4

Denseness Questions

Overview

We give an extended treatment of when various Markov spaces are dense. In particular, we show that denseness, in many situations, is equivalent to denseness of the zeros of the associated Chebyshev polynomials. This is the principal theorem of the first section. Various versions of Weierstrass' classical approximation theorem are then considered. The most important is in Section 4.2 where Müntz's theorem concerning the denseness of $\text{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}$ is analyzed in detail. The third section concerns the equivalence of denseness of Markov spaces and the existence of unbounded Bernstein inequalities. In the final section we consider when rational functions derived from Markov systems are dense. Included is the surprising result that rational functions from a fixed infinite Müntz system are always dense.

4.1 Variations on the Weierstrass Theorem

Much of the utility of polynomials stems from the fact that all continuous functions on a finite closed interval are uniform limits of them. This is the well-known Weierstrass approximation theorem. There are numerous proofs of this; several are presented in the exercises. Another proof follows from the main theorem of this section.

Associated with a Markov system $\mathcal{M} := (f_0, f_1, \dots)$ on $[a,b]$ we define, as in Section 3.3, the Chebyshev polynomials

$$T_n := T_n\{f_0, f_1, \dots, f_n; [a,b]\}.$$

Denote the zeros of T_n by $(a \le) x_1 < x_2 < \cdots < x_n (\le b)$. Let $x_0 := a$ and $x_{n+1} := b$. The *mesh* of T_n is defined by

$$M_n := M_n(T_n : [a,b]) := \max_{1 \le i \le n+1} |x_i - x_{i-1}|. \tag{4.1.1}$$

This is a measure of the maximal gap between two consecutive zeros of T_n with respect to the interval $[a,b]$.

For a sequence $(T_n)_{n=0}^{\infty}$ of Chebyshev polynomials associated with a fixed Markov system on $[a,b]$, we have

$$\lim_{n\to\infty} M_n = 0 \qquad \text{if and only if} \qquad \liminf_{n\to\infty} M_n = 0.$$

This follows from the fact that if $m < n$, then T_m cannot have more than one zero between any two consecutive zeros of T_n.

Our main result shows the strong connection between the denseness of the real span of an infinite Markov system $\mathcal{M}$ of C^1 functions on $[a,b]$ in $C[a,b]$ and the density of the zeros of the associated Chebyshev polynomials.

Theorem 4.1.1. *Suppose $\mathcal{M} := (1, f_1, f_2, \dots)$ is an infinite Markov system on $[a,b]$ with each $f_i \in C^1[a,b]$. Then* span $\mathcal{M}$ *is dense in $C[a,b]$ if and only if*

$$\lim_{n\to\infty} M_n = 0,$$

where M_n is the mesh of the associated Chebyshev polynomials.

Proof. The *only if* part of this result is the easier part and we offer the following proof. Suppose span $\mathcal{M}$ is dense in $C[a,b]$, while $\liminf_{n\to\infty} M_n > 0$. Then there exists an interval $[c,d] \subset [a,b]$ that contains no zero of T_n for infinitely many n, say, for $n_1 < n_2 < \cdots$. Consider the piecewise linear function F defined as follows. Let $c < y_1 < y_2 < y_3 < y_4 < d$, and let

$$F(x) := \begin{cases} 0, & x \in \{a, c, d, b\} \\ 2, & x \in \{y_1, y_3\} \\ -2, & x \in \{y_2, y_4\} \end{cases}$$

and be linear elsewhere. Since span $\mathcal{M}$ is dense in $C[a,b]$, there exists a $k \in \mathbb{N}$ and a $p \in \text{span}\{1, f_1, \dots, f_{n_k}\}$ with

$$\|p - F\|_{[a,b]} < 1. \tag{4.1.2}$$

Now $p - T_{n_k}$ has at least $n_k - 2$ zeros on $[a,c] \cup [d,b]$ because T_{n_k} has at least n_k extrema on these intervals. The four extrema of F on (c,d) together with (4.1.2) guarantee at least three more zeros of $p - T_{n_k}$ on (c,d). Hence $p - T_{n_k}$ has at least $n_k + 1$ zeros and vanishes identically. This contradicts (4.1.2).

The *if* part of the theorem follows from the next theorem and E.8 a] of Section 3.2. This exercise shows that $(f_1', f_2', \dots)$ is a weak Markov system on $[a,b]$. □

The phenomenon formulated in Theorem 4.1.1 is quite general, and we prove a rather more general result than is needed for the preceding theorem. The *modulus of continuity* ω_f of a function $f : [a,b] \mapsto \mathbb{R}$ is defined by

$$\omega_f(\delta) := \sup_{\substack{|x-y|<\delta \\ x,y\in[a,b]}} |f(x) - f(y)| . \tag{4.1.3}$$

Theorem 4.1.2. *Suppose that*

$$H_n := \operatorname{span}\{1, g_1, g_2, \dots, g_n\}$$

is a Chebyshev space on $[a,b]$ with associated Chebyshev polynomial T_n. Suppose each $g_i \in C^1[a,b]$ and $(g_1', \dots, g_n')$ is a weak Chebyshev system on $[a,b]$ (weak Chebyshev systems are defined in E.8 of Section 3.2). Let $H_n' := \operatorname{span}\{g_1', \dots, g_n'\}$. If $f \in C[a,b]$, then there exists an $h_n \in H_n$ such that

$$\|h_n - f\|_{[a,b]} \le C\omega_f\big(\sqrt{\delta_n}\big) ,$$

where

$$\delta_n := M_n(T_n : [a,b]) .$$

Here C is a constant depending only on a and b.

Proof. Suppose $a < c < d < b$ and $S_n \in H_n$ is the best uniform approximation from H_n to F on $[a,c] \cup [d,b]$, where

$$F(x) := \begin{cases} 0, & x \in [a,c] \\ 1, & x \in [d,b] . \end{cases}$$

We claim the following:

$$S_n \text{ is monotone on } [c,d] \tag{4.1.4}$$

and

$$\|S_n - F\|_{[a,c]\cup[d,b]} \le \frac{5\delta_n}{(d-c)} . \tag{4.1.5}$$

Let $\eta := n+1$ be the dimension of the Chebyshev space H_n. Since S_n is a best approximation to F on $[a,c]\cup[d,b]$, there exist $\eta+1$ points in this set where the maximum error

$$\epsilon_n := \|F - S_n\|_{[a,c]\cup[d,b]} \tag{4.1.6}$$

occurs with alternating sign (see Theorem 3.1.6). Suppose $m+1$ of these points $y_0 < \cdots < y_m$ lie in $[a,c]$, and $\eta - m$ of these points $y_{m+1} < \cdots < y_\eta$ lie in $[d,b]$. Then S_n' has at least $m-1$ sign changes in (a,c) (one at each alternation point in $[a,c]$ except possibly at the endpoints a and c). Likewise, S_n' has at least $\eta - m - 2$ sign changes in (d,b). So S_n' has at least $\eta - 3$ sign changes in $(a,c)\cup(d,b)$. Note that this count excludes y_m and y_{m+1}. Thus S_n' has at most one more sign change in (a,b) unless S_n' vanishes identically (which is not possible for $\eta \geq 2$). Now suppose S_n' has a sign change on (c,d). Then, since there is at most one sign change of S_n' in (c,d), it cannot be the case that both $y_m = c$ and $y_{m+1} = d$ and S_n' changes sign at neither c nor d, otherwise

$$\operatorname{sign}(S_n(c) - f(c)) = \operatorname{sign}(S_n(d) - f(d))$$

as a consideration of the two cases shows. But if $y_m \neq c$ or $y_{m+1} \neq d$ or S_n' changes sign at either c or d, then we have accounted for all the sign changes of S_n' by accounting for the (possible) one additional sign change (either S_n' vanishes with sign change at c or d or one of y_m or y_{m+1} is an interior alternation point of S_n where S_n' vanishes). Thus S_n' has no zeros with sign change in (c,d) and (4.1.4) is proved.

To prove (4.1.5) we proceed as follows. With ϵ_n defined by (4.1.6),

$$D_n := \epsilon_n T_n - S_n$$

has at least m zeros on $[a,c]$ and

$$D_n^* := D_n + 1 = 1 + \epsilon_n T_n - S_n$$

has at least $\eta - m - 1$ zeros on $[d,b]$ (counting each internal zero without sign change twice). Thus D_n' has at least $\eta - 3$ sign changes on $[a,c]\cup[d,b]$. Suppose T_n has at least four alternation points on an interval $[\gamma,\delta] \subset (c,d)$, and suppose that

$$S_n(\delta) - S_n(\gamma) < 2\epsilon_n\,.$$

Then, because of (4.1.4) and the oscillation of T_n on $[\gamma,\delta]$,

$$D_n + \frac{S_n(\gamma)+S_n(\delta)}{2} = \epsilon_n T_n - \left[S_n - \frac{S_n(\gamma)+S_n(\delta)}{2}\right]$$

has at least three zeros on $[\gamma,\delta]$ and hence

$$D_n' = \left(D_n + \frac{S_n(\gamma) + S_n(\delta)}{2} \right)'$$

has at least two sign changes on $[\gamma, \delta]$. This, however, gives that $D_n' \in H_n'$ has a total of at least $\eta - 1 = n$ sign changes, which is impossible. In particular,

$$S_n(\delta) - S_n(\gamma) \geq 2\epsilon_n$$

on any interval $[\gamma, \delta] \subset (c, d)$ where T_n has at least 4 alternation points. Thus,

$$S_n(d) - S_n(c) \geq \frac{(d-c)}{5\delta_n}\, 2\epsilon_n \,.$$

However, since S_n is a best approximation to F on $[a, c] \cup [d, b]$,

$$S_n(d) - S_n(c) \leq 1 + 2\epsilon_n$$

and we can deduce (4.1.5) on comparing these last two inequalities and noting that $\epsilon_n \leq \frac{1}{2}$.

The proof is now a routine argument, which for simplicity, is presented on the interval $[a, b] := [0, 1]$. Let

$$V(x) := f(0) + \sum_{i=0}^{m-1} \left(f\left(\tfrac{i+1}{m}\right) - f\left(\tfrac{i}{m}\right) \right) S_{n,i}(x) \,,$$

where, for $i = 0, 1, \ldots, m-1$, $S_{n,i} \in H_n$ is the best uniform approximation to

$$F_{n,i}(x) := \begin{cases} 0 \,, & x \in \left[0, \frac{i+1}{m}\right) \\ 1 \,, & x \in \left[\frac{i+1}{m}, 1\right] \end{cases}$$

on $\left[0, \frac{i}{m}\right] \cup \left[\frac{i+1}{m}, 1\right]$. Let

$$\widetilde{f}(x) := f(0) + \sum_{i=0}^{m-1} \left(f\left(\tfrac{i+1}{m}\right) - f\left(\tfrac{i}{m}\right) \right) F_{n,i}(x) \,.$$

Then repeated applications of (4.1.5) with the intervals $[a, c] := \left[0, \frac{i}{m}\right]$ and $[d, b] := \left[\frac{i+1}{m}, 1\right]$ yield for every $x \in [0, 1]$ that

$$\begin{aligned} |V(x) - f(x)| &\leq |V(x) - \widetilde{f}(x)| + |\widetilde{f}(x) - f(x)| \\ &\leq \sum_{i=0}^{m-1} \left(f\left(\tfrac{i+1}{m}\right) - f\left(\tfrac{i}{m}\right) \right) \left(S_{n,i}(x) - F_{n,i}(x) \right) + \omega_f\left(\tfrac{1}{m}\right) \\ &\leq (m-1)(5\delta_n m)\, \omega_f\left(\tfrac{1}{m}\right) + 2\,\omega_f\left(\tfrac{1}{m}\right) + \omega_f\left(\tfrac{1}{m}\right) \,. \end{aligned}$$

Hence, with $m := \lfloor \delta_n^{-1/2} \rfloor$,

$$\|V - f\|_{[0,1]} \leq C\, \omega_f\left(\sqrt{\delta_n}\right) .$$

□

An immediate corollary to Theorem 4.1.1 is the Weierstrass theorem.

Corollary 4.1.3. *The polynomials are dense in* $C[-1,1]$.

Proof. $\mathcal{M} = (1, x, x^2, \dots)$ is an infinite Markov system of C^1 functions on $[-1,1]$. The associated Chebyshev polynomials are just the usual Chebyshev polynomials T_n (see Section 2.1) and

$$M_n \le \frac{\pi}{n}, \qquad n = 1, 2, \dots$$

is obvious from E.1 of Section 2.1. □

Also from the last part of the proof of Theorem 4.1.2 we have the following corollary.

Corollary 4.1.4. *Suppose* $\mathcal{M} := (1, f_1, f_2, \dots)$ *is an infinite Markov system on* $[a,b]$ *with each* $f_i \in C^1[a,b]$. *Then for each* $n \in \mathbb{N}$, *there exists a*

$$p_n \in \operatorname{span}\{1, f_1, f_2, \dots, f_n\}$$

such that

$$\|p_n - f\|_{[a,b]} \le C(1 + m^2 M_n)\, \omega_f\left(\tfrac{1}{m}\right)$$

for every $m \in \mathbb{N}$, *where* C *is a constant depending only on* a *and* b.

Comments, Exercises, and Examples.

The Weierstrass approximation theorem of 1885 (see Weierstrass [15]) is one of the very basic theorems of approximation theory. It, of course, requires that clear distinctions be made about the nature of convergence (pointwise versus uniform) and the region of convergence (intervals versus complex domains). Weierstrass, the preeminent analyst of the last third of the nineteenth century, was principal in insisting that such distinctions be clearly made. His famous and profoundly surprising example of a nowhere differentiable continuous function dates from 1872. A number of proofs of his approximation theorem and its many generalizations are explored in the exercises. Theorem 4.1.1 was proved by Borwein [90]. The *only if* part of this theorem can be found in Kroó and Peherstorfer [92].

Applications of the methods and results of this section can be found in Borwein [91b], Borwein and Saff [92], and Lorentz, Golitschek, and Makovoz [92]. The last two papers give an application to weighted incomplete polynomials, where the zeros of the Chebyshev polynomials are often dense in a subinterval (see also Mhaskar and Saff [85]).

E.1 The Weierstrass Approximation Theorem. *Every real-valued continuous function on a finite closed interval* $[a,b]$ *can be uniformly approximated by polynomials with real coefficients.*

Every complex-valued continuous function on a finite closed interval $[a,b]$ *can be uniformly approximated by polynomials with complex coefficients.*

More precisely, in the real case, let

$$E_n := E_n(f : [a,b]) := \inf_{p \in \mathcal{P}_n} \|f - p\|_{[a,b]} .$$

The Weierstrass approximation theorem asserts that

$$\lim_{n\to\infty} E_n(f : [a,b]) = 0 , \qquad f \in C[a,b] .$$

The following steps outline an elementary proof basically due to Lebesgue [1898]. Parts a] to d] deal with the real version (first statement) of the theorem. The complex version (second statement) of the theorem can easily be reduced to the real version; see part e].

a] Every continuous function on $[a,b]$ can be uniformly approximated by piecewise linear functions.

Hint: Consider the piecewise linear function that interpolates f at n equally spaced points and use the uniform continuity of f. □

b] It suffices to prove that $|x|$ can be uniformly approximated by polynomials on $[-1,1]$.

Hint: Use part a]. □

c] Approximation to $|x|$. Show that

$$\lim_{n\to\infty} E_n(|x| : [-1,1]) = 0 .$$

Hint: The Taylor series expansion of $f(z) := \sqrt{1-z}$ yields

$$\sqrt{1-z} = 1 - \frac{1}{2}\, z + \frac{1}{2\cdot 4}\, z^2 - \frac{1\cdot 3}{2\cdot 4\cdot 6}\, z^3 + \cdots$$

and the convergence is uniform for $0 \le z \le 1$. (By Abel's theorem, a power series converges uniformly on every closed subinterval of the set of points in $\mathbb{R}$ where it converges; see, for example, Stromberg [81]). Thus,

$$\begin{aligned} |x| &= \sqrt{x^2} = \sqrt{1-(1-x^2)} \\ &= 1 - \frac{1}{2}(1-x^2) + \frac{1}{2\cdot 4}(1-x^2)^2 - \frac{1\cdot 3}{2\cdot 4\cdot 6}(1-x^2)^3 + \cdots \end{aligned}$$

and the convergence is uniform for $-1 \le x \le 1$. □

d] An Alternative to c]. Let

$$Q_0(x) := 1 \qquad \text{and} \qquad Q_{n+1}(x) := \frac{1}{2}(1 - x^2 + Q_n^2(x)) .$$

Show that

$$0 \le Q_{n+1}(x) \le Q_n(x) \le 1, \qquad n = 0, 1, \ldots , \quad x \in [-1,1]$$

and $Q_n(x) \to 1 - |x|$ uniformly on $[-1,1]$ as $n \to \infty$.

