frac{|a_0| + |a_1| + \cdots + |a_n|}{\sqrt{|a_0 a_n|}} \right)^2$$

whenever $|z| = 1$.

b] Since p has m positive real roots q has m roots at 1. Use the change of variables $x := z + z^{-1}$ applied to $z^n q(z^{-1}) q(z)$ to show that

$$\begin{aligned}
&\|q\|^2_{\{|z|=1\}} \\
&\quad \geq \min_{\{b_k\}} \|(z-1)^m (z^{n-m} + b_{n-m-1} z^{n-m-1} + \cdots + b_1 z + b_0)\|^2_{\{|z|=1\}} \\
&\quad \geq \min_{\{c_k\}} \|x^m (x^{n-m} + c_{n-m-1} x^{n-m-1} + \cdots + c_1 x + c_0)\|_{[0,4]} \\
&\quad = 4^n \min_{\{d_k\}} \|x^m (x^{n-m} + d_{n-m-1} x^{n-m-1} + \cdots + d_1 x + d_0)\|_{[0,1]} \\
&\quad \geq \frac{4^n}{\sqrt{2n+1}\binom{2n}{n+m}} ,
\end{aligned}$$

where the last inequality follows by E.2 c] of Section 4.2.

c] Show that

$$\log \left(\frac{4^n}{\sqrt{2n+1}\binom{2n}{n+m}} \right) \geq m^2/n$$

and finish the proof of the main result.

1.3 Zeros of the Derivative

The most basic and important theorem linking the zeros of the derivative of a polynomial to the zeros of the polynomial is variously attributed to Gauss, Lucas, Grace, and others, but is usually called Lucas' theorem [1874].

Theorem 1.3.1 (Lucas' Theorem). *Let $p \in \mathcal{P}_n^c$. All the zeros of p' are contained in the closed convex hull of the set of zeros of p.*

The proof of this theorem follows immediately from the following lemma by considering the intersection of the halfplanes containing the convex hull of the zeros of p.

Lemma 1.3.2. *Let $p \in \mathcal{P}_n^c$. If p has all its zeros in a closed halfplane, then p_n' also has all its zeros in the same closed halfplane.*

Proof. On consideration of the effect of the transformation $z \mapsto \alpha z + \beta$, by which any closed halfplane may be mapped to $\overline{H}_l := \{z : \mathrm{Re}(z) \leq 0\}$, it suffices to prove the lemma under the assumption that p has all its zeros in $\overline{H}_l$. If p has all its zeros in $\overline{H}_l$, then

$$\frac{p'(z)}{p(z)} = \sum_{k=1}^{n} \frac{1}{z - \alpha_k}, \qquad \alpha_k \in \overline{H}_l \,.$$

But if $z \in H_r := \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$, then

$$\frac{1}{z - \alpha_k} \in H_r \quad \text{for each} \quad \alpha_k \in \overline{H}_l \,,$$

and it follows that

$$\sum_{k=1}^{n} \frac{1}{z - \alpha_k} \in H_r \,.$$

In particular,

$$\sum_{k=1}^{n} \frac{1}{z - \alpha_k} \neq 0 \,,$$

which finishes the proof. □

There is a sharpening of Lucas' theorem for real polynomials formulated by Jensen. We need to introduce the notion of *Jensen circles* for a polynomial $p \in \mathcal{P}_n$. For $p \in \mathcal{P}_n$ the nonreal roots of p come in conjugate pairs. For each such pair, $\alpha + i\beta$, $\alpha - i\beta$, form the circle centered at α with radius $|\beta|$. So this circle centered on the x-axis at α has $\alpha + i\beta$ and $\alpha - i\beta$ on the opposite ends of its perpendicular diameter. The collection of all such circles are called the Jensen circles for p.

Theorem 1.3.3 (Jensen's Theorem). *Let $p \in \mathcal{P}_n$. Each nonreal zero of p' lies in or on some Jensen circle for p.*

The proof, which is similar to the proof of Lucas' theorem, is left for the reader as E.3.

We state the following pretty generalization of Lucas' theorem due to Walsh [21]. The proof is left as E.4. Proofs can also be found in Marden [66] and Pólya and Szegő [76].