Hint: First show the pointwise convergence and then use Dini's theorem (see, for example, Royden [88]). □

e] **Complex Version of the Weierstrass Approximation Theorem.** *Every complex-valued continuous function on a finite closed interval $[a,b]$ can be uniformly approximated by polynomials with complex coefficients.*

It can be shown that

$$E_n(|x| : [-1,1]) \sim \frac{c}{n},$$

where $0.280168 < c < 0.280174$. Bernstein [13] established the above asymptotic with weaker bounds on c, namely, $0.278 < c < 0.286$, and observed that $\frac{1}{2}\pi^{-1/2} = 0.282$ is roughly the average of these bounds. The stronger bounds on c, due to Varga and Carpenter, show that $c \neq \frac{1}{2}\pi^{-1/2}$, but it is open whether or not c is some familiar constant; see Varga [90].

E.2 The Stone-Weierstrass Theorem. *If X is a compact Hausdorff space, then a subalgebra $\mathcal{A}$ of $C(X)$, which contains $f = 1$ and separates points, is dense in $C(X)$.*

A subalgebra $\mathcal{A}$ of $C(X)$ is a vector space of functions that is closed under multiplication (here, addition and multiplication are pointwise). Separating points means that for any two distinct x, $y \in X$, there exists an $f \in \mathcal{A}$ such that $f(x) \neq f(y)$.

a] Observe that the set $\mathcal{P} := \cup_{n=0}^{\infty} \mathcal{P}_n$ of all polynomials with real coefficients is a subalgebra of $C[a,b]$ that separates points, and hence the Stone-Weierstrass theorem implies the Weierstrass approximation theorem.

b] Observe that the real polynomials in x^2 form a subalgebra of $C[-1,1]$ that does not separate points.

We outline a standard proof of the Stone-Weierstrass theorem. Let $\overline{\mathcal{A}}$ denote the closure of a subalgebra $\mathcal{A} \subset C(X)$ in the uniform norm.

c] If $f \in \overline{\mathcal{A}}$, then $|f| \in \overline{\mathcal{A}}$.

Proof. If $f \in \mathcal{A}$, then $p(f) \in \mathcal{A}$ for any polynomial p. Now choose p_n such that $p_n(x) \to |x|$ on the interval $[-\|f\|, \|f\|]$. □

d] Let

$$(f \vee g)(x) := \max\{f(x), g(x)\} \quad \text{and} \quad (f \wedge g)(x) := \min\{f(x), g(x)\}.$$

Show that if f, $g \in \overline{\mathcal{A}}$, then so are $f \vee g$ and $f \wedge g$.

Hint:

$$f \vee g = \frac{1}{2}(f + g + |f - g|)\,, \quad \text{and} \quad f \wedge g = \frac{1}{2}(f + g - |f - g|)\,.$$

□

e] If $p, q \in X$ are distinct and $\lambda, \mu \in \mathbb{R}$, then there exists $f \in \mathcal{A}$ with

$$f(p) = \lambda \qquad \text{and} \qquad f(q) = \mu\,.$$

Hint: Let $g \in \mathcal{A}$ be such that $g(p) \neq g(q)$ and consider

$$f := \frac{\lambda - \mu}{g(p) - g(q)} \cdot g + \frac{\mu g(p) - \lambda g(q)}{g(p) - g(q)} \cdot 1\,.$$

□

f] Completion of Proof. Let $f \in C(X)$. For each $p, q \in X$, let f_{pq} be an element of $\mathcal{A}$ with $f_{pq}(p) = f(p)$ and $f_{pq}(q) = f(q)$. Fix $\epsilon > 0$ and define open sets

$$V_{pq} := \{x \in X : f_{pq}(x) < f(x) + \epsilon\}\,.$$

Now $\{V_{pq} : p \in X\}$ is an open cover of the compact Hausdorff space X, so for each $q \in X$ we can pick a finite subcover

$$\{V_{p_1 q}, V_{p_2 q}, \ldots, V_{p_n q}\}$$

of X. We let

$$f_q := \min\{f_{p_1 q}, f_{p_2 q}, \ldots, f_{p_n q}\}\,.$$

Observe that $f_q \in \overline{\mathcal{A}}$ by part e], and

$$f_q(x) < f(x) + \epsilon\,, \qquad x \in X\,.$$

g] Continued. Let

$$V_q := \{x \in X : f_q(x) > f(x) - \epsilon\}\,,$$

where f_q is defined in part f] for every $q \in X$. Then $\{V_q : q \in X\}$ is an open cover of the compact Hausdorff space X, so we can extract a finite subcover

$$\{V_{q_1}, V_{q_2}, \ldots, V_{q_m}\}$$

of X. Now let

$$g := \max\{f_{q_1}, f_{q_2}, \ldots, f_{q_m}\}\,.$$

Note that $g \in \overline{\mathcal{A}}$ by part e], and

$$f(x) = \epsilon < g(x) < f(x) + \epsilon\,, \qquad x \in X\,,$$

which finishes the proof. □

The next exercise presents pretty theorems due to Bohman [52] and Korovkin [53] on the convergence of sequences of positive linear operators. The exercise after that gives some applications that include different proofs of the Weierstrass theorem via convergence of special polynomials, such as the Bernstein polynomials.

An operator L on $C(X)$ is called *monotone* if

$$f \le g \quad \text{implies} \quad L(f) \le L(g)$$

(here $f \le g$ means $f(x) \le g(x)$ for all $x \in X$).

E.3 Monotone Operator Theorems.

Korovkin's First Theorem. *Let $(L_n)_{n=1}^{\infty}$ be a sequence of monotone linear operators on $C(K)$ (the set of continuous, 2π periodic, real-valued functions on $\mathbb{R}$). Let*

$$f_0(x) := 1\,, \quad f_1(x) := \sin x\,, \quad f_2(x) := \cos x\,.$$

Then

$$\lim_{n\to\infty} \|L_n(f) - f\|_K = 0$$

for all $f \in C(K)$ if and only if

$$\lim_{n\to\infty} \|L_n(f_i) - f_i\|_K = 0\,, \qquad i = 0,1,2\,.$$

Korovkin's Second Theorem. *Let $(L_n)_{n=1}^{\infty}$ be a sequence of monotone linear operators on $C[a,b]$. Let*

$$f_0(x) := 1\,, \quad f_1(x) := x\,, \quad f_2(x) := x^2\,.$$

Then

$$\lim_{n\to\infty} \|L_n(f) - f\|_{[a,b]} = 0$$

for all $f \in C[a,b]$ if and only if

$$\lim_{n\to\infty} \|L_n(f_i) - f_i\|_{[a,b]} = 0\,, \qquad i = 0,1,2\,.$$

Korovkin's theorem in a more general setting can be found in Lorentz [86a].

a] **Proof of Korovkin's Second Theorem.** The *only if* part of the theorem is trivial. For the *if* part, observe that the pointwise convergence of $(L_n)_{n=1}^{\infty}$ can be easily proved since, for any preassigned $\epsilon > 0$ at any fixed x_0, one can

find parabolas $y = p_1(x) := a_1x^2 + b_1x + c_1$ and $y = p_2(x) := a_2x^2 + b_2x + c_2$ such that

$$p_1(x) < f(x) < p_2(x)\,, \qquad x \in [a,b]$$

with

$$|f(x_0) - p_1(x_0)| < \epsilon \qquad \text{and} \qquad |f(x_0) - p_2(x_0)| < \epsilon\,.$$

Now use the continuity of f and the compactness of $[a,b]$ to make the above argument uniform on the interval $[a,b]$. □

b] Proof of Korovkin's First Theorem.

Hint: Modify the proof of Korovkin's second theorem. □

E.4 Bernstein Polynomials. The nth *Bernstein polynomial* for a function $f \in C[0,1]$ is defined by

$$B_n(f)(x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}\,, \qquad n = 1,2,\ldots\,.$$

a] Let

$$f_0(x) := 1\,, \quad f_1(x) := x\,, \quad f_2(x) := x^2\,.$$

Show that

$$B_n(f_0) = f_0\,, \quad B_n(f_1) = f_1\,, \quad B_n(f_2) = \tfrac{n-1}{n}f_2 + \tfrac{1}{n}f_1$$

for every $n = 0,1,2,\ldots$.

b] Use Korovkin's second theorem and part a] to show that

$$\lim_{n\to\infty} \|B_n(f) - f\|_{[0,1]} = 0$$

for every $f \in C[0,1]$.

For more on Bernstein polynomials, see Lorentz [86b].

E.5 The Fourier and Fejér Operators. For $f \in C(K)$, let

$$S_n(f)(x) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t+x)\left(\frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t}\right)dt\,, \qquad n = 0,1,\ldots$$

and

$$F_n(f)(x) := \frac{1}{2\pi n}\int_{-\pi}^{\pi} f(t+x)\left(\frac{\sin\frac{1}{2}nt}{\sin\frac{1}{2}t}\right)^2 dt\,, \qquad n = 0,1,\ldots\,.$$

The operator S_n is called the *Fourier operator*, while the operator F_n is called the *Fejér operator*.

a] Show that $S_n(f)$ is the nth partial sum of the Fourier series of f, that is,

$$S_n(f)(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)\,,$$

where

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos kt \, dt$$

and

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin kt \, dt\,.$$

Hint:

$$\frac{\sin\left(n + \frac{1}{2}\right) t}{2 \sin \frac{1}{2} t} = \frac{1}{2} + \sum_{k=1}^{n} \cos kt$$

and

$$S_n(f)(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t+x) \left(\frac{1}{2} + \sum_{k=1}^{n} \cos kt \right) dt\,.$$

□

b] $F_n(f)$ is the Cesàro mean of $S_0, S_1, \ldots S_{n-1}$, that is,

$$F_n(f) = \frac{S_0(f) + S_1(f) + \cdots + S_{n-1}(f)}{n}\,.$$

Hint:

$$\sum_{k=0}^{n-1} \frac{\sin\left(k + \frac{1}{2}\right) t}{\sin \frac{1}{2} t} = \left(\frac{\sin \frac{1}{2} nt}{\sin \frac{1}{2} t} \right)^2\,.$$

□

c] **Fejér's Theorem.** *For every $f \in C(K)$, $F_n(f) \to f$ uniformly on $\mathbb{R}$.*

Hint: Each F_n is obviously a monotone operator on $C(K)$, so it suffices to prove the uniform convergence of $(F_n)_{n=1}^{\infty}$ on $\mathbb{R}$ only for f_i, $i = 0, 1, 2$, as defined in Korovkin's first theorem. However, this is obvious, since

$$F_n(f_0) = f_0\,, \quad F_n(f_1) = \tfrac{n-1}{n} f_1\,, \quad F_n(f_2) = \tfrac{n-1}{n} f_2$$

for every $n = 1, 2, \ldots$. □

d] The set $\mathcal{T} := \cup_{n=0}^{\infty} \mathcal{T}_n$ of all real trigonometric polynomials is dense in $C(K)$, the set of all continuous, 2π periodic, real-valued functions. The set $\mathcal{T}^c := \cup_{n=0}^{\infty} \mathcal{T}_n^c$ of all complex trigonometric polynomials is dense in $C(K)$, the set of all continuous, 2π periodic, complex-valued functions.

Hint: This follows from Fejér's theorem. This is also a corollary of the Stone-Weierstrass theorem (see E.2). □

The remaining parts of the exercise follow Lorentz [86a]. Suppose that $L_n : C(K) \mapsto \mathcal{T}_n$ is a linear operator. We say that L_n preserves the elements of $\mathcal{T}_n$ if $L_n(t) = t$ for every $t \in \mathcal{T}_n$. A canonical example for such a linear operator L_n is the Fourier operator S_n. The purpose of the remaining part of the exercise is to show that the Fourier operator S_n is extremal among linear operators preserving the elements of $\mathcal{T}_n$ in the sense that it has the smallest norm. This leads to the result of Faber, Nikolaev, and Lozinskii (see part g]) that for arbitrary linear operators L_n preserving the elements of $\mathcal{T}_n$, $n = 1, 2, \ldots$, the sequence $(L_n(f))_{n=1}^{\infty}$ cannot converge for every $f \in C(K)$.

e] Berman's Generalization of a Formula of Faber and Marcinkiewicz. Let f_a denote the *a-translation* of a function $f \in C(K)$, that is, $f_a(x) := f(x + a)$. Suppose L_n is a linear operator preserving $\mathcal{T}_n$. Show that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} L_n(f_t)(x - t)\, dt = S_n(f)(x)$$

for every $f \in C(K)$ and $x \in K$.

Hint: Let

$$A_n(x) := \frac{1}{2\pi} \int_{-\pi}^{\pi} L_n(f_t)(x - t)\, dt\,.$$

Show that $A_n(f) = S_n(f)$ for every $f \in \mathcal{T}_n$. Prove that $A_n(f) = S_n(f)$ for every f of the form $f(x) = \cos mx$ or $f(x) = \sin mx$, where m is an integer greater than n. Conclude that $A_n(f) = S_n(f)$ for every $f \in \mathcal{T} := \cup_{n=0}^{\infty} \mathcal{T}_n$. Note that $\mathcal{T}$ is dense in $C(K)$. This means that to complete the proof, it is sufficient to show that $A_n : C(K) \to \mathcal{T}_n$ and $S_n : C(K) \to \mathcal{T}_n$ are continuous. Observe that $\|A_n\| \le \|L_n\|$ and $\|S_n\| \le c \log n$ for some $c > 0$; see also part f]. □

f] The Norm of the Fourier Operator S_n. Show that

$$\|S_n\| := \sup \left\{ \frac{\|S_n(f)\|_K}{\|f\|_K} : \quad f \in C(K) \right\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\sin\left(n + \frac{1}{2}\right) t}{2 \sin \frac{1}{2} t} \right| \, dt\,.$$

Use this to prove that there exist two constants $c_1 > 0$ and $c_2 > 0$ independent of n such that

$$c_1 \log n \le \|S_n\| \le c_2 \log n\,, \qquad n = 2, 3, \ldots\,.$$

Actually, it can be proved that

$$\|S_n\| = \frac{4}{\pi^2} \log n + O(1)\,, \qquad n = 2, 3, \ldots\,.$$

See, for example, Lorentz [86a].

g] The Norm of Operators that Preserve Trigonometric Polynomials. Let $L_n : C(K) \to \mathcal{T}_n$ be a linear operator preserving the elements of $\mathcal{T}_n$. Show that

$$\|L_n\| \geq \|S_n\| \geq c_1 \log n\,,$$

where $c_1 > 0$ is a constant independent of n.

Hint: Use parts e] and f]. □

E.6 Polynomials in x^{λ_n}. Given $n \in \mathbb{N}$ and $\lambda_n \in \mathbb{R}$, let

$$\mathcal{P}_n(\lambda_n) := \{p_n(x^{\lambda_n}) : p_n \in \mathcal{P}_n\}.$$

Suppose $\delta \in (0,1)$ and $\lambda_n \geq 1$ for all $n \in \mathbb{N}$. Then $\cup_{n=1}^{\infty} \mathcal{P}_n(\lambda_n)$ is dense in $C[\delta,1]$ if and only if

$$\limsup_{n\to\infty} \frac{\log n}{\lambda_n} \geq \frac{1}{2} \log \frac{1}{\delta}\,.$$

To prove the above statement, proceed as follows (see also Borwein [91b]). Denote the Chebyshev polynomial for $\mathcal{P}_n(\lambda_n)$ on $[\delta,1]$ by $T_{n,\delta}$. Denote the zeros of $T_{n,\delta}$ in $[\delta,1]$ by

$$x_{1,n}^{(\delta)} < x_{2,n}^{(\delta)} < \cdots < x_{n,n}^{(\delta)}\,.$$

Let $x_{0,n}^{(\delta)} := \delta$ and $x_{n+1,n}^{(\delta)} := 1$. Let

$$M_n(\delta) := \max_{1\leq i\leq n+1} \left(x_{i,n}^{(\delta)} - x_{i-1,n}^{(\delta)}\right)\,.$$

a] Show that

$$T_{n,\delta}(x) = T_n\left(\frac{2}{1-\delta^{\lambda_n}}\, x^{\lambda_n} - \frac{1+\delta^{\lambda_n}}{1-\delta^{\lambda_n}}\right)\,, \qquad x \in [\delta,1]\,,$$

where T_n is the Chebyshev polynomial of degree n as defined by (2.1.1).

b] Let $\delta := \liminf_{n\to\infty} x_{1,n}^{(0)}$. Show that if

$$\liminf_{n\to\infty} \left(\max_{2\leq i\leq n+1} \left(x_{i,n}^{(0)} - x_{i-1,n}^{(0)}\right)\right) = 0\,,$$

then $\liminf_{n\to\infty} M_n(\delta) = 0$.

Hint: Count the zeros of $T_{n,\delta} - T_{n,0} \in \mathcal{P}_n(\lambda_n)$ in $[\delta,1]$. □

c] Let $\delta := \liminf_{n\to\infty} x_{1,n}^{(0)}$, as in part b]. Show that

$$\frac{1}{2}\log\frac{1}{\delta} = \limsup_{n\to\infty}\frac{\log n}{\lambda_n}$$

whenever the right-hand side is finite.

Hint: Use the explicit formula for $T_{n,0}$ given in part a]. □

d] Let δ be defined by

$$\frac{1}{2}\log\frac{1}{\delta} = \limsup_{n\to\infty}\frac{\log n}{\lambda_n}\,.$$

Suppose $\delta > 0$. Show that

$$\liminf_{n\to\infty}(x_{2,n}^{(0)} - x_{1,n}^{(0)}) = 0\,.$$

Hint: Use parts a] and c]. □

e] Let $\delta := \liminf_{n\to\infty} x_{1,n}^{(0)}$, as in parts b] and c]. Suppose $\delta > 0$. Show that

$$x_{i,n}^{(0)} - x_{i-1,n}^{(0)} \le \frac{2}{\delta}\left(x_{2,n}^{(0)} - x_{1,n}^{(0)}\right)$$

for every sufficiently large $n \in \mathbb{N}$ and for every $i = 2, 3, \ldots, n+1$.

Hint: Count the zeros of

$$T_{n,0}(x) - T_{n,0}(\alpha_{i,n}x) \in \mathcal{P}_n(\lambda_n)\,,$$

where

$$\alpha_{i,n} := \frac{x_{i-1,n}^{(0)}}{x_{1,n}^{(0)}} < \frac{2}{\delta}$$

for every sufficiently large $n \in \mathbb{N}$ and for every $i = 2, 3, \ldots, n+1$. □

f] Let δ be defined by

$$\frac{1}{2}\log\frac{1}{\delta} = \limsup_{n}\frac{\log n}{\lambda_n}\,.$$

Suppose $\delta > 0$. Show that $\cup_{n=1}^{\infty}\mathcal{P}_n(\lambda_n)$ is dense in $C[\delta, 1]$.

Hint: By parts a] to e], $\liminf_{n\to\infty} M_n(\delta) = 0$. Now apply Theorem 4.1.2. □

g] Let $0 \le y < \delta$, where δ is as in part f]. Show that $\cup_{n=1}^{\infty}\mathcal{P}_n(\lambda_n)$ is not dense in $C[y, 1]$.

Hint: Show that there exists a constant c depending only on δ (and not on n or y) such that

$$|p(y)| \le c\|p\|_{[\delta,1]}$$

for every $p \in \cup_{n=1}^{\infty}\mathcal{P}_n(\lambda_n)$ and $y \in [0, \delta]$. Now use E.4 c] of Section 3.3 and part a]. □

E.10 of Section 6.2 extends part g] of the above exercise. Namely, if $0 \le \widetilde{\delta} < \delta$, where δ is the same as in part f] and $A \subset [0, 1]$ is a set of Lebesgue measure at least $1 - \widetilde{\delta}$, then $\cup_{n=1}^{\infty}\mathcal{P}_n(\lambda_n)$ is not dense in $C(A)$.

E.7 The Weierstrass Theorem in L_p. *Let $[a,b]$ be a finite interval and $p \in (0,\infty)$. Show that both $C[a,b] \cap L_p[a,b]$ and the set $\mathcal{P} := \cup_{n=0}^{\infty} \mathcal{P}_n$ of all real algebraic polynomials are dense in $L_p[a,b]$.*

Hint: The proof of the first statement is a routine measure theoretic argument; see Rudin [87]. The second statement follows from the first and the Weierstrass approximation theorem; see E.1. □

E.8 Density of Polynomials with Integer Coefficients.

a] Suppose $f \in C[0,1]$ and $f(0)$ and $f(1)$ are integers. Show that for every $\epsilon > 0$ there is a polynomial p with integer coefficients such that

$$\|f - p\|_{[0,1]} < \epsilon \,.$$

Outline. By E.4, there is an integer $n > 2/\epsilon$ so that

$$\|f - B_n(f)\|_{[0,1]} < \frac{\epsilon}{2} \,.$$

Let

$$\widetilde{B}_n(f) := \sum_{k=0}^{n} \left\lfloor f\left(\frac{k}{n}\right)\binom{n}{k}\right\rfloor x^k (1-x)^{n-k} \,.$$

Show that if $x \in [0,1]$, then

$$\begin{aligned} 0 \le B_n(f)(x) - \widetilde{B}_n(f)(x) &\le \sum_{k=1}^{n-1} x^k (1-x)^{n-k} \\ &\le \frac{1}{n}\sum_{k=0}^{n}\binom{n}{k} x^k (1-x)^{n-k} = \frac{1}{n} < \frac{\epsilon}{2} \,. \end{aligned}$$

Note that $\widetilde{B}_n(f)$ is a polynomial with integer coefficients, and

$$\begin{aligned} \|f - \widetilde{B}_n(f)\|_{[0,1]} &\le \|f - B_n(f)\|_{[0,1]} + \|B_n(f) - \widetilde{B}(f)\|_{[0,1]} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \,. \end{aligned}$$

□

b] Suppose the interval $[a,b]$ does not contain an integer. Show that polynomials with integer coefficients form a dense set in $C[a,b]$.

Proof 1. This is an immediate consequence of part a]. □

Proof 2. Assume, without loss of generality, that $[a,b] \subset (0,1)$. By the Weierstrass approximation theorem, it is sufficient to prove that for every $\epsilon > 0$ there is a polynomial p with integer coefficients such that

$$\left\| \tfrac{1}{2} - p \right\|_{[a,b]} < \epsilon \,,$$

since then all real numbers, and hence all $p \in \mathcal{P}_n$, can be approximated by polynomials with integer coefficients.

The existence of such a polynomial p follows from the identity

$$\frac{1}{2} = \frac{1-x}{1-(1-2(1-x))} = \sum_{k=0}^{\infty} (1-x)(1-2(1-x))^k \,,$$

where the infinite sum converges uniformly on $[a,b] \subset (0,1)$. □

E.9 Weierstrass Theorem on Arcs. Let ∂D denote the unit circle of the complex plane.

a] Show that the set $\mathcal{P}^c := \cup_{n=0}^{\infty} \mathcal{P}_n^c$ of all polynomials with complex coefficients is not dense in $C(\partial D)$.

Hint: Use the orthonormality of the system $((2\pi)^{-1/2} e^{in\theta})_{n=-\infty}^{\infty}$ on $[-\pi, \pi]$ to show that if k is a positive integer and $p \in \mathcal{P}^c$, then

$$2\pi \| z^{-k} - p(z) \|_{C(\partial D)} \geq \| e^{-ik\theta} - p(e^{i\theta}) \|_{L_2[-\pi,\pi]} \geq 2\pi \,.$$

So none of the functions $z^{-1}, z^{-2}, \ldots$ is in the uniform closure of $\mathcal{P}^c$ on ∂D. □

b] Let $A \subset \partial D$ be an arc of length less than 2π. Then the set $\mathcal{P}^c$ of all polynomials with complex coefficients is dense in $C(A)$.

This is a special case of Mergelyan's theorem (see, for example, Rudin [87]).

Proof. Without loss of generality, we may assume that A is symmetric with respect to the real line. By E.5 d], it is sufficient to prove that $f(z) := z^{-1}$ is in the uniform closure $\overline{\mathcal{P}^c}$ of $\mathcal{P}^c$ on ∂D (this already implies that each z^k, $k \in \mathbb{Z}$, is in $\overline{\mathcal{P}^c}$). By E.11 j] of Section 2.1, $\operatorname{cap}(A) < 1$. By E.11 g] of Section 2.1, $\mu(A) = \operatorname{cap}(A)$, where $\mu(A)$ denotes the Chebyshev constant of A. Hence

$$0 \leq \mu(A) < \alpha < 1$$

with some α. Recalling the definition of $\mu(A)$, we can deduce that there are monic polynomials $p_n \in \mathcal{P}_n^c$ such that

$$\| p_n \|_A \leq \alpha^n \,, \qquad n = 1, 2, \ldots \,.$$

For $n = 1, 2, \ldots$, let

$$q_n(z) := z^{n-1} p_n(1/z) = z^{-1} + r_{n-1}(z) \,,$$

where $r_{n-1} \in \mathcal{P}_{n-1}^c$. Since A is symmetric with respect to the real line,

$$\| z^{-1} + r_{n-1}(z) \|_A = \| q_n \|_A = \| p_n \|_A \leq \alpha^n \underset{n \to \infty}{\longrightarrow} 0 \,.$$

Hence $f(z) = z^{-1}$ is in $\overline{\mathcal{P}^c}$, which finishes the proof. □

4.2 Müntz's Theorem

A very attractive variant of the Weierstrass theorem characterizes exactly when the linear span of a system of monomials

$$\mathcal{M} := (x^{\lambda_0}, x^{\lambda_1}, \dots)$$

is dense in $C[0,1]$ or $L_2[0,1]$.

Theorem 4.2.1 (Full Müntz Theorem in $C[0,1]$). *Suppose $(\lambda_i)_{i=1}^{\infty}$ is a sequence of distinct positive numbers. Then*

$$\operatorname{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}$$

is dense in $C[0,1]$ if and only if

$$\sum_{i=1}^{\infty} \frac{\lambda_i}{\lambda_i^2 + 1} = \infty\,.$$

Note that when $\inf_i \lambda_i > 0$,

$$\sum_{i=1}^{\infty} \frac{\lambda_i}{\lambda_i^2 + 1} = \infty \quad \text{if and only if} \quad \sum_{i=1}^{\infty} \frac{1}{\lambda_i} = \infty\,.$$

Müntz studied only this case, and his theorem is usually given in terms of the second condition.

When $\lambda_i \geq 1$ for each $i = 1, 2, \dots$, the above theorem follows by a simple trick from the L_2 version of Müntz's theorem. The proof of the general case is left as a guided exercise. The difficult case to deal with is the one where 0 and ∞ are both cluster points of the sequence $(\lambda_i)_{i=0}^{\infty}$; see E.18.

Theorem 4.2.2 (Full Müntz Theorem in $L_2[0,1]$). *Suppose $(\lambda_i)_{i=0}^{\infty}$ is a sequence of distinct real numbers greater than $-1/2$. Then*

$$\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

is dense in $L_2[0,1]$ if and only if

$$\sum_{i=0}^{\infty} \frac{2\lambda_i + 1}{(2\lambda_i + 1)^2 + 1} = \infty\,.$$

The proof of the following full L_1 version of Müntz's theorem is presented as E.19.

Theorem 4.2.3 (Full Müntz Theorem in $L_1[0,1]$). *Suppose $(\lambda_i)_{i=0}^{\infty}$ is a sequence of distinct real numbers greater than -1. Then*

$$\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

is dense in $L_1[0,1]$ if and only if

$$\sum_{i=0}^{\infty} \frac{\lambda_i + 1}{(\lambda_i + 1)^2 + 1} = \infty .$$

Now we formulate a general Müntz-type theorem in $L_p[0,1]$, that contains the above $C[0,1]$, $L_2[0,1]$, and $L_1[0,1]$ results as special cases. The proof of this theorem is outlined in E.20.

Theorem 4.2.4 (Full Müntz Theorem in $L_p[0,1]$). *Let $p \in [1,\infty)$. Suppose $(\lambda_i)_{i=0}^{\infty}$ is a sequence of distinct real numbers greater than $-1/p$. Then*

$$\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

is dense in $L_p[0,1]$ if and only if

$$\sum_{i=0}^{\infty} \frac{\lambda_i + \frac{1}{p}}{\left(\lambda_i + \frac{1}{p}\right)^2 + 1} = \infty .$$

The full version of Müntz's theorem for arbitrary distinct real exponents on an interval $[a,b]$, $0 < a < b$, is given in E.7 and E.9.

Proof of Theorem 4.2.1 assuming Theorem 4.2.2 and each $\lambda_i \geq 1$. We need the following two inequalities:

$$\begin{aligned} (4.2.1) \quad \left| x^m - \sum_{i=0}^{n} a_i x^{\lambda_i} \right| &= \left| \int_0^x \left(m t^{m-1} - \sum_{i=0}^{n} a_i \lambda_i t^{\lambda_i - 1} \right) dt \right| \\ &\leq \int_0^1 \left| m t^{m-1} - \sum_{i=0}^{n} a_i \lambda_i t^{\lambda_i - 1} \right| dt \\ &\leq \left(\int_0^1 \left| m t^{m-1} - \sum_{i=0}^{n} a_i \lambda_i t^{\lambda_i - 1} \right|^2 dt \right)^{1/2} \end{aligned}$$

for every $x \in [0,1]$ and $m = 1, 2, \dots$, and

$$(4.2.2) \qquad \left(\int_0^1 \left| t^m - \sum_{i=0}^{n} a_i t^{\lambda_i} \right|^2 dt \right)^{1/2} \leq \left\| x^m - \sum_{i=0}^{n} a_i x^{\lambda_i} \right\|_{[0,1]}$$

for every $m = 0, 1, 2, \ldots$. The assumption that $\lambda_i \geq 1$ for each i implies that

$$\sum_{i=0}^{\infty} \frac{\lambda_i}{\lambda_i^2 + 1} = \infty \quad \text{if and only if} \quad \sum_{i=0}^{\infty} \frac{2(\lambda_i - 1) + 1}{(2(\lambda_i - 1) + 1)^2 + 1} = \infty$$

and

$$\sum_{i=0}^{\infty} \frac{\lambda_i}{\lambda_i^2 + 1} = \infty \quad \text{if and only if} \quad \sum_{i=0}^{\infty} \frac{2\lambda_i + 1}{(2\lambda_i + 1)^2 + 1} = \infty .$$

If $\sum_{i=0}^{\infty} \lambda_i/(\lambda_i^2 + 1) = \infty$, then (4.2.1), together with Theorem 4.2.2 and the Weierstrass approximation theorem (see E.1 of Section 4.1), shows that

$$\operatorname{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \ldots\}$$

is dense in $C[0,1]$.

If the above span is dense in $C[0,1]$, then (4.2.2), together with E.7 of Section 4.1, shows that it is also dense in $L_2[0,1]$. Hence Theorem 4.2.2 implies $\sum_{i=1}^{\infty} \lambda_i/(\lambda_i^2 + 1) = \infty$. □

Proof of Theorem 4.2.2. We consider the approximation to x^m by elements of $\operatorname{span}\{x^{\lambda_0}, \ldots, x^{\lambda_{n-1}}\}$ in $L_2[0,1]$, and we assume $m > -\frac{1}{2}$ and $m \neq \lambda_i$ for any i. In the notation of Section 3.4 we define

$$\Lambda := (\lambda_0, \lambda_1, \ldots, \lambda_{n-1}, m)$$

and study L_n^*, the nth orthonormal Müntz-Legendre polynomial associated with Λ. By (3.4.8) and (3.4.6) we have (with $\lambda_n := m$)

$$L_n^*(x) = a_n x^m + \sum_{i=0}^{n-1} a_i x^{\lambda_i} ,$$

where

$$|a_n| = \sqrt{1 + 2m} \prod_{i=0}^{n-1} \left| \frac{m + \lambda_i + 1}{m - \lambda_i} \right| .$$

It follows from $\|L_n^*\|_{L^2[0,1]} = 1$ and orthogonality that L_n^*/a_n is the error term in the best $L_2[0,1]$ approximation to x^m from $\operatorname{span}\{x^{\lambda_0}, \ldots, x^{\lambda_{n-1}}\}$ (why?). Therefore

$$\min_{b_i \in \mathbb{C}} \left\| x^m - \sum_{i=0}^{n-1} b_i x^{\lambda_i} \right\|_{L^2[0,1]} = \frac{1}{|a_n|} = \frac{1}{\sqrt{1 + 2m}} \prod_{i=0}^{n-1} \left| \frac{m - \lambda_i}{m + \lambda_i + 1} \right| .$$

So, for a nonnegative integer m different from any of the exponents λ_i,

$$x^m \in \overline{\text{span}}\{x^{\lambda_0}, x^{\lambda_1}, \dots\} \tag{4.2.3}$$

(where $\overline{\text{span}}$ denotes the $L_2[0,1]$ closure of the span) if and only if

$$\lim_{n\to\infty} \prod_{i=0}^{n-1} \left| \frac{m-\lambda_i}{m+\lambda_i+1} \right| = 0\,.$$

That is, (4.2.3) holds if and only if

$$\lim_{n\to\infty} \prod_{\substack{i=0 \\ \lambda_i > m}}^{n-1} \left| 1 - \frac{2m+1}{m+\lambda_i+1} \right| \prod_{\substack{i=0 \\ -1/2 < \lambda_i \le m}}^{n-1} \left| 1 - \frac{2\lambda_i+1}{m+\lambda_i+1} \right| = 0\,.$$

Hence (4.2.3) holds if and only if either

$$\sum_{\substack{i=0 \\ \lambda_i > m}}^{\infty} \frac{1}{2\lambda_i+1} = \infty \quad \text{or} \quad \sum_{\substack{i=0 \\ -1/2 < \lambda_i \le m}}^{\infty} (2\lambda_i+1) = \infty\,,$$

which is the case if and only if

$$\sum_{i=0}^{\infty} \frac{2\lambda_i+1}{(2\lambda_i+1)^2+1} = \infty\,,$$

and the proof can be finished by the Weierstrass approximation theorem (see E.1 of Section 4.1). □

Comments, Exercises, and Examples.