Theorem 1.3.4 (Walsh's Two-Circle Theorem). *Suppose $p \in \mathcal{P}_n^c$ has all its n zeros in the disk D_1 with center c_1 and radius r_1. Suppose $q \in \mathcal{P}_m^c$ has all its m zeros in the disk D_2 with center c_2 and radius r_2. Then*

a] *All the zeros of $(pq)'$ lie in $D_1 \cup D_2 \cup D_3$, where D_3 is the disk with center c_3 and radius r_3 given by*

$$c_3 := \frac{nc_2 + mc_1}{n+m}, \qquad r_3 := \frac{nr_2 + mr_1}{n+m}.$$

b] *Suppose $(n \neq m)$. Then all the zeros of $(p/q)'$ lie in $D_1 \cup D_2 \cup D_3$, where D_3 is the disk with center c_3 and radius r_3 given by*

$$c_3 := \frac{nc_2 - mc_1}{n-m}, \qquad r_3 := \frac{nr_2 + mr_1}{|n-m|}.$$

Comments, Exercises, and Examples.

Lucas proved his theorem in 1874, although it is an easy and obvious consequence of an earlier result of Gauss. Jensen's theorem is formulated in Jensen [13] and proved in Walsh [20]. Much more concerning the geometry of zeros of the derivative can be found in Marden [66].

E.1 A Remark on Lucas' Theorem. Show that $p' \in \mathcal{P}_n^c$ has a zero α on the boundary of the convex hull of the zeros of p if and only if α is a multiple zero of p.

E.2 Laguerre's Theorem. *Suppose $p \in \mathcal{P}_n^c$ has all its zeros in a disk D. Let $\zeta \in \mathbb{C}$. Let w be any zero of*

$$q(z) := np(z) + (\zeta - z)p'(z)$$

(q is called the polar derivative of p with respect to ζ).

a] If $\zeta \notin D$, then w lies in D.

Hint: Consider $r(z) := p(z)(z-\zeta)^{-n}$, where p has all its zeros in D and $\zeta \notin D$. Then

$$\frac{r'(z)}{r(z)} = \frac{p'(z)}{p(z)} + \frac{n}{\zeta - z}$$

and if $q(w) = 0$ with $w \notin D$, then $r'(w) = 0$. Now observe that r is of the form

$$r(z) = s\left(\frac{1}{z-\zeta}\right), \qquad s \in \mathcal{P}_n^c,$$

where $s'((w-\zeta)^{-1}) = 0$. Note that $\zeta \notin D$ implies that

$$\widetilde{D} := \{(z-\zeta)^{-1} : z \in D\}$$

is a disk. Then s has all its zeros in $\widetilde{D}$ and so does s' by Lucas' theorem. However, $w \notin D$ implies $(w-\zeta)^{-1} \notin \widetilde{D}$, so $s'((w-\zeta)^{-1}) \neq 0$, a contradiction. □

b] If $p(w) \neq 0$, then any circle through w and ζ either passes through all the zeros of p_n or separates them.

E.3 Proof of Jensen's Theorem. Prove Theorem 1.3.3.

Hint: Suppose $p \in \mathcal{P}_n \setminus \mathcal{P}_{n-1}$ and denote the zeros of p by $z_1, z_2, \ldots, z_n$. Then

$$\frac{p'(z)}{p(z)} = \sum_{k=1}^{n} \frac{1}{z - z_k}\,.$$

If $z_k = \alpha_k + i\beta_k$ with $\alpha_k, \beta_k \in \mathbb{R}$, and $z = x + iy$ with $x, y \in \mathbb{R}$, then

$$\begin{aligned}&\operatorname{Im}\left(\frac{1}{x + iy - \alpha_k - i\beta_k} + \frac{1}{x + iy - \alpha_k + i\beta_k}\right)\\ &\quad = \frac{-2y((x-\alpha_k)^2 + y^2 - \beta_k^2)}{((x-\alpha_k)^2 + (y-\beta_k)^2)\cdot((x-\alpha_k)^2 + (y+\beta_k)^2)}\,,\end{aligned}$$

and so outside all the Jensen circles and off the x-axis,

$$\operatorname{sign}\left(\operatorname{Im}\left(\frac{p'(z)}{p_n(z)}\right)\right) = -\operatorname{sign}(y) \neq 0\,.$$

□

E.4 Proof of Walsh's Theorem. Prove Theorem 1.3.4.

a] Prove Theorem 1.3.4 a].