Theorem 4.2.1 (in the case when $\inf\{\lambda_i : i \in \mathbb{N}\} > 0$) and Theorem 4.2.2 were proved independently by Müntz [14] and Szász [16]. Szász [16] proved the full version of Theorem 4.2.2. Theorem 4.2.1 is to be found in Borwein and Erdélyi [to appear 5]. Much of Theorem 4.2.4 is stated in Schwartz [59] without proof and may be deduced by his methods. Indeed, Schwartz's method appears to give Theorem 4.2.4 for $p \in [1,2]$. Johnson (private communication) and Operstein [to appear] show how to derive the full Theorem 4.2.4 from Theorem 4.2.1 as does E.20; see also E.7 of the next section.

Less complete versions of the results presented in this section are often called the Müntz-Szász Theorems. A 1912 version due to Bernstein can be found in his collected works.

A variant on our proof of Müntz's theorem is presented in E.2. A distinct proof based on possible zero sets of analytic functions may be found in Feinerman and Newman [76]; see also E.10, where this method is explored for denseness questions for $\{\cos \lambda_k \theta\}$.

Extensions of Müntz's theorem abound. For example, generalizations to complex exponents are considered in Luxemburg and Korevaar [71], to angular regions in Anderson [72] and with an exponential weight on $[0,\infty)$ in Fuchs [46]. It is a nontrivial problem to establish a Müntz-type theorem on an interval $[a,b]$, $a>0$, in which case the elements of the sequence $\Lambda=(\lambda_i)_{i=0}^{\infty}$ are allowed to be *arbitrary distinct real* numbers. This is the content of E.7; it is due to Clarkson and Erdős [43] (in the case when each λ_i is a nonnegative integer) and Schwartz [59] (in the general case). It is shown in Section 6.2 that if $\Lambda=(\lambda_i)_{i=0}^{\infty}$ is an increasing sequence of nonnegative real numbers, then the interval $[0,1]$ in Müntz's theorem (Theorem 4.2.1) can be replaced by an arbitrary compact set $A\subset[0,\infty)$ of positive Lebesgue measure.

The exercises also explore in detail the closure of Müntz spaces in the nondense cases. This study was initiated by Clarkson and Erdős [43], who treated the case when the exponents are nonnegative integers. The considerably harder general case is due to Schwartz [59].

Denseness questions about quotients and products of Müntz polynomials from a given Müntz space are discussed in Sections 4.4 and 6.2, respectively.

Some of the literature on the multivariate versions of Müntz's theorem can be found in Ogawa and Kitahara [87], Bloom [90], and Kroó and Szabados [94].

E.1 Another Proof of Some Cases of Müntz's Theorem.

a] Golitsckek's Proof of Müntz's Theorem when $\sum_{i=1}^{\infty}1/\lambda_i=\infty$. Suppose that $(\lambda_i)_{i=1}^{\infty}$ is a sequence of distinct, positive real numbers satisfying $\sum_{i=1}^{\infty}1/\lambda_i=\infty$. Golitschek [83] gives the following simple argument to show that $\operatorname{span}\{1,x^{\lambda_1},x^{\lambda_2},\dots\}$ is dense in $C[0,1]$.

Proof. Assume that $m\neq\lambda_k$, $k=0,1,\dots$, and define the functions Q_n inductively: $Q_0(x):=x^m$ and

$$Q_n(x):=(\lambda_n-m)x^{\lambda_n}\int_x^1 Q_{n-1}(t)t^{-1-\lambda_n}\,dt\,,\qquad n=1,2,\dots\,.$$

Show, by induction on n, that each Q_n is of the form

$$Q_n(x)=x^m-\sum_{i=0}^{n}a_{n,i}x^{\lambda_i}\,,\qquad a_{n,i}\in\mathbb{R}\,.$$

Show also that

$$\|Q_0\|_{[0,1]}=1\qquad\text{and}\qquad\|Q_n\|_{[0,1]}\le\left|1-\frac{m}{\lambda_n}\right|\cdot\|Q_{n-1}\|_{[0,1]}\,,$$

so

$$\|Q_n\|_{[0,1]}\le\prod_{i=0}^{n}\left|1-\frac{m}{\lambda_i}\right|\to 0\qquad\text{as } n\to\infty\,.$$

□

b] Another Proof of Müntz's Theorem when $\lambda_i \to c > 0$. Suppose that $(\lambda_i)_{i=1}^{\infty}$ is a sequence of distinct positive real numbers that converges to $c > 0$. Show, without using the arguments given in the proof of Müntz's theorem, that $\operatorname{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}$ is dense in $C[a,b]$, $a > 0$.

Hint: Let k be a nonnegative integer. Use divided differences to approximate $x^c \log^k x$ uniformly on $[a,b]$. Finish the proof by using the Weierstrass approximation theorem (see E.1 of Section 4.1). □

E.2 Another Proof of Müntz's Theorem in $L_2[0,1]$.

a] Gram's Lemma. *Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space, and let $g \in V$. Suppose $\{f_1, \dots, f_n\}$ is a basis for an n-dimensional subspace P of V. Then the distance d_n from g to P is given by*

$$d_n := \inf\{\langle g-p, g-p\rangle^{1/2} : p \in P\} = \left(\frac{G(f_1, f_2, \dots, f_n, g)}{G(f_1, f_2, \dots, f_n)}\right)^{1/2},$$

where G is the Gram determinant

$$G(f_1, f_2, \dots, f_m) := \begin{vmatrix} \langle f_1, f_1\rangle & \dots & \langle f_1, f_m\rangle \\ \vdots & \ddots & \vdots \\ \langle f_m, f_1\rangle & \dots & \langle f_m, f_m\rangle \end{vmatrix}.$$

Proof. As in Theorem 2.2.3, the best approximation to g from P is given by

$$f^* = \sum_{i=1}^{n} c_i f_i,$$

where the c_i are uniquely determined by the orthogonality conditions

$$\langle f^* - g, f_k\rangle = 0, \qquad k = 1, 2, \dots, n.$$

Since

$$d_n^2 = \langle g - f^*, g - f^*\rangle,$$

we are led to a system of $n+1$ equations

$$\sum_{i=1}^{n} c_i \langle f_i, f_k\rangle = \langle g, f_k\rangle, \qquad k = 1, 2, \dots, n$$

and

$$\sum_{i=1}^{n} c_i \langle f_i, g\rangle + d_n^2 = \langle g, g\rangle.$$

Solving this system by using Cramer's rule, we get the desired result. □

b] As in E.3 of Section 3.2,

$$\begin{vmatrix} \frac{1}{\alpha_1+\beta_1} & \cdots & \frac{1}{\alpha_1+\beta_n} \\ \vdots & \ddots & \vdots \\ \frac{1}{\alpha_n+\beta_1} & \cdots & \frac{1}{\alpha_n+\beta_n} \end{vmatrix} = \frac{\prod\limits_{1\le i<j\le m}(\alpha_j-\alpha_i)(\beta_j-\beta_i)}{\prod\limits_{1\le i,j\le n}(\alpha_i+\beta_j)}$$

for arbitrary complex numbers α_i and β_j with $\alpha_i+\beta_j\neq 0$.

c] Let $\gamma,\lambda_0,\ldots,\lambda_n$ be distinct real numbers greater than $-1/2$. Then the $L_2[0,1]$ distance d_n from x^γ to $\operatorname{span}\{x^{\lambda_0},\ldots,x^{\lambda_n}\}$ is given by

$$d_n=\frac{1}{\sqrt{2\gamma+1}}\prod_{i=0}^{n}\left|\frac{\gamma-\lambda_i}{\gamma+\lambda_i+1}\right|.$$

Hint: In $L_2[0,1]$,

$$\langle x^a,x^b\rangle=\int_0^1 x^a x^b\,dx=\frac{1}{a+b+1},\qquad a,b\in(-1/2,\infty).$$

Now apply parts a] and b]. □

d] Complete the proof of Müntz's theorem in $L_2[0,1]$.

E.3 More on Müntz's Theorem in the Nondense Case. We assume throughout this exercise that $(\lambda_i)_{i=0}^{\infty}$ is a sequence of nonnegative real numbers satisfying

$$\sum_{i=1}^{\infty}\frac{1}{\lambda_i}<\infty$$

and the gap condition

$$\inf\{\lambda_i-\lambda_{i-1}:i\in\mathbb{N}\}>0$$

holds. Some of the results of this exercise hold even if the above gap condition is removed (see the later exercises).

a] Show that

$$0<\prod_{\substack{i=0\\ i\neq m}}^{\infty}\left|\frac{\lambda_i+\lambda_m}{\lambda_i-\lambda_m}\right|=\exp(\gamma_m\lambda_m),$$

where $\gamma_m\to 0$ as $m\to\infty$.

Hint: First show that the above infinite product exists. Write the above product as

$$\prod_{\substack{i=0\\ \lambda_i<\lambda_m}}^{\infty}\left|\frac{\lambda_i+\lambda_m}{\lambda_i-\lambda_m}\right|\prod_{\substack{i=0\\ \lambda_i\in(\lambda_m,2\lambda_m)}}^{\infty}\left|\frac{\lambda_i+\lambda_m}{\lambda_i-\lambda_m}\right|\prod_{\substack{i=0\\ \lambda_i\ge 2\lambda_m}}^{\infty}\left|\frac{\lambda_i+\lambda_m}{\lambda_i-\lambda_m}\right|$$

and estimate the three factors above separately. □

b] Deduce that if $\lambda_i \neq \lambda_m$ for each i, then

$$\begin{aligned}\left\|x^{\lambda_m} - p(x)\right\|_{L_2[0,1]} &\geq \frac{1}{\sqrt{2\lambda_m + 1}} \prod_{i=0}^{\infty} \left| \frac{(\lambda_i + \frac{1}{2}) - (\lambda_m + \frac{1}{2})}{(\lambda_i + \frac{1}{2}) + (\lambda_m + \frac{1}{2})} \right| \\ &= \exp(-\gamma_m(\lambda_m))\end{aligned}$$

for every $p \in \operatorname{span}\{x^{\lambda_0}, \ldots, x^{\lambda_n}\}$, where $\gamma_m \to 0$ as $m \to \infty$.

c] Show that for every $\epsilon > 0$ there is a constant c_ϵ depending only on ϵ and $(\lambda_i)_{i=0}^{\infty}$ (but not on the number of terms in p) such that

$$|a_i| \leq c_\epsilon (1+\epsilon)^{\lambda_i} \|p\|_{L_2[0,1]}$$

for every $p \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \ldots\}$ of the form $p(x) = \sum_{i=0}^{n} a_i x^{\lambda_i}$.

Hint: Use part b]. □

d] Bounded Bernstein-Type Inequality. Let $\lambda_0 = 0$ and $\lambda_1 \geq 1$. Show that for every $\epsilon \in (0,1)$ there is a constant c_ϵ depending only on ϵ and $(\lambda_i)_{i=0}^{\infty}$ (but not on the number of terms in p) such that

$$\|p'\|_{[0,1-\epsilon]} \leq c_\epsilon \|p\|_{L_2[0,1]}$$

and hence

$$\|p'\|_{[0,1-\epsilon]} \leq c_\epsilon \|p\|_{[0,1]}$$

for every $p \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \ldots\}$.

Hint: Use part c]. □

The result of the next part is due to Clarkson and Erdős [43].

e] The Closure of a Nondense Müntz Space. Suppose $f \in C[0,1]$ and there exist $p_n \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \ldots\}$ of the form

$$p_n(x) = \sum_{i=0}^{k_n} a_{i,n} x^{\lambda_i}, \qquad a_{i,n} \in \mathbb{R}, \quad n = 1, 2, \ldots$$

such that $\lim_{n\to\infty} \|p_n - f\|_{[0,1]} = 0$. Show that f is of the form

$$f(x) = \sum_{i=0}^{\infty} a_i x^{\lambda_i}, \qquad a_{i,n} \in \mathbb{R}, \quad x \in [0,1).$$

Show also that f can be extended analytically throughout the region

$$\{z \in \mathbb{C} \backslash (-\infty, 0] : |z| < 1\}$$

and

$$\lim_{n\to\infty} a_{i,n} = a_i, \qquad i = 0, 1, \ldots.$$

If $(\lambda_i)_{i=0}^{\infty}$ is a sequence of distinct nonnegative integers, then f can be extended analytically throughout the open unit disk.

Hint: Use part c]. □

If $(\lambda_i)_{i=0}^{\infty}$ is *lacunary* (that is, $\inf\{\lambda_{i+1}/\lambda_i : i \in \mathbb{N}\} > 1$), then the uniform closure of $\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ on $[0,1]$ is exactly

$$\left\{ f \in C[0,1] : f(x) = \sum_{i=0}^{\infty} a_i x^{\lambda_i}, \quad x \in [0,1] \right\}.$$

If $(\lambda_i)_{i=0}^{\infty}$ is not lacunary, then this fails, namely, there exists a function f of the form

$$f(x) = \sum_{i=0}^{\infty} a_i x^{\lambda_i}, \qquad x \in [0,1)$$

in the uniform closure of $\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ on $[0,1]$ such that the right-hand side does not converge at the endpoint 1; see Clarkson and Erdős [43].

f] Bounded Chebyshev-Type Inequality. Show that for every $\epsilon \in (0,1)$ there exists a constant c_ϵ depending only on ϵ and $(\lambda_i)_{i=0}^{\infty}$ (but not on the number of terms in p) such that

$$\|p\|_{[0,1]} \le c_\epsilon \|p\|_{[1-\epsilon,1]}$$

for every $p \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_i}, \dots\}$.

Outline. Using the scaling $x \to x^{1/\lambda_1}$, without loss of generality we may assume that $\lambda_1 = 1$. Suppose there exists a sequence

$$(p_m)_{m=1}^{\infty} \subset \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, \}, \qquad m = 1, 2, \dots$$

such that

$$0 < A_m := \|p_m\|_{[0,1]} \to \infty$$

while

$$\|p_m\|_{[1-\epsilon,1]} = 1, \qquad m = 1, 2, \dots.$$

Let $q_m := p_m/A_m$. Note that $\|q_m\|_{[0,1]} = 1$ for each m, and $\|q_m\|_{[1-\epsilon,1]} \to 0$ as $m \to \infty$. Then, by part d],

$$\|q_m'\|_{[0,1-\delta]} \le c_\delta$$

for every $\delta \in (0,1)$. Hence $(q_m)_{m=0}^{\infty}$ is a sequence of uniformly bounded and equicontinuous functions on closed subintervals of $[0,1)$, and by the Arzela-Ascoli theorem (see, for example, Rudin [87]) we may extract a uniformly convergent subsequence on $[0, 1-\epsilon/2]$. This subsequence, by part e], converges uniformly to a function F analytic on $(0, 1-\epsilon/2)$, but since $\|q_m\|_{[1-\epsilon,1]} \to 0$, F must be identically zero. This is a contradiction since $\|q_m\|_{[0,1]} = 1$ and $\|q_m\|_{[0,1-\epsilon]} = \|q_m\|_{[0,1]}$ for every sufficiently large m. □

g] Suppose $(q_m)_{m=1}^{\infty} \subset \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ and $\|q_m\|_{[a,b]} \le 1$ for each m, where $0 \le a < b$. Show that there is a subsequence of $(q_m)_{m=1}^{\infty}$ that converges uniformly on every closed subinterval of $[0,b)$.

Hint: Use parts f] and d] and the Arzela-Ascoli theorem. □

E.4 Müntz's Theorem with Real Exponents on $[a,b]$, $a > 0$. Suppose $(\lambda_i)_{i=-\infty}^{\infty}$ is a set of distinct real numbers satisfying

$$\sum_{\substack{i=-\infty \\ \lambda_i \neq 0}}^{\infty} \frac{1}{|\lambda_i|} < \infty$$

with $\lambda_i < 0$ for $i < 0$ and $\lambda_i \geq 0$ for $i \geq 0$. Suppose that the gap condition

$$\inf\{\lambda_i - \lambda_{i-1} : i \in \mathbb{Z}\} > 0$$

holds. Associated with

$$p(x) := \sum_{i=-n}^{n} a_i x^{\lambda_i}, \qquad n = 0, 1, \ldots$$

let

$$p^-(x) := \sum_{i=-n}^{-1} a_i x^{\lambda_i} \qquad \text{and} \qquad p^+(x) := \sum_{i=0}^{n} a_i x^{\lambda_i}.$$

Let $0 < a < b$.

a] Show that there exists a constant c depending only on a, b, and $(\lambda_i)_{i=-\infty}^{\infty}$ (but not on the number of terms in p) such that

$$\|p^+\|_{[a,b]} \leq c\|p\|_{[a,b]} \qquad \text{and} \qquad \|p^-\|_{[a,b]} \leq c\|p\|_{[a,b]}$$

for every $p \in \operatorname{span}\{x^{\lambda_i}\}_{i=-\infty}^{\infty}$.

Outline. It is sufficient to prove only the first inequality; the second inequality follows from the first by the substitution $y = x^{-1}$. If the first inequality fails to hold, then there exists a sequence $(p_n)_{n=1}^{\infty} \subset \operatorname{span}\{x^{\lambda_i}\}_{i=-\infty}^{\infty}$ such that

$$\|p_n^+\|_{[a,b]} = 1, \qquad n = 1, 2, \ldots, \qquad \text{and} \qquad \lim_{n\to\infty} \|p_n\|_{[a,b]} = 0.$$

Since $p = p^+ + p^-$, the above relations imply that

$$\|p_n^-\|_{[a,b]} \leq K < \infty, \qquad n = 1, 2, \ldots.$$

By E.3 g] and E.3 e], there exists a subsequence $(n_i)_{i=1}^{\infty}$ such that $(p_{n_i}^+)_{i=1}^{\infty}$ converges uniformly on every closed subinterval of $[0, b)$ to a function f analytic on

$$D_b := \{z \in \mathbb{C}\backslash(-\infty, 0] : |z| < b\}$$

of the form

$$f(z) = \sum_{i=0}^{\infty} a_i z^{\lambda_i}, \qquad z \in D_b,$$

while $(p^-_{n_i})_{i=1}^{\infty}$ converges uniformly on every closed subinterval of (a,∞) to a function g analytic on

$$E_a := \{z \in \mathbb{C} \setminus (-\infty, 0] : |z| > a\}$$

of the form

$$g(z) = \sum_{i=-\infty}^{-1} a_i z^{\lambda_i}, \qquad z \in E_a, \qquad \lim_{\substack{x\to\infty \\ x\in\mathbb{R}}} g(x) = 0.$$

Now $\lim_{i\to\infty} \|p_{n_i}\|_{[a,b]} = 0$ and $p_{n_i} = p^+_{n_i} + p^-_{n_i}$ imply that $f + g = 0$ on (a,b). Show that

$$h(z) := \begin{cases} f(e^z), & \mathrm{Re} z < \log b \\ -g(e^z), & \mathrm{Re} z > \log a \end{cases}$$

is a well-defined bounded entire function, and hence $h = 0$ on $\mathbb{C}$ by Liouville's theorem. From this, deduce that

$$f = 0 \text{ on } [0,b) \qquad \text{and} \qquad g = 0 \text{ on } (a,\infty).$$

Hence, for every $y \in (a,b)$,

$$\lim_{i\to\infty} \|p^+_{n_i}\|_{[a,y]} = 0$$

and

$$\lim_{i\to\infty} \|p^+_{n_i}\|_{[y,b]} = \lim_{i\to\infty} \|p_{n_i} - p^-_{n_i}\|_{[y,b]} = 0.$$

Therefore

$$\lim_{i\to\infty} \|p^+_{n_i}\|_{[a,b]} = 0,$$

which contradicts $\|p^+_n\|_{[a,b]} = 1$, $n = 1, 2, \ldots$. □

b] The Closure of Müntz Polynomials. Let $f \in C[0,1]$, and suppose there exist Müntz polynomials $p_n \in \mathrm{span}\{x^{\lambda_i}\}_{i=-\infty}^{\infty}$ of the form

$$p_n(x) = \sum_{i=-k_n}^{k_n} a_{i,n} x^{\lambda_i}, \qquad n = 1, 2, \ldots$$

such that $\lim_{n\to\infty} \|p_n - f\|_{[a,b]} = 0$. Show that f is of the form

$$f(x) = \sum_{i=-\infty}^{\infty} a_i x^{\lambda_i}, \qquad x \in (a,b),$$

where

$$f^+(x) := \sum_{i=1}^{\infty} a_i x^{\lambda_i}\,, \qquad x \in [0,b)\,,$$

$$f^-(x) := \sum_{i=-\infty}^{-1} a_i x^{\lambda_i}\,, \qquad x \in (a,\infty)\,, \qquad \lim_{x\to\infty} f^-(x) = 0\,,$$

f can be extended analytically throughout the region

$$\{z \in \mathbb{C} \setminus (-\infty, 0] : a < |z| < b\}\,,$$

and

$$\lim_{n\to\infty} a_{i,n} = a_i\,, \qquad i \in \mathbb{Z}\,.$$

Hint: Use part a] and E.3 e]. □

E.5 Removing the Gap Conditions. Assume throughout this exercise that $0 \le \lambda_0 < \lambda_1 < \cdots$ and $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$.

a] Bounded Chebyshev-Type Inequality. Show that for every $\epsilon \in (0,1)$ there is a constant c_ϵ depending only on ϵ and $(\lambda_i)_{i=0}^{\infty}$ (but not on the number of terms in p) such that

$$\|p\|_{[0,1]} \le c_\epsilon \|p\|_{[1-\epsilon,1]}$$

for every $p \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$. (This is the inequality of E.3 f] without the gap condition $\inf\{\lambda_i - \lambda_{i-1} : i \in \mathbb{N}\} > 0$.)

Hint: Assume, without loss of generality, that $\lambda_0 = 0$. Observe that $\lim_{i\to\infty} \lambda_i/i = \infty$. Choose $m \in \mathbb{N}$ such that $\lambda_i > 2i$ whenever $i > m$. Define $\Gamma := (\gamma_i)_{i=1}^{\infty}$ by

$$\gamma_i := \begin{cases} \min\{\lambda_i, i\}\,, & i = 0, 1, \dots, m \\ \frac{1}{2}\lambda_i + i\,, & i = m+1, m+2, \dots\,. \end{cases}$$

Then

$$0 = \gamma_0 < \gamma_1 < \cdots\,, \qquad \sum_{i=1}^{\infty} \frac{1}{\gamma_i} < \infty\,, \qquad \gamma_i \le \lambda_i\,, \qquad i = 0, 1, \dots$$

and $\inf\{\gamma_i - \gamma_{i-1} : i \in \mathbb{N}\} > 0$. Now use E.3 g] of Section 3.3 with $[a,b] = [1-\epsilon, 1]$ and E.3 f] of this section. □

b] Bounded Bernstein-Type Inequality. Suppose $\lambda_0 = 0$ and $\lambda_1 \ge 1$. Prove that for every $\epsilon \in (0,1)$ there is a constant c_ϵ depending only on ϵ and $(\lambda_i)_{i=0}^{\infty}$ (but not on the number of terms in p) such that

$$\|p'\|_{[0,1-\epsilon]} \le c_\epsilon \|p\|_{[0,1]}$$

for every $p \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$. (This is the second inequality of E.3 d] without the gap condition $\inf\{\lambda_i - \lambda_{i-1} : i \in \mathbb{N}\} > 0$.)

Hint: Define the sequence Γ as in the hint given to part a]. Now use E.3 f] of Section 3.3 with $[a,b]$, $a \in (0, 1-\epsilon]$, E.3 g] of this section, and part a] of this exercise. □

c] Let $0 \le a < b$. Show that $\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ is not dense in $C[a,b]$.

Hint: Use part a] and Theorem 4.2.1 (full Müntz's theorem in $C[0,1]$). □

d] Let

$$\lambda_{2k-1} := k^2\,, \qquad \lambda_{2k} := k^2 + 2^{-k^2}\,, \qquad k = 1, 2, \dots\,,$$
$$a_{2k-1} := 2^{k^2}\,, \qquad a_{2k} := -2^{k^2}\,, \qquad k = 1, 2, \dots\,.$$

Show that the function $f(x) := \sum_{i=1}^{\infty} \left(a_{2i-1} x^{\lambda_{2i-1}} + a_{2i} x^{\lambda_{2i}}\right)$ is a well-defined continuous function on $[0,1]$. Show also that $\sum_{i=1}^{\infty} a_i x^{\lambda_i}$ does not converge for any $x \in (0,\infty)$ (hence the conclusions of E.3 e] are not valid without a gap condition).

E.6 A Comparison Theorem. *Let $0 \le k \le n$ be fixed integers. Assume*

$$\lambda_0 < \lambda_1 < \cdots < \lambda_k < 0 < \lambda_{k+1} < \lambda_{k+2} < \cdots < \lambda_n\,,$$
$$\gamma_0 < \gamma_1 < \cdots < \gamma_k < 0 < \gamma_{k+1} < \gamma_{k+2} < \cdots < \gamma_n\,,$$

and

$$|\gamma_i| \le |\lambda_i|\,, \qquad i = 0, 1, \dots, n$$

with strict inequality for at least one index i. Let

$$H_n := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\} \quad \text{and} \quad G_n := \operatorname{span}\{x^{\gamma_0}, x^{\gamma_1}, \dots, x^{\gamma_n}\}$$

and let $0 < a < b$. Then

$$\min_{p \in G_n} \|1 - p\|_{[a,b]} < \min_{p \in H_n} \|1 - p\|_{[a,b]}\,.$$

Hint: Let $q^* \in H_n$ be the best approximation to $x^0 \equiv 1$ on $[a,b]$. Let

$$r(x) = (-1)x^0 + \sum_{i=0}^{n} x^{\gamma_i} \in \operatorname{span}\{x^{\gamma_0}, \dots, x^{\gamma_k}, x^0, x^{\gamma_{k+1}}, \dots, x^{\gamma_n}\}$$

interpolate

$$q^* - 1 \in \operatorname{span}\{x^{\lambda_0}, \dots, x^{\lambda_k}, x^0, x^{\lambda_{k+1}}, \dots, x^{\lambda_n}\}$$

at the $n+1$ distinct zeros, $x_1, x_2, \dots, x_{n+1}$, of $q^* - 1$ on $[a,b]$ (see Theorem 3.1.6). Use Theorem 3.2.5 to show that

$$|r(x)| \le |q(x)|\,, \qquad x \in [a,b]$$

with strict inequality for $x \ne x_i$. Finally show that if $p^* := r + 1$, then

$$\min_{p \in G_n} \|1 - p\|_{[a,b]} \le \|1 - p^*\|_{[a,b]} < \|1 - q^*\|_{[a,b]} = \min_{p \in H_n} \|1 - p\|_{[a,b]}\,.$$

□

E.7 Full Müntz Theorem on $[a,b]$, $a>0$. Let $(\lambda_i)_{i=0}^{\infty}$ be a sequence of distinct real numbers, and let $0<a<b$. Show that $\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ is dense in $C[a,b]$ if and only if

$$\sum_{\substack{i=0\\ \lambda_i\neq 0}}^{\infty} \frac{1}{|\lambda_i|} = \infty\,.$$

Hint: Distinguish the following cases.

Case 1: The sequence $(\lambda_i)_{i=0}^{\infty}$ has a cluster point $0\neq\lambda\in\mathbb{R}$. Use Theorem 4.2.1 to show that $\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ is dense in $C[a,b]$.

Case 2: The point 0 is a cluster point of $(\lambda_i)_{i=1}^{\infty}$. Use Case 1 to show first that $\operatorname{span}\{x^{\lambda_0+1}, x^{\lambda_1+1}, \dots\}$ is dense in $C[a,b]$, and recall that $a>0$.

Case 3: The sequence $(\lambda_i)_{i=0}^{\infty}$ does not have any (finite) cluster points, and either

$$\sum_{\substack{i=0\\ \lambda_i>0}}^{\infty} \frac{1}{\lambda_i} = \infty \qquad \text{or} \qquad \sum_{\substack{i=0\\ \lambda_i<0}}^{\infty} \frac{1}{|\lambda_i|} = \infty\,.$$

Use Theorem 4.2.1 to show that $\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ is dense in $C[a,b]$.

Case 4:

$$\sum_{\substack{i=0\\ \lambda_i\neq 0}}^{\infty} \frac{1}{|\lambda_i|} < \infty\,.$$

Without loss of generality we may assume that $0\notin\{\lambda_i\}_{i=0}^{\infty}$ (why?). By a change of scaling, we may also assume that $[a,b]=[1-\epsilon,1]$. Let

$$\{\widetilde{\lambda}_i\}_{i=-\infty}^{\infty} = \{\lambda_i\}_{i=0}^{\infty}\,, \quad \text{where} \quad \cdots < \widetilde{\lambda}_{-2} < \widetilde{\lambda}_{-1} < 0 < \widetilde{\lambda}_0 < \widetilde{\lambda}_1 < \cdots\,.$$

Show that there is a sequence $(\gamma_i)_{i=-\infty}^{\infty}$ satisfying

$$\cdots < \gamma_{-2} < \gamma_{-1} < 0 < \gamma_0 < \gamma_1 < \cdots\,,$$

$$|\gamma_i| < |\widetilde{\lambda}_i|\,, \qquad i\in\mathbb{Z}\,, \qquad \sum_{i=-\infty}^{\infty} \frac{1}{|\gamma_i|} < \infty\,,$$

and the gap condition

$$\inf\{\gamma_i - \gamma_{i-1} : i\in\mathbb{Z}\} > 0\,.$$

Use E.4 a], E.5 a], and E.2 c] to show that

$$1\notin \overline{\operatorname{span}}\{x^{\gamma_i}\}_{i=-\infty}^{\infty}\,,$$

where $\overline{\operatorname{span}}\{x^{\gamma_i}\}_{i=-\infty}^{\infty}$ denotes the uniform closure of the span on $[a,b]$. Finally use E.6 to show that

$$1\notin \overline{\operatorname{span}}\{x^{\bar{\lambda}_i}\}_{i=-\infty}^{\infty} = \overline{\operatorname{span}}\{x^{\lambda_i}\}_{i=0}^{\infty}\,.$$

□

E.8 Further Results for Nonnegative Sequences with No Gap Condition. Assume throughout this exercise that $0 \le \lambda_0 < \lambda_1 < \cdots$, $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$, and $0 \le a < b$.

a] Show that for every $\epsilon \in (0, b)$ there is a constant c_ϵ depending only on ϵ, a, b, and $(\lambda_i)_{i=0}^{\infty}$ (but not on the number of terms in p) such that

$$\|p\|_{[0,b-\epsilon]} \le c_\epsilon \int_a^b |p(x)|\,dx$$

for every $p \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$.

Hint: Assume that $b = 1$; the general case can be reduced to this by scaling. Use parts a] and b] of E.5 with

$$\widetilde{p}(x) := \int_0^x p(t)\,dt \in \operatorname{span}\{x^{\lambda_0+1}, x^{\lambda_1+1}, \dots\}.$$

□

b] Assume

$$(p_n)_{n=1}^{\infty} \subset \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

converges to an $f \in C[a, b]$ uniformly on $[a, b]$. Show that f can be extended analytically throughout the region

$$D_b := \{z \in \mathbb{C} \setminus (-\infty, 0] : 0 < |z| < b\}$$

and the convergence is uniform on every closed subset of D_b.

Hint: This part of the exercise is difficult. A proof of a more general statement can be found in Schwartz [59, pp. 38–48]. □

c] Suppose $(p_n)_{n=1}^{\infty} \subset \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ and $\|p_n\|_{[a,b]} \le 1$ for each n. Show that there is a subsequence of $(p_n)_{n=1}^{\infty}$ that converges uniformly on every closed subinterval of $[0, b)$. (So the conclusion of E.3 g] holds without the gap condition $\inf\{\lambda_i - \lambda_{i-1} : i \in \mathbb{N}\} > 0$.)

Hint: Use parts a] and b] of E.5 and the Arzela-Ascoli theorem. □

d] Let K be a closed subset of D_b defined in part b]. Show that there is a constant c_K depending only on K, a, b, and $(\lambda_i)_{i=0}^{\infty}$ (but not on the number of terms in p) such that

$$\|p\|_K \le c_K \|p\|_{[a,b]}$$

for every $p \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$.

Hint: Use parts c] and b]. (If the gap condition $\inf\{\lambda_i - \lambda_{i-1} : i \in \mathbb{N}\} > 0$ holds, then the simpler result of E.3 e] can be used instead of part b] of this exercise.) □

E.9 Full Müntz Theorem on $[a,b]$, $a > 0$, in L_q Norm. Schwartz [59] gives the following results: Suppose $(\lambda_i)_{i=-\infty}^{\infty}$ is a sequence of distinct real numbers. For a finite set Γ of integers and

$$p(x) = \sum_{i \in \Gamma} a_i x^{\lambda_i}, \qquad a_i \in \mathbb{R},$$

let

$$p^-(x) := \sum_{\substack{i \in \Gamma \\ \lambda_i < 0}} a_i x^{\lambda_i} \qquad \text{and} \qquad p^+(x) := \sum_{\substack{i \in \Gamma \\ \lambda_i \geq 0}} a_i x^{\lambda_i}.$$

Let $0 < a < b$ and $1 \leq q \leq \infty$.