Hint: Let z_0 be a zero of $p'q + q'p$ outside D_1 and D_2. Let

$$\zeta_1 := z_0 - \frac{np(z_0)}{p'(z_0)} \quad \text{and} \quad \zeta_2 := z_0 - \frac{mq(z_0)}{q'(z_0)}$$

($p'(z_0) \neq 0$ and $q'(z_0) \neq 0$ by Lucas' theorem). Observe that $\zeta_1 \in D_1$ and $\zeta_2 \in D_2$ by E.2, and

$$z_0 = \frac{n\zeta_2 + m\zeta_1}{n+m}\,.$$

□

b] Prove Theorem 1.3.4 b].

Hint: Proceed as in the hint to part a], starting from a zero z_0 of $p'q - q'p$ outside D_1 and D_2. □

c] If in Theorem 1.3.4 a] D_1, D_2, and D_3 are disjoint, then D_1 contains $n-1$ zeros, D_2 contains $m-1$ zeros, and D_3 contains 1 zero of $(pq)'$.

Hint: By a continuity argument we may reduce the general case to the case where $p(z) = (z - c_1)^n$ and $q(z) = (z - c_2)^m$. □

d] If in Theorem 1.3.4 b] $n = m$ and D_1 and D_2 are disjoint, then $D_1 \cup D_2$ contains all the zeros of $(p/q)'$.

E.5 Real Zeros and Poles.

a] If all the zeros of $p \in \mathcal{P}_n$ are real, then all the zeros of p_n' are also real.

b] Suppose all the zeros of both $p \in \mathcal{P}_n$ and $q \in \mathcal{P}_m$ are real, and all the zeros of p_n are smaller than any of the zeros of q_n. Show that all the zeros of $(p/q)'$ are real.

Hint: Consider the graph of

$$\frac{(p/q)'}{(p/q)} = \frac{p'}{p} - \frac{q'}{q}\,.$$

□

Define $W(p)$, the *Wronskian* of p, by

$$\begin{aligned} W(p)(z) &= p(z)p''(z) - (p'(z))^2 \\ &= \begin{vmatrix} p(z) & p'(z) \\ p'(z) & p''(z) \end{vmatrix} \\ &= p^2(z)\left(\frac{p'(z)}{p(z)}\right)'\,. \end{aligned}$$

c] Prove that if $p \in \mathcal{P}_n$ has only distinct real zeros, then $W(p)$ has no real zeros.

In Craven, Csordas, and Smith [87] it is conjectured that, for $p \in \mathcal{P}_n$, the number of real zeros of $W(p)/p^2$ does not exceed the number of nonreal zeros of p (a question they attribute to Gauss).

d] Let $p \in \mathcal{P}_n$. Show that any real zero of $W(p)$ lies in or on a Jensen circle of p.

Proof. See Dilcher [91]. □

e] Show that Lucas' theorem does not hold for rational functions.

Hint: Consider $r(x) = x/(\alpha^2 - x^2)$. □

The next exercise is a weak form of Descartes' rule of signs.

E.6 Positive Zeros of Müntz Polynomials. Suppose $\delta_0 < \delta_1 < \cdots < \delta_n$ and

$$f(x) := a_0 x^{\delta_0} + a_1 x^{\delta_1} + \cdots + a_n x^{\delta_n}, \quad a_k \in \mathbb{R}\,.$$

Show that either $f = 0$ or f has at most n zeros in $(0, \infty)$.