Theorem 4.2.5. *Suppose*

$$\sum_{\substack{i=-\infty \\ \lambda_i \neq 0}}^{\infty} \frac{1}{|\lambda_i|} < \infty.$$

Then there exists a constant c depending only on a, b, q, and $(\lambda_i)_{i=-\infty}^{\infty}$ (but not on the number of terms in p) such that

$$\|p^+\|_{L_q[a,b]} \leq c\|p\|_{L_q[a,b]} \qquad \textit{and} \qquad \|p^-\|_{L_q[a,b]} \leq c\|p\|_{L_q[a,b]}$$

for every $p \in \operatorname{span}\{x^{\lambda_i}\}_{i=-\infty}^{\infty}$.

Theorem 4.2.6. *Suppose that $0 < a < b$. Then $\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ is dense in $L_q[a,b]$ if and only if*

$$\sum_{\substack{i=0 \\ \lambda_i \neq 0}}^{\infty} \frac{1}{|\lambda_i|} = \infty. \tag{4.2.4}$$

a] Prove the two above results under the gap condition

$$\inf\{\lambda_i - \lambda_{i-1} : i \in \mathbb{Z}\} > 0.$$

Hint to Theorem 4.2.5: When $q = \infty$ see E.4 a]. If $1 \leq q < \infty$, then modify the proof suggested in the hints to E.4 a], by using E.8 a]. □

Hint to Theorem 4.2.6: If (4.2.4) holds, then the fact that $\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ is dense in $L_q[a,b]$ follows from E.7 and the obvious inequality

$$\|p\|_{L_q[a,b]} \leq (b-a)^{1/q}\|p\|_{[a,b]}.$$

Now suppose (4.2.4) does not hold. Use Theorem 4.2.5 and E.8 a] to show that for every $\epsilon \in (0, \frac{1}{2}(b-a))$ there exists a constant c_ϵ depending only on a, b, q, and $(\lambda_i)_{i=-\infty}^{\infty}$ (but not on the number of terms in p) such that

$$\|p\|_{[a+\epsilon,b-\epsilon]} \leq c_\epsilon \|p\|_{L_q[a,b]}$$

for every $p \in \operatorname{span}\{x^{\lambda_i}\}_{i=-\infty}^{\infty}$. Now show that the above inequality implies that $\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ is not dense in $L_q[a,b]$. □

E.10 Denseness of $\operatorname{span}\{\cos\lambda_k\theta\}$ **and** $\operatorname{span}\{z^{\lambda_k}\}$**.** Throughout this exercise the span is assumed to be over $\mathbb{C}$. Let

$$D_R := \{z \in \mathbb{C} : |z| < R\} \qquad \text{and} \qquad C_R := \{z \in \mathbb{C} : |z| = R\}.$$

a] Show that $(\cos n\theta)_{n=0}^{\infty}$ is a complete orthogonal system in $L_2[0,\pi]$. So no term, $\cos n\theta$, can be removed if we wish to preserve denseness of the span in $L_2[0,\pi]$.

b] Let $A := \{e^{i\theta} : \theta \in [0,\delta]\}$. Suppose $(\lambda_k)_{k=0}^{\infty}$ is a sequence of distinct complex numbers satisfying

$$|\lambda_k| \le ke^{1-\delta}, \qquad k = 1,2,\dots .$$

If $\delta \in [1,2\pi]$, then $\operatorname{span}\{z^{\lambda_0}, z^{\lambda_1}, \dots\}$ is dense in $L_2(A)$.

The proof of part b] is outlined in parts c], d], and e].

c] **Jensen's Formula**. Suppose h is a nonnegative integer and

$$f(z) = \sum_{k=h}^{\infty} c_k z^k, \qquad c_h \neq 0$$

is analytic on a disk of radius greater than R, and suppose that the zeros of f in $\overline{D}_R \setminus \{0\}$ are $a_1, a_2, \dots, a_n$, where each zero is listed as many times as its multiplicity. Then

$$\log|c_h| + h\log R + \sum_{k=1}^{n} \log\frac{R}{|a_k|} = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta .$$

Proof. This is a simple consequence of Poisson's formula (see, for example, Ahlfors [53]), which states that

$$\log|F(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|F(Re^{i\theta})|\,d\theta$$

whenever the function F is analytic and zero-free in an open region containing the closed disk $\overline{D}_R$. Now, in the above notation, if we let

$$F(z) := f(z)\left(\frac{R}{z}\right)^h \prod_{k=1}^{n} \frac{R^2 - \overline{a}_k z}{R(z-a_k)}$$

and apply Poisson's formula to F, we get the required result by noting that $|F(z)| = |f(z)|$ whenever $|z| = R$. (The case where f has zeros on the boundary of D requires an additional limiting argument.) □

d] Let $-\infty \le a < b \le \infty$. Suppose $(f_k)_{k=0}^{\infty}$ is a sequence in $L_2(a,b)$ and $\operatorname{span}\{f_0, f_1, \dots\}$ is not dense in $L_2(a,b)$. Then there exists a nonzero $g \in L_2(a,b)$ such that

$$\int_a^b f_k(x)g(x)\,dx = 0\,, \qquad k = 0,1,\dots\,.$$

This is an immediate consequence of the Riesz representation theorem (see E.7 g] of Section 2.2) and the Hahn-Banach theorem. The second theorem says that if $\operatorname{span}\{f_0, f_1, \dots\}$ is not dense in a Banach space, then there exists a nonzero continuous linear functional vanishing on $\{f_0, f_1, \dots\}$. The first theorem gives the form of the functional; see Rudin [73].

e] Prove b] as follows: Suppose $\operatorname{span}\{z^{\lambda_0}, z^{\lambda_1}, \dots\}$ is not dense in $L_2(A)$. Then by d] there exists a $g \in L_2[0,\delta]$ such that

$$f(z) := \int_0^{\delta} \exp(i(z+1)\theta)\,g(\theta)\,d\theta$$

vanishes at $z = \lambda_k$, $k = 0,1,\dots$. Also observe that f is an entire function, and there is an absolute constant $\alpha > 0$ such that

$$|f(z)| \le \alpha\, \exp(\delta|z|)\,.$$

Use E.8 a] of Section 2.2 to show that $f \not\equiv 0$. Let $R > |\lambda_0|$ be an integer. Applying c] on D_R and exponentiating, we obtain

$$|c_h| \exp((\delta-1)R)\frac{R^{R+1}}{R!} \le |c_h| R^h \prod_{\substack{k=0 \\ \lambda_k \neq 0}}^{R} \frac{R}{|\lambda_k|} \le \alpha \exp(\delta R)\,,$$

where c_h is the first nonvanishing coefficient of the Taylor series expansion of f around 0. However,

$$\lim_{R\to\infty} \frac{R^{R+1}}{R!\,e^R} = \infty\,,$$

which is a contradiction and finishes the proof. □

f] Let $A := [0,\delta]$. Suppose $(\lambda_k)_{k=0}^{\infty}$ is a sequence of distinct complex numbers satisfying

$$|\lambda_k| \le k e^{1-\delta}\,, \qquad k = 1,2,\dots\,.$$

Show that if $\delta \in [1,\pi]$, then $\operatorname{span}\{\cos\lambda_0\theta, \cos\lambda_1\theta, \dots\}$ is dense in $L_2(A)$.

Hint: Proceed as in the proof of part b]. □

g] Suppose $(\lambda_k)_{k=0}^{\infty}$ is a sequence of distinct real numbers satisfying

$$0 \le \lambda_k \le k\,, \qquad k = 0, 1, \dots\,.$$

Then $\operatorname{span}\{\cos\lambda_0\theta,\, \cos\lambda_1\theta,\, \dots\}$ is dense in $L_2[0, \pi - \epsilon]$ for any $\epsilon > 0$.

Proof. This is harder; see Boas [54, p. 235]. □

h] Suppose $(\lambda_k)_{k=0}^{\infty}$ is a sequence of distinct complex numbers satisfying $0 \le |\lambda_k| \le k$. Suppose f is an entire function such that $\|f\|_{D_R} \le \alpha e^R$ for all $R > 0$ with an absolute constant $\alpha > 0$, and $\operatorname{span}\{f(\beta z) : \beta \in \mathbb{C}\}$ is dense in $L_2(C_1)$. Then $\operatorname{span}\{f(\lambda_0 z), f(\lambda_1 z), \dots\}$ is dense in $L_2(C_1)$.

E.11 On the Hardy Space $\mathcal{H}_\infty$. We denote by $\mathcal{H}_\infty$ the class of functions that are analytic and bounded on $D := \{z \in \mathbb{C} : |z| < 1\}$. We let

$$\|f\|_{\mathcal{H}_\infty} := \|f\|_D = \sup_{u \in D} |f(u)|\,.$$

a] If $f \in \mathcal{H}_\infty$, then

$$f(z) = \sum_{n=0}^{\infty} a_n z^n\,, \qquad z \in D\,,$$

where

$$|a_n| \le \|f\|_{\mathcal{H}_\infty}\,.$$

Hint: By Cauchy's integral formula

$$|a_n| = \frac{1}{n!}|f^{(n)}(0)| \le \frac{1}{2\pi}\int_{|t|=R}\left|\frac{f(t)}{t^{n+1}}\right||dt| \le R^{-n}\|f\|_{\mathcal{H}_\infty}$$

holds for every $R \in (0, 1)$. □

b] If $f \in \mathcal{H}_\infty$, then

$$|f'(z)| \le \left(\frac{1}{1-|z|}\right)^2 \|f\|_{\mathcal{H}_\infty}\,, \qquad |z| < 1$$

and

$$|f^{(n)}(z)| \le n!\left(\frac{1}{1-|z|}\right)^{n+1} \|f\|_{\mathcal{H}_\infty}\,, \qquad |z| < 1\,.$$

c] $\mathcal{H}_\infty$ is a Banach algebra.

Hint: See Rudin [73]. □

E.12 Blaschke Products. A product of the form

$$B(z) := z^k \prod_{i=1}^{\infty} \left(\frac{\alpha_i - z}{1 - \overline{\alpha}_i z} \right) \frac{|\alpha_i|}{\alpha_i}, \qquad \alpha_i \in \mathbb{C} \setminus \{0\}$$

with $k \in \mathbb{Z}$ is called a *Blaschke product.* Let $D := \{z \in \mathbb{C} : |z| < 1\}$.

a] Let

$$\varphi_\alpha(z) := \frac{\alpha - z}{1 - \overline{\alpha} z}, \qquad \alpha \in \mathbb{C}.$$

Show that $|\varphi_\alpha(z)| = 1$ whenever $|z| = 1$ and

$$\varphi'_\alpha(z) = \frac{|\alpha|^2 - 1}{(1 - \overline{\alpha} z)^2}.$$

b] Show that if $|\alpha| < 1$, then $\varphi_\alpha(z)$ maps the closed unit disk $\overline{D}$ one-to-one onto itself.

c] A Minimization Property. Let $\beta_1, \beta_2, \ldots, \beta_n$ be fixed complex numbers with $|\beta_i| > 1$, $i = 1, \ldots n$. Show that

$$\min_{a_i \in \mathbb{C}} \left\| 1 - \sum_{i=1}^{n} \frac{a_i}{z - \beta_i} \right\|_D = \prod_{i=1}^{n} |\beta_i|^{-1}$$

and that the minimum is attained by the normalized finite Blaschke product

$$1 - \sum_{i=1}^{n} \frac{a_i^*}{z - \beta_i} = \left(\prod_{i=1}^{n} \beta_i \right)^{-1} \prod_{i=1}^{n} \left(\frac{\overline{\beta}_i^{-1} - z}{1 - \beta_i^{-1} z} \right).$$

Hint: Suppose that the statement is false. Then there are some $a_i \in \mathbb{C}$ such that

$$\left| 1 - \sum_{i=1}^{n} \frac{a_i}{z - \beta_i} \right| < \left(\prod_{i=1}^{n} |\beta_i| \right)^{-1} = \left| 1 - \sum_{i=1}^{n} \frac{a_i^*}{z - \beta_i} \right|$$

for all $z \in \mathbb{C}$ with $|z| = 1$. Now Rouché's theorem implies that

$$\sum_{i=1}^{n} \frac{a_i - a_i^*}{z - \beta_i}$$

has n zeros in the disk D, which is a contradiction. □

d] Suppose $(\beta_n)_{n=1}^{\infty}$ is a sequence in D satisfying

$$\beta_1 = \beta_2 = \cdots = \beta_k = 0, \qquad \beta_n \neq 0, \qquad n = k+1, k+2, \ldots$$

and

$$\sum_{n=1}^{\infty}(1-|\beta_n|)<\infty\,.$$

Then

$$B(z):=z^k\prod_{n=k+1}^{\infty}\left(\frac{\beta_n-z}{1-\overline{\beta}_n z}\right)\frac{|\beta_n|}{\beta_n}$$

defines a bounded analytic function on D (that is, $B\in\mathcal{H}_\infty$), which vanishes at z if and only if $z=\beta_j$ for some $j=1,2,\dots$, in which case the multiplicity of z_0 in $B(z)$ is the same as the multiplicity of β_j in $(\beta_n)_{n=1}^{\infty}$.

e] Suppose $(\beta_n)_{n=1}^{\infty}$ is a sequence in D satisfying

$$\sum_{n=1}^{\infty}(1-|\beta_n|)=\infty\,.$$

Denote the multiplicity of β_j in $(\beta_n)_{n=1}^{\infty}$ by m_j. Suppose $f\in\mathcal{H}_\infty$ has a zero at each β_j with multiplicity m_j. Then $f=0$.

Hint: Suppose $\|f\|_D>0$. Without loss of generality we may assume that $f(0)\neq 0$. By Jensen's formula (see E.10 c]),

$$\sum_{\substack{n=1\\|\beta_n|<R}}^{\infty}\log\frac{R}{|\beta_n|}+\log|f(0)|=\frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{i\theta}|\,d\theta\le\log\|f\|_D$$

for every $R\in(0,1)$. Letting R tend to 1, we obtain

$$\sum_{n=1}^{\infty}\log\frac{1}{|\beta_n|}<\log\|f\|_D-\log|f(0)|<\infty\,.$$

Hence

$$\sum_{n=1}^{\infty}(1-|\beta_n|)<\infty\,,$$

which contradicts the assumption. □

Note that the conclusion of part e] holds for the larger *Nevanlinna class* N, which is defined as the set of those analytic functions f on D for which

$$\sup_{R\in(0,1)}\ \frac{1}{2\pi}\int_0^{2\pi}\log^+|f(Re^{i\theta})|d\theta<\infty\,,$$

where $\log^+ x:=\max\{\log x,0\}$.

f] Let

$$B(z) := z^k \prod_{i=1}^{n} \left(\frac{\alpha_i - z}{1 - \overline{\alpha}_i z} \right) \frac{|\alpha_i|}{\alpha_i}, \qquad \alpha_i \in \mathbb{C} \setminus \{0\}$$

be a finite Blaschke product. Show that

$$|B'(z)| = k + \sum_{i=1}^{n} \frac{1 - |\alpha_i|^2}{|z - \alpha_i|^2}, \qquad |z| = 1 .$$

Hint: Consider B'/B, where $|B(z)| = 1$ whenever $|z| = 1$. □

g] Suppose

$$\sum_{i=1}^{\infty} (1 - |\alpha_i|) < \infty, \qquad \alpha_i \in (0,1)$$

and

$$B(z) := \prod_{i=1}^{\infty} \left(\frac{\alpha_i - z}{1 - \overline{\alpha}_i z} \right) \frac{|\alpha_i|}{\alpha_i} .$$

Show that

$$\|((1-z)^2 B(z))'\|_D \leq \left(4 + 2 \sum_{i=1}^{\infty} (1 - \alpha_i^2) \right) \|B\|_D .$$

Hint: Use f]. □

E.13 Yet Another Proof of Müntz's Theorem when $\inf\{\lambda_i : i \in \mathbb{N}\} > 0$**.** As in E.10, this proof requires a consequence of the Hahn-Banach theorem and the Riesz representation theorem which we state in a]. For details, the reader is referred to Feinerman and Newman [76] and Rudin [87]. We assume throughout the exercise that $\lambda_0 := 0$ and that $(\lambda_k)_{k=1}^{\infty}$ is a sequence of distinct positive numbers satisfying $\inf\{\lambda_k : k \in \mathbb{N}\} > 0$.

a] $\operatorname{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}$ is not dense if and only if there exists a nonzero finite Borel measure μ on $[0,1]$ with

$$\int_0^1 t^{\lambda_k} \, d\mu(t) = 0, \qquad k = 0, 1, 2 \dots .$$

b] Show that $\sum_{k=1}^{\infty} 1/\lambda_k = \infty$ implies that $\operatorname{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}$ is dense in $C[0,1]$.

Outline. Suppose there is a nonzero finite Borel measure μ on $[0,1]$ such that

$$\int_0^1 t^{\lambda_k} \, d\mu(t) = 0, \qquad k = 0, 1, \dots .$$

Let

$$f(z) = \int_0^1 t^z \, d\mu(t)\,, \qquad \mathrm{Re}(z) > 0\,.$$

Show that

$$g(z) := f\left(\frac{1+z}{1-z}\right) \in \mathcal{H}_\infty$$

and

$$g\left(\frac{\lambda_k - 1}{\lambda_k + 1}\right) = 0 \qquad \text{with} \qquad \left|\frac{\lambda_k - 1}{\lambda_k + 1}\right| < 1\,, \qquad k = 1, 2, \ldots .$$

Note that $\sum_{k=1}^{\infty} 1/\lambda_k = \infty$ and $\inf\{\lambda_k : k \in \mathbb{N}\} > 0$ imply

$$\sum_{k=1}^{\infty} \left(1 - \left|\frac{\lambda_k - 1}{\lambda_k + 1}\right|\right) = \infty\,.$$

Hence E.12 e] yields that $g = 0$ on the open unit disk. Therefore $f(z) = 0$ whenever $\mathrm{Re}(z) > 0$, so

$$f(n) = \int_0^1 t^n \, d\mu(t) = 0\,, \qquad n = 1, 2, \ldots .$$

Note that

$$\int_0^1 t^0 \, d\mu(t) = 0$$

also holds because of the choice of μ. Now the Weierstrass approximation theorem yields that

$$\int_0^1 f(t) \, d\mu(t) = 0$$

for every $f \in C[0,1]$, which contradicts the fact that the Borel measure μ is nonzero. So part a] implies that $\mathrm{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \ldots\}$ is dense in $C[0,1]$. □

c] Show that $\sum_{k=1}^{\infty} 1/\lambda_k < \infty$ implies that $\mathrm{span}\{1, x^{\lambda_1}, x^{\lambda_2} \ldots\}$ is not dense in $C[0,1]$.

Outline. Show under the above assumption that

$$f(z) = \int_0^1 t^z \left\{\frac{1}{2\pi} \int_{-\infty}^{\infty} f(-1+is) e^{-is \log t} \, ds\right\} dt\,, \qquad \mathrm{Re}(z) > -1$$

if f is defined by

$$f(z) := \frac{z}{(2+z)^3} \prod_{n=1}^{\infty} \frac{\lambda_n - z}{2 + \lambda_n + z}\,.$$

Show that

$$d\mu(t) = \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} f(-1+is) e^{-is\log t}\, ds \right\} dt$$

defines a nonzero finite Borel measure, μ, on $[0,1]$ such that

$$\int_0^1 t^{\lambda_k}\, d\mu(t) = 0\,, \qquad k = 0, 1, 2, \ldots$$

as is required by part a]. For the above, show that

$$f(z) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{f(is-1)}{is-1-z}\, ds\,, \qquad \mathrm{Re}(z) > -1$$

and use that

$$\frac{1}{1+z-is} = \int_0^1 t^{z-is}\, dt\,.$$

□

E.14 Another Proof of Denseness of Müntz Spaces when $\lambda_i \to 0$. Suppose $\Lambda := (\lambda_i)_{i=1}^{\infty}$ is a sequence of distinct positive numbers with $\lim\limits_{i\to\infty} \lambda_i = 0$. Show that

$$M(\Lambda) := \mathrm{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \ldots\}$$

is dense in $C[0,1]$ if and only if $\sum_{i=1}^{\infty} \lambda_i = \infty$.

Hint: If $\sum_{i=1}^{\infty} \lambda_i = \infty$, then $\lim\limits_{i\to\infty} \lambda_i = 0$ implies that

$$\sum_{i=1}^{\infty} \left(1 - \left|\frac{\lambda_i - 1}{\lambda_i + 1}\right|\right) = \infty\,.$$

So the outline of the proof of E.13 b] yields that $M(\Lambda)$ is dense in $C[0,1]$.

If $\eta := \sum_{i=1}^{\infty} \lambda_i < \infty$, then, by Theorem 6.1.1, the inequality

$$\|xp'(x)\|_{[0,1]} \le 9\eta\, \|p\|_{[0,1]}$$

holds for every $p \in M(\Lambda)$. Use this inequality to show that $M(\Lambda)$ fails to be dense in $C[0,1]$. □

E.15 Denseness of Müntz Spaces with Complex Exponents. Suppose $\Lambda := (\lambda_i)_{i=1}^{\infty}$ is a sequence of complex numbers satisfying

$$\mathrm{Re}(\lambda_i) > 0\,, \qquad i = 1, 2, \ldots\,.$$

Show that if

$$\sum_{n=1}^{\infty} \left(1 - \left|\frac{\lambda_n - 1}{\lambda_n + 1}\right|\right) = \infty\,,$$

then $\mathrm{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \ldots\}$ is dense in $C[0,1]$. (In this exercise the span is taken over $\mathbb{C}$, and $C[0,1]$ denotes the set of all complex-valued continuous functions on $[0,1]$.)

E.16 Christoffel Functions for Nondense Müntz Spaces. Let $\Lambda = (\lambda_i)_{i=0}^{\infty}$ be a sequence of distinct complex numbers with $\mathrm{Re}(\lambda_i) > -\frac{1}{2}$ for each i. As in Section 3.4, let $L_k^* := L_k^*\{\lambda_0, \dots, \lambda_k\}$ denote the associated orthonormal Müntz-Legendre polynomials on $[0,1]$.

a] Let K_n be defined by

$$\frac{1}{K_n(y)} := \inf\left\{ \int_0^1 |p(t)|^2\, dt : \; p \in \; \mathrm{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}, \;\; p(y) = 1 \right\}.$$

Show that

$$K_n(y) = \sum_{k=0}^{n} |L_k^*(y)|^2 .$$

The function $1/K_n$ is called the nth *Christoffel function associated with* Λ.

Hint: Proceed as in the hint to E.13 of Section 2.3. □

In the rest of the exercise we assume that $(\lambda_i)_{i=0}^{\infty}$ is a sequence of nonnegative integers. We use this assumption for treating (higher) derivatives, although some weaker assumptions would lead to the same conclusions.

b] Suppose $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$. Show that for every $\epsilon \in (0,1)$ and $m \in \mathbb{N}$, there exists a constant $c_{\epsilon,m}$ depending only on Λ, ϵ, and m such that

$$\|p^{(m)}\|_{[0,1-\epsilon]} \le c_{\epsilon,m} \|p\|_{L_2[0,1]}$$

for every $p \in \mathrm{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$.

Hint: Use E.3 c]. □

c] Show that the following statements are equivalent:

(1) $\mathrm{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ is not dense in $C[0,1]$.

(2) $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$.

(3) $\sum_{k=0}^{\infty} (L_k^*)^2$ converges uniformly on $[0, 1-\epsilon]$ for all $\epsilon \in (0,1)$.

(4) There exists an $x \in [0,1)$ so that $\sum_{k=0}^{\infty} (L_k^*(x))^2 < \infty$.

Outline. The equivalence of (1) and (2) is the content of Müntz's theorem (Theorem 4.2.1). To see that (2) implies (3), first observe that

$$\sum_{k=0}^{\infty} \left|(L_k^*)^{(m)}(y)\right|^2 = \sup\left\{ \left|p^{(m)}(y)\right|^2 : \; p \in \mathrm{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}, \quad \|p\|_{L_2[0,1]} = 1 \right\},$$

which can be proved similarly to part a]. Hence by part b], for every $\epsilon \in (0,1)$, there exists a constant c_ϵ depending only on ϵ such that

$$\sum_{k=0}^{\infty} (L_k^*(x))^2 \le c_\epsilon \qquad \text{and} \qquad \sum_{k=0}^{\infty} (L_k^{*\prime}(x))^2 \le c_\epsilon\,, \qquad x \in [0, 1-\epsilon]\,.$$

Since $\left(\sum_{k=0}^{n} (L_k^*)^2\right)' = 2\sum_{k=0}^{n} L_k^* L_k^{*\prime}$ on $[0,\infty)$, applying the Cauchy-Schwarz inequality, we obtain that

$$\left|\left(\sum_{k=0}^{n} (L_k^*)^2\right)'(x)\right| \le 2c_\epsilon\,, \qquad x \in [0, 1-\epsilon]\,.$$

Therefore the functions $\sum_{k=0}^{n} (L_k^*)^2$, $n = 1, 2, \dots$, are uniformly bounded and equicontinuous on $[0, 1-\epsilon]$, which implies the uniform convergence of the functions K_n on $[0, 1-\epsilon]$ by the Arzela-Ascoli theorem. Since (3) obviously implies (4), what remains to be proven is that (4) implies (1). This can be easily done by part a]. □

d] Let $\epsilon \in (0,1)$ and $m \in \mathbb{N}$ be fixed. Show that if $\sum_{i=0}^{\infty} 1/\lambda_i < \infty$, then $\sum_{k=0}^{\infty} ((L_k^*)^{(m)})^2$ converges uniformly on $[0, 1-\epsilon]$.

Hint: Modify the argument given in the hints to part c]. □

e] Show that if $\sum_{i=0}^{\infty} 1/\lambda_i < \infty$, then

$$\lim_{k\to\infty} \|(L_k^*)^{(m)}\|_{[0,1-\epsilon]} = 0$$

for every $\epsilon \in (0,1)$ and $m \in \mathbb{N}$.

E.17 Chebyshev-Type Inequality with Explicit Bound via the Paley-Wiener Theorem. The method outlined in this exercise was suggested by Halász. A function f is called *entire* if it is analytic on the complex plane. An entire function f is called a function of *exponential type* δ if there exists a constant c depending only on f such that

$$|f(z)| \le c\, \exp(\delta|z|)\,, \qquad z \in \mathbb{C}.$$

The collection of all such entire functions of exponential type δ is denoted be E^δ. The Paley-Wiener theorem characterizes the functions F that can be written as the Fourier transform of some function $f \in L_2[-\delta, \delta]$.

Theorem (Paley-Wiener). *Let $\delta \in (0,\infty)$. Then $F \in E^\delta \cap L_2(\mathbb{R})$ if and only if there exists an $F \in L_2[-\delta,\delta]$ such that*

$$F(z) = \int_{-\delta}^{\delta} f(t) e^{itz}\, dt\,.$$

For a proof see, for example, Rudin [87].

In the rest of the exercise let $\Lambda = (\lambda_k)_{k=0}^{\infty}$ be an increasing sequence with $\lambda_0 = 0$ and $\sum_{k=1}^{\infty} 1/\lambda_k < \infty$.

E.5 says that

$$C(\epsilon, \Lambda) := \sup \left\{ \frac{\|p\|_{[0,1]}}{\|p\|_{[1-\epsilon,1]}} : \ p \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\} \right\} < \infty$$

holds for every $\epsilon \in (0,1)$. In this exercise we establish an explicit bound for $C(\epsilon, \Lambda)$.

a] Show that

$$C(\epsilon, \Lambda) = \sup \left\{ \frac{|p(0)|}{\|p\|_{[1-\epsilon,1]}} : \ p \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\} \right\}$$

for every $\epsilon \in (0,1)$.

Hint: Use $\lambda_0 = 0$, E.4 c] of Section 3.3, and the monotonicity of the Chebyshev polynomial

$$T_n\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}; [1-\epsilon, 1]\}$$

on $[0, 1-\epsilon]$. □

b] Assume that

(1) $F \in E^\delta \cap L_2(\mathbb{R})$;

(2) $F(i\lambda_k) = 0, \ k = 1, 2, \dots$ (i is the imaginary unit); and

(3) $F(0) = 1$.

Show that

$$|P(\infty)| \le \|F\|_{L_2(\mathbb{R})} \, \|P\|_{L_2[-\delta,\delta]}$$

for every $P \in \operatorname{span}\{e^{-\lambda_0 t}, e^{-\lambda_1 t}, \dots\}$.

Outline. By the Paley-Wiener theorem

$$F(z) = \int_{-\delta}^{\delta} f(t) e^{itz} \, dt$$

for some $f \in L_2[-\delta, \delta]$. Now if

$$P(t) = a_0 + \sum_{k=1}^{n} a_k e^{-\lambda_k t},$$

then

$$\begin{aligned} \int_{-\delta}^{\delta} f(t) P(t) \, dt &= a_0 \int_{-\delta}^{\delta} f(t) \, dt + \sum_{k=1}^{n} a_k \int_{-\delta}^{\delta} f(t) e^{-\lambda_k t} \, dt \\ &= a_0 F(0) + \sum_{k=1}^{n} a_k F(i\lambda_k) = a_0 = P(\infty). \end{aligned}$$

Hence by the Cauchy-Schwarz inequality and the L_2 inversion theorem of Fourier transforms, we obtain that

$$|P(\infty)| \le \|f\|_{L_2[-\delta,\delta]} \, \|P\|_{L_2[-\delta,\delta]} \le \|F\|_{L_2(\mathbb{R})} \, \|P\|_{L_2[-\delta,\delta]} \, .$$

□

Given $\delta \in (0,1)$, let $N \in \mathbb{N}$ be chosen so that

$$\sum_{k=N+1}^{\infty} \frac{1}{\lambda_k} \le \frac{\delta}{3} \, .$$

Let

$$\sigma_k := A\lambda_k \qquad \text{with} \qquad A := \frac{\delta}{3N} \, .$$

Let

$$F(z) := \frac{\sin(\delta z/3)}{\delta z/3} \times \prod_{k=1}^{N} \left(\left(1 - \frac{z}{i\lambda_k} \right) \frac{\sin(\sigma_k z/\lambda_k)}{\sigma_k z/\lambda_k} \right) \prod_{k=N+1}^{\infty} \left(1 - \left(\frac{\sin(z/\lambda_k)}{\sin i} \right)^4 \right),$$

where i is the imaginary unit.

c] Show that $F \in E^{\delta}$.

d] Observe that $F(0) = 1$, $F(i\lambda_k) = 0$, $k = 1, 2, \dots$, and

$$|F(t)| \le \frac{\sin(\delta t/3)}{\delta t/3} \prod_{k=1}^{N} \left(2 + \frac{1}{\sigma_k} \right) , \qquad t \in \mathbb{R} \, .$$

e] Show that

$$|P(\infty)| \le \frac{3c}{\delta} \prod_{k=1}^{N} \left(2 + \frac{1}{\sigma_k} \right) \|P\|_{[-\delta,\delta]}$$

for every $P \in \operatorname{span}\{e^{-\lambda_0 t}, e^{-\lambda_1 t}, \dots\}$ with $c := \|t^{-1} \sin t\|_{L_2(\mathbb{R})}$.

Hint: Use parts b], c], and d]. □

f] Let $\lambda_k := k^{\alpha}$, $\alpha > 1$. Show that there exists a constant c_α depending only on α such that

$$\|p\|_{[0,1]} \le \exp\left(c_\alpha \epsilon^{1/(1-\alpha)} \right) \|p\|_{[1-\epsilon,1]}$$

for every $p \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ and for every $\epsilon \in (0, 1/2]$.

Proof. Let

$$\delta := -\frac{1}{2}\log(1-\epsilon)\,. \tag{4.2.5}$$

Observe that N in part e] can be chosen so that

$$N := \left\lfloor \left(\frac{\delta(\alpha-1)}{3}\right)^{1/(1-\alpha)} \right\rfloor + 1\,. \tag{4.2.6}$$

Also, σ_k in part d] is of the form $\sigma_k = \delta k^\alpha (3N)^{-1}$. Let $M+1$ be the smallest value of $k \in \mathbb{N}$ for which

$$\frac{1}{\sigma_k} < 1\,, \qquad \text{that is}\,, \qquad \frac{3N}{k^\alpha \delta} \le 1\,.$$

Note that

$$M := \left\lfloor \left(\frac{3N}{\delta}\right)^{1/\alpha} \right\rfloor .$$

If $0 < M < N$, then

$$\begin{aligned}
\prod_{k=1}^{N}\left(2+\frac{1}{\sigma_k}\right) &= \prod_{k=1}^{N}\left(2+\frac{3N}{\delta k^\alpha}\right) \\
&\le \left(\prod_{k=1}^{M}\frac{9N}{\delta k^\alpha}\right)\left(\prod_{k=M+1}^{N} 3\right) \le \left(\frac{9N}{\delta}\right)^M \left(\frac{M}{e}\right)^{-\alpha M} 3^{N-M} \\
&= \left(\frac{9e^\alpha N}{\delta}\right)^M M^{-\alpha M} 3^{N-M} \\
&\le \left(\frac{9e^\alpha N}{\delta}\right)^M \left(\frac{1}{2}\left(\frac{3N}{\delta}\right)^{1/\alpha}\right)^{-\alpha M} 3^{N-M} \\
&\le (3(2e)^\alpha)^M 3^{N-M} \le (3(2e)^\alpha)^N\,,
\end{aligned}$$

and the theorem follows by (4.2.5), (4.2.6), and part e].