Hint: Proceed by induction on n. □

E.7 Apolar Polynomials and Szegő's Theorem. Two polynomials

$$f(x) := \sum_{k=0}^{n} a_k \binom{n}{k} x^k\,, \qquad a_n \neq 0$$

and

$$g(x) := \sum_{k=0}^{n} b_k \binom{n}{k} x^k\,, \qquad b_n \neq 0$$

are called *apolar* if

$$\sum_{k=0}^{n} (-1)^k a_k b_{n-k} \binom{n}{k} = 0\,.$$

a] A Theorem of Grace [02]. *Suppose that f and g are apolar polynomials. If f has all its zeros in a (closed or open) disk D, then g has at least one zero in D.*

Hint: Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\beta_1, \beta_2, \ldots, \beta_n$ denote the zeros of f and g, respectively. Suppose that the zeros of g are all outside D. Let

$$f_1(x) := nf(x) + (\beta_1 - x)f'(x)$$

and for $k = 2, 3, \ldots, n$, let

$$f_k(x) := (n-k+1)f_{k-1}(x) + (\beta_k - x)f_k'(x)\,.$$

Then, by E.2, each f_k has all its zeros in D. Now compute

$$f_{n-1}(\beta_n) = \frac{n!}{b_n}\left(\binom{n}{0} a_0 b_n - \binom{n}{1} a_1 b_{n-1} + \cdots + (-1)^n \binom{n}{n} a_n b_0\right) = 0\,,$$

where the vanishing follows by apolarity. This is a contradiction. □

b] If f and g are apolar, then the closed convex hull of the zeros of f intersects the closed convex hull of the zeros of g.

c] A Theorem of Szegő [22]. *Suppose*

$$f(x) := \sum_{k=0}^{n} a_k \binom{n}{k} x^k\,, \qquad a_n \neq 0\,,$$

$$g(x) := \sum_{k=0}^{n} b_k \binom{n}{k} x^k\,, \qquad b_n \neq 0\,,$$

and

$$h(x) := \sum_{k=0}^{n} a_k b_k \binom{n}{k} x^k\,.$$

Suppose f has all its zeros in a closed disk $\overline{D}$, and g has zeros $\beta_1, \ldots, \beta_n$. Then all the zeros of h are of the form $\beta_i \gamma_i$ with $\gamma_i \in \overline{D}$.

Hint: Suppose δ is a zero of h. Then

$$\sum_{k=0}^{n} a_k b_k \binom{n}{k} \delta^k = 0\,.$$

So the polynomial

$$r(x) := \sum_{k=0}^{n} (-1)^k \binom{n}{k} b_k \delta^k x^{n-k}$$

is apolar to f, and thus has a zero α in D. But then $\alpha = -\delta/\beta_i$ for some i since $r(x) = x^n g(-\delta/x)$. □

E.8 Zeros of the Integral. Suppose $p \in \mathcal{P}_n \setminus \mathcal{P}_{n-1}$ has all its zeros in $\overline{D}_1 := \{z \in \mathbb{C} : |z| \le 1\}$.

a] Show that the polynomial q defined by $q(x) := \int_0^x p(t)\,dt$ has all its zeros in $\overline{D}_2 := \{z \in \mathbb{C} : |z| \le 2\}$.

Hint: Apply E.7 c]. Take

$$f(x) := p(x), \quad g(x) := \sum_{k=0}^{n} \binom{n}{k} \frac{x^k}{k+1}\,.$$

Then

$$h(x) = \frac{1}{x} \int_0^x p(t)\,dt\,.$$

Note that $g(x) = (n+1)^{-1} x^{-1}((1+x)^{n+1} - 1)$ has all its zeros in D_2. □

b] Show that

$$q(x) := \int_0^x \int_0^{t_{m-1}} \int_0^{t_{m-2}} \cdots \int_0^{t_1} p(t)\,dt\,dt_1 \cdots dt_{m-2}\,dt_{m-1}$$

has all its zeros in $\overline{D}_{r_{m,n}} := \{z \in \mathbb{C} : |z| \le r_{m,n}\}$, where $r_{m,n} \le m+1$ is the zero of

$$\sum_{k=0}^{n} \binom{m+n}{m+k} x^k$$

with the largest modulus. Note that q is the mth integral of p normalized so that the constants of integration are all zero.

Proof. See Borwein, Chen, and Dilcher [95]. □

E.9 Grace's Complex Version of Rolle's Theorem. *Suppose α and β are zeros of $p \in \mathcal{P}_n \setminus \mathcal{P}_{n-1}$. Then p' has at least one zero in the disk*

$$\overline{D}(c,r) := \{z \in \mathbb{C} : |z-c| \le r\},$$

where

$$c := \frac{\alpha+\beta}{2} \qquad \textit{and} \qquad r := \frac{|\alpha-\beta|}{2}\cot\frac{\pi}{n}.$$

Hint: Assume, without loss of generality, that $\alpha = -1$ and $\beta = 1$. Let

$$p'(x) = \sum_{k=0}^{n-1} a_k x^k, \quad \text{that is}, \quad p(x) = c + \sum_{k=0}^{n-1} a_k \frac{x^{k+1}}{k+1}.$$

Apply E.7 a]. Note that

$$0 = \frac{p(1)-p(-1)}{2} = a_0 + \frac{a_2}{3} + \frac{a_4}{5} + \cdots.$$

So

$$f(z) := (z-1)^n - (z+1)^n$$

and p' are apolar. □

E.10 Corollaries of Szegő's Theorem. *Suppose*

$$f(z) := \binom{n}{0}a_0 + \binom{n}{1}a_1 z + \cdots + \binom{n}{n}a_n z^n,$$
$$g(z) := \binom{n}{0}b_0 + \binom{n}{1}b_1 z + \cdots + \binom{n}{n}b_n z^n,$$

and

$$h(z) := \binom{n}{0}a_0 b_0 + \binom{n}{1}a_1 b_1 z + \cdots + \binom{n}{n}a_n b_n z^n$$

with $a_n b_n \neq 0$.

a] *If f has all its zeros in a convex set S containing 0 and g has all its zeros in $[-1,0]$, then h has all its roots in S.*

b] *If f and g have all their zeros in $[-1,0]$, then so does h.*

E.11 Another Corollary of Szegő's Theorem. *If $\sum_{k=0}^{n} a_k z^k$ has all its zeros in $\overline{D}_1 := \{z \in \mathbb{C} : |z| \le 1\}$, then so does $\sum_{k=0}^{n} \frac{a_k z^k}{\binom{n}{k}}$. In particular, $\sum_{k=0}^{n} \frac{z^k}{\binom{n}{k}}$ has all its zeros in $\overline{D}_1$.*

The results of the next exercise were first proved by M. Riesz (see, for example, Mignotte [92]) and were rediscovered by Walker [93].

E.12 Consecutive Zeros of p' for $p \in \mathcal{P}_n$ with Real Zeros. For a polynomial

$$p(x) := \prod_{i=1}^{n} (x - \alpha_i), \quad \alpha_1 < \alpha_2 < \cdots < \alpha_n, \quad n \geq 2$$

with only real zeros, let

$$\Delta(p) := \min_{1 \leq i \leq n-1} (\alpha_{i+1} - \alpha_i).$$

By Rolle's theorem

$$p'(x) = n \prod_{i=1}^{n-1} (x - \beta_i), \quad \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \beta_{n-1} < \alpha_n.$$

a] Suppose $n \geq 3$. Prove that $\Delta(p) < \Delta(p')$.

Outline. It is required to show that $\beta_j - \beta_{j-1} > \Delta(p)$ for each $j \geq 2$. Let $2 \leq j \leq n$ be fixed. Since

$$\frac{f'(\beta_j)}{f(\beta_j)} = \frac{f'(\beta_{j-1})}{f(\beta_{j-1})} = 0$$

we have

$$\sum_{i=1}^{n} \frac{1}{(\beta_{j-1} - \alpha_i)(\beta_j - \alpha_i)} = 0.$$

Now let $u_j := \alpha_j - \beta_{j-1}$, $v_j := \beta_j - \alpha_j$. Also for each i, let $d_i := \alpha_j - \alpha_{j-i}$, $e_i := \alpha_{j+i} - \alpha_j$. Then the above can be rewritten as

$$\sum_{i=1}^{j-1} \frac{1}{(d_i - u_j)(d_i + v_j)} + \frac{1}{(-u_j v_j)} + \sum_{i=1}^{n-j} \frac{1}{(e_i + u_j)(e_i - v_j)} = 0.$$

Define

$$F(u, v) := \sum_{i=1}^{j-1} \frac{uv}{(d_i - u)(d_i + v)} + \sum_{i=1}^{n-j} \frac{uv}{(e_i + u)(e_i - v)}.$$

Note that F is increasing in each variable ($0 \leq u < d_1$, $0 \leq v < e_1$) and observe that

$$F(u_j, v_j) = 1.$$

To prove the result, it suffices to show that if u and v are nonnegative numbers satisfying $u + v = \Delta(p)$, then $F(u, v) < 1$.

Now show that

$$\begin{aligned} F(u,v) \le & uv\left(\sum_{i=1}^{j-1}\frac{1}{(d_i-u)(d_{i+1}-u)}+\sum_{i=1}^{n-j}\frac{1}{(e_{i+1}-v)(e_i-v)}\right) \\ \le & \frac{uv}{\Delta(p)}\left(\sum_{i=1}^{j-1}\left(\frac{1}{d_i-u}-\frac{1}{d_{i+1}-u}\right)+\sum_{i=1}^{n-j}\left(\frac{1}{e_i-v}-\frac{1}{e_{i+1}-v}\right)\right) \\ < & \frac{uv}{\Delta(p)}\left(\frac{1}{d_1-u}+\frac{1}{e_1-v}\right) \le \frac{uv}{\Delta(p)}\left(\frac{1}{v}+\frac{1}{u}\right) \le 1 \end{aligned}$$

whenever u and v are nonnegative numbers satisfying $u+v=\Delta(p)$. □

b] Suppose $n \ge 3$ and $\gamma \in \mathbb{R}$. Show that $\Delta(p'-\gamma p)$ has only real zeros and $\Delta(p'-\gamma p) > \Delta(p)$.

c] What happens when p has only real zeros but they are not necessarily distinct?

E.13 Fejér's Theorem on the Zeros of Müntz Polynomials. The following pretty results of Fejér may also be found in Pólya and Szegő [76]:

Suppose that $(\lambda_k)_{k=0}^{\infty}$ is an increasing sequence of nonnegative integers with $\lambda_0 = 0$.

a] Let

$$p(z) := \sum_{k=0}^{n} a_k z^{\lambda_k}, \qquad a_k \in \mathbb{C}, \quad a_0 a_1 \neq 0.$$

Then p has at least one zero $z_0 \in \mathbb{C}$ so that

$$|z_0| \le \left(\frac{\lambda_2\lambda_3\cdots\lambda_n}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)\cdots(\lambda_n-\lambda_1)}\right)^{1/\lambda_1}\left|\frac{a_0}{a_1}\right|^{1/\lambda_1}.$$

Outline. We say that $z_1 \in \mathbb{C}$ is not less than $z_2 \in \mathbb{C}$ if $|z_2| \le |z_1|$. Studying $q(z) := z^{\lambda_n}p(z^{-1})$, we need to show that the largest zero of

$$q(z) = a_0 x^{\lambda_n} + \sum_{k=1}^{n} a_k x^{\lambda_n-\lambda_k}$$

is not less than

$$\left(\frac{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)\cdots(\lambda_n-\lambda_1)}{\lambda_2\lambda_3\cdots\lambda_n}\right)^{1/\lambda_1}\left|\frac{a_1}{a_0}\right|^{1/\lambda_1}.$$

We prove this statement by induction on n. The statement is obviously true for $n=1$. Now assume that the statement is true for $n-1$. It follows from Lucas' theorem that if q is a polynomial with complex coefficients, then the largest zero of q' is not greater than the largest zero of q.

By the above corollary of Lucas' theorem, it is sufficient to prove that the largest zero of

$$z^{\lambda_{n-1}-\lambda_n+1}q'(z) = \lambda_n a_0 z^{\lambda_{n-1}} + \sum_{k=1}^{n-1}(\lambda_n - \lambda_k)a_k z^{\lambda_{n-1}-\lambda_k}$$

is not less than

$$\left(\frac{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)\cdots(\lambda_n-\lambda_1)}{\lambda_2\lambda_3\cdots\lambda_n}\right)^{1/\lambda_1}\left|\frac{a_1}{a_0}\right|^{1/\lambda_1}.$$

However, this is true by the inductive hypothesis. □

b] Suppose

$$f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}, \qquad a_k \in \mathbb{C}$$

is an entire function so that $\sum_{k=1}^{\infty} 1/\lambda_k < \infty$, that is, the entire function f satisfies the *Fejér gap condition*. Show that there is a $z_0 \in \mathbb{C}$ so that $f(z_0) = 0$.

Hint: Use part a]. □