If $N \le M$, then

$$\begin{aligned}
\prod_{k=1}^{N}\left(2+\frac{1}{\sigma_k}\right) &= \prod_{k=1}^{N}\left(2+\frac{3N}{\delta k^\alpha}\right) \\
&\le \left(\prod_{k=1}^{N}\frac{9N}{\delta k^\alpha}\right) \le \left(\frac{9N}{\delta}\right)^N \left(\frac{N}{e}\right)^{-\alpha N} = \left(\frac{9e^\alpha N^{(1-\alpha)}}{\delta}\right)^N \\
&\le \left(\frac{9e^\alpha}{\delta}\right)^N \left(\left(\frac{\delta(\alpha-1)}{3}\right)^{1/(1-\alpha)}\right)^{(1-\alpha)N} \\
&\le \left(\frac{9e^\alpha}{\delta}\right)^N \left(\frac{\delta(\alpha-1)}{3}\right)^N \le (3e^\alpha(\alpha-1))^N\,,
\end{aligned}$$

and the theorem follows by (4.2.5), (4.2.6), and part e].

If $M = 0$, then

$$\prod_{k=1}^{N} \left(2 + \frac{1}{\sigma_k}\right) \leq \prod_{k=1}^{N} 3 = 3^N ,$$

and the theorem follows by (4.2.5), (4.2.6), and part e]. □

The next part of the exercise shows that the result of part f] is close to sharp.

g] Let $\lambda_k := k^\alpha$, $\alpha > 1$. Let $\epsilon \in (0, 1/2]$. Show that there exists a constant c_α depending only on $\alpha > 0$ so that

$$\sup \left\{ \frac{|p(0)|}{\|p\|_{[1-\epsilon,1]}} : \ p \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\} \right\} \geq \exp\left(c_\alpha \epsilon^{1/(1-\alpha)}\right) .$$

Proof. Let $n \in \mathbb{N}$ be a fixed. We define $\gamma_k := kn^{\alpha-1}$, $k = 0, 1, \dots$. Let $T_n(x) := ((x-1)/2)^n$ and

$$Q_n(x) := T_n \left(\frac{2x^{n^{\alpha-1}}}{1-(1-\epsilon)^{n^{\alpha-1}}} - \frac{1+(1-\epsilon)^{n^{\alpha-1}}}{1-(1-\epsilon)^{n^{\alpha-1}}} \right) .$$

Then $Q_n \in \operatorname{span}\{x^{\gamma_0}, \dots x^{\gamma_n}\}$, and by E.3 g] of Section 3.3 we obtain that

$$\begin{aligned} \sup \left\{ \frac{|p(0)|}{\|p\|_{[1-\epsilon,1]}} : \ p \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\} \right\} &\geq \frac{|Q_n(0)|}{\|Q_n\|_{[1-\epsilon,1]}} = |Q_n(0)| \\ &= \left(\frac{1}{1-(1-\epsilon)^{n^{\alpha-1}}} \right)^n . \end{aligned}$$

Now let n be the smallest integer satisfying $n^{\alpha-1} \geq \epsilon^{-1}$. Since $(1-\epsilon)^{1/\epsilon}$ is bounded away from 0 on $(0, 1/2]$, the result follows. □

E.18 Completion of the Proof of Theorem 4.2.1. The case when $\lambda_i \geq 1$ for each i has already been proved. The only real remaining difficulty is part d].

a] Prove Theorem 4.2.1 in the case when $\inf\{\lambda_i : i \in \mathbb{N}\} > 0$.

Hint: Use the scaling $x \to x^{1/\delta}$ and the already proved case. □

b] Show that if $(\lambda_i)_{i=1}^\infty \subset (0, \infty)$ has a cluster point $\lambda \in (0, \infty)$, then $\operatorname{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}$ is dense in $C[0, 1]$.

Hint: Use part a]. □

c] Suppose $(\lambda_i)_{i=1}^\infty \subset (0, \infty)$ and $\lambda_i \to 0$. Then $\operatorname{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}$ is dense in $C[0, 1]$ if and only if $\sum_{i=1}^\infty \lambda_i = \infty$.

Hint: This is the content of E.14. □

d] Suppose

$$\{\lambda_i : i \in \mathbb{N}\} = \{\alpha_i : i \in \mathbb{N}\} \cup \{\beta_i : i \in \mathbb{N}\}$$

with

$$\lim_{i\to\infty} \alpha_i = 0 \qquad \text{and} \qquad \lim_{i\to\infty} \beta_i = \infty\,.$$

Show that $\operatorname{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}$ is dense in $C[0,1]$ if and only if

$$\sum_{i=1}^{\infty} \alpha_i + \sum_{i=1}^{\infty} \frac{1}{\beta_i} = \infty\,. \tag{4.2.7}$$

Outline. If (4.2.7) holds, then the denseness of $\operatorname{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}$ in $C[0,1]$ follows from parts a] and c]. Now assume that (4.2.7) does not hold, so

$$\sum_{i=1}^{\infty} \alpha_i < \infty \qquad \text{and} \qquad \sum_{i=1}^{\infty} \frac{1}{\beta_i} < \infty\,.$$

For notational convenience, let

$$\begin{aligned} T_{n,\alpha} &:= T_n\{1, x^{\alpha_1}, \dots, x^{\alpha_n} : [0,1]\}\,, \\ T_{n,\beta} &:= T_n\{1, x^{\beta_1}, \dots, x^{\beta_n} : [0,1]\}\,, \\ T_{2n,\alpha,\beta} &:= T_{2n}\{1, x^{\alpha_1}, \dots, x^{\alpha_n}, x^{\beta_1}, \dots, x^{\beta_n} : [0,1]\} \end{aligned}$$

(we use the notation introduced in Section 3.3).

It follows from Theorem 6.1.1 (Newman's inequality) and the Mean Value Theorem that for every $\epsilon > 0$ there exists a $k_1(\epsilon) \in \mathbb{N}$ depending only on $(\alpha_i)_{i=1}^{\infty}$ and ϵ (and not on n) such that $T_{n,\alpha}$ has at most $k_1(\epsilon)$ zeros in $[\epsilon, 1)$ and at least $n - k_1(\epsilon)$ zeros in $(0, \epsilon)$.

Similarly, E.5 b] and the Mean Value Theorem imply that for every $\epsilon > 0$ there exists a $k_2(\epsilon) \in \mathbb{N}$ depending only on $(\beta_i)_{i=1}^{\infty}$ and ϵ (and not on n) so that $T_{n,\beta}$ has at most $k_2(\epsilon)$ zeros in $(0, 1-\epsilon]$ and at least $n - k_2(\epsilon)$ zeros $(1-\epsilon, 1)$.

Now, on counting the zeros of $T_{n,\alpha} - T_{2n,\alpha,\beta}$ and $T_{n,\beta} - T_{2n,\alpha,\beta}$, we can deduce that for every $\epsilon > 0$ there exists a $k(\epsilon) \in \mathbb{N}$ depending only on $(\lambda_i)_{i=1}^{\infty}$ and ϵ (and not on n) so that $T_{2n,\alpha,\beta}$ has at most $k(\epsilon)$ zeros in $[\epsilon, 1-\epsilon]$.

Let $\epsilon := \frac{1}{4}$ and $k := k\left(\frac{1}{4}\right)$. Pick $k+4$ points

$$\tfrac{1}{4} < \eta_0 < \eta_1 < \cdots < \eta_{k+3} < \tfrac{3}{4}$$

and a function $f \in C[0,1]$ such that $f(x) = 0$ for all $x \in \left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right]$, while

$$f(\eta_i) := 2 \cdot (-1)^i\,, \qquad i = 0, 1, \dots\,.$$

Assume that there exists a $p \in \operatorname{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}$ such that

$$\|f - p\|_{[0,1]} < 1\,.$$

Then $p - T_{2n,\alpha,\beta}$ has at least $2n+1$ zeros in $(0,1)$. However, for sufficiently large n,

$$p - T_{2n,\alpha,\beta} \in \operatorname{span}\{1, x^{\lambda_1}, \dots, x^{\lambda_{2n}}\}$$

so it can have at most $2n$ zeros in $[0,\infty)$. This contradiction shows that $\operatorname{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}$ is not dense in $C[0,1]$. □

e] Prove Theorem 4.2.1 in full generality.

Outline. Combine parts a] to d]. □

E.19 Proof of Theorem 4.2.3. Prove Theorem 4.2.3.

Proof. Assume that

$$\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

is dense in $L_1[0,1]$. Let m be a fixed nonnegative integer. Let $\epsilon > 0$. Choose a

$$p \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

such that

$$\|x^m - p(x)\|_{L_1[0,1]} < \epsilon\,.$$

Now let

$$q(x) := \int_0^x p(t)\,dt \in \operatorname{span}\{x^{\lambda_0+1}, x^{\lambda_1+1}, \dots\}\,.$$

Then

$$\left\| \frac{x^{m+1}}{m+1} - q(x) \right\|_{[0,1]} < \epsilon\,.$$

So the Weierstrass approximation theorem yields that

$$\operatorname{span}\{1, x^{\lambda_0+1}, x^{\lambda_1+1}, \dots\}$$

is dense in $C[0,1]$, and Theorem 2.1 implies that

$$\sum_{i=0}^{\infty} \frac{\lambda_i + 1}{(\lambda_i+1)^2 + 1} = \infty\,.$$

Now assume that

$$\sum_{i=0}^{\infty} \frac{\lambda_i + 1}{(\lambda_i+1)^2 + 1} = \infty. \tag{4.2.8}$$

By the Hahn-Banach theorem and the Riesz representation theorem

$$\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

is not dense in $L_1[0,1]$ if and only if there exists a $0 \neq h \in L_\infty[0,1]$ satisfying

$$\int_0^1 t^{\lambda_i} h(t)\, dt = 0, \qquad i = 0, 1, \dots .$$

Suppose there exists a $0 \neq h \in L_\infty[0,1]$ such that

$$\int_0^1 t^{\lambda_i} h(t)\, dt = 0\,, \qquad i = 0, 1, \dots .$$

Let

$$f(z) := \int_0^1 t^z h(t)\, dt\,, \qquad \operatorname{Re}(z) > -1\,.$$

Then

$$g(z) := f\left(\frac{1+z}{1-z} - 1\right)$$

is a bounded analytic function on the open unit disk that satisfies

$$g\left(\frac{\lambda_i}{\lambda_i + 2}\right) = 0 \qquad \text{with} \qquad \left|\frac{\lambda_i}{\lambda_i + 2}\right| < 1\,, \qquad i = 0, 1, \dots .$$

Note that (4.2.8) implies

$$\sum_{i=1}^{\infty} \left(1 - \left|\frac{\lambda_i}{\lambda_i + 2}\right|\right) = \infty\,.$$

Hence Blaschke's theorem (E.12 e]) yields that $g = 0$ on the open unit disk. Therefore $f(z) = 0$ whenever $\operatorname{Re}(z) > -1$, so

$$f(n) = \int_0^1 t^n h(t)\, dt = 0\,, \qquad n = 0, 1, \dots .$$

Now the Weierstrass approximation theorem yields

$$\int_0^1 u(t) h(t)\, dt = 0$$

for every $u \in C[0,1]$, which contradicts the fact that $0 \neq h$. So

$$\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

is dense in $L_1[0,1]$. □

E.20 Proof of Theorem 4.2.4.

a] Show that if

$$\sum_{i=0}^{\infty} \frac{\lambda_i + \frac{1}{p}}{\left(\lambda_i + \frac{1}{p}\right)^2 + 1} = \infty\,, \tag{4.2.9}$$

then $\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ is dense in $L_p[0,1]$.

Outline. By the Hahn-Banach theorem and the Riesz representation theorem

$$\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

is not dense in $L_p[0,1]$ if and only if there exists a $0 \not\equiv h \in L_q[0,1]$ satisfying

$$\int_0^1 t^{\lambda_i} h(t)\, dt = 0\,, \qquad i = 0, 1, \dots\,,$$

where q is the conjugate exponent of p defined by $p^{-1} + q^{-1} = 1$.

Suppose there exists a $0 \not\equiv h \in L_q[0,1]$ such that

$$\int_0^1 t^{\lambda_i} h(t)\, dt = 0\,, \qquad i = 0, 1, \dots\,.$$

Let

$$f(z) := \int_0^1 t^z h(t)\, dt\,, \qquad \operatorname{Re}(z) > -\tfrac{1}{p}\,.$$

Use Hölder's inequality to show that

$$g(z) := f\left(\frac{1+z}{1-z} - \frac{1}{p}\right)$$

is a bounded analytic function on the open unit disk that satisfies

$$g\left(\frac{\lambda_i + \frac{1}{p} - 1}{\lambda_i + \frac{1}{p} + 1}\right) = 0 \qquad \text{with} \qquad \left|\frac{\lambda_i + \frac{1}{p} - 1}{\lambda_i + \frac{1}{p} + 1}\right| < 1\,, \qquad i = 0, 1, \dots\,.$$

Note that (4.2.9) implies

$$\sum_{i=1}^{\infty} \left(1 - \left|\frac{\lambda_i + \frac{1}{p} - 1}{\lambda_i + \frac{1}{p} + 1}\right|\right) = \infty\,.$$

Hence Blaschke's theorem (E.12 e]) yields that $g = 0$ on the open unit disk. Therefore $f(z) = 0$ whenever $\operatorname{Re}(z) > -\frac{1}{p}$, so

$$f(n) = \int_0^1 t^n h(t)\,dt = 0\,, \qquad n = 0, 1, \ldots\,.$$

Now the Weierstrass approximation theorem yields

$$\int_0^1 u(t)h(t)\,dt = 0$$

for every $u \in C[0,1]$, which contradicts the fact that $0 \neq h$. So

$$\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \ldots\}$$

is dense in $L_p[0,1]$. □

b] Show that if

$$\sum_{i=0}^{\infty} \frac{\lambda_i + \frac{1}{p}}{\left(\lambda_i + \frac{1}{p}\right)^2 + 1} < \infty\,,$$

then $\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \ldots\}$ is not dense in $L_p[0,1]$.

Outline. This follows from E.7 of Section 4.3. □

c] Suppose $(\lambda_i)_{i=0}^{\infty}$ is a sequence of distinct positive numbers. Let $p \in [1, \infty)$. Show that $\operatorname{span}\{e^{-\lambda_0 t}, e^{-\lambda_1 t}, \ldots\}$ is dense in $L_p[0,\infty)$ if and only if

$$\sum_{i=0}^{\infty} \frac{\lambda_i}{\lambda_i^2 + 1} = \infty\,.$$

Outline. Use parts a] and b] and the substitution $x = e^{-t}$. □

E.21 Müntz Theorem on $[a,b]$ with $a < 0 < b$. *Suppose $\Lambda := (\lambda_i)_{i=1}^{\infty}$ is a sequence of distinct nonnegative integers, and suppose $a < 0 < b$. Then* $\operatorname{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \ldots\}$ *is dense in $C[a,b]$ if and only if*

$$\sum_{\substack{i=1 \\ \lambda_i \text{ is even}}}^{\infty} \frac{1}{\lambda_i} = \infty \qquad \textit{and} \qquad \sum_{\substack{i=1 \\ \lambda_i \text{ is odd}}}^{\infty} \frac{1}{\lambda_i} = \infty\,.$$

E.22 The Zeros of the Chebyshev Polynomials in Nondense Müntz Spaces. Let $(\lambda_i)_{i=0}^{\infty}$ be a sequence of distinct nonnegative real numbers with $\lambda_0 := 0$ and $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$. Let

$$T_n := T_n\{\lambda_0, \lambda_1, \ldots, \lambda_n; [0,1]\}$$

be the Chebyshev polynomials for $\operatorname{span}\{x^{\lambda_0}, \ldots, x^{\lambda_n}\}$ on $[0,1]$. Let

$$Z := \{x \in [0,1] : T_n(x) = 0 \text{ for some } n \in \mathbb{N}\}\,.$$

Let Z' be the set of all limit points of Z and Z'' be the set of all limit points of Z'. Show that $Z'' = \{1\}$.

Hint: Use the bounded Bernstein-type inequality of E.5 b] and the interlacing property of the zeros of the Chebyshev polynomials T_n. □

4.3 Unbounded Bernstein Inequalities

In Section 4.1 we characterized the denseness of C^1 Markov spaces by the behavior of the zeros of their associated Chebyshev polynomials. The principal result of this section is a characterization of denseness of Markov spaces by whether or not they have an unbounded Bernstein inequality.

Definition 4.3.1 (Unbounded Bernstein Inequality). Let $\mathcal{A}$ be a subset of $C^1[a,b]$. We say that $\mathcal{A}$ has an *everywhere unbounded Bernstein inequality* if

$$\sup\left\{\frac{\|p'\|_{[\alpha,\beta]}}{\|p\|_{[a,b]}}: \quad 0 \neq p \in \mathcal{A}\right\} = \infty$$

for every $[\alpha,\beta] \subset [a,b], \quad \alpha \neq \beta$.

The subset

$$\mathcal{A} := \{x^2 p(x) : p \in \mathcal{P}_n, \quad n = 0,1,\dots\}$$

has an everywhere unbounded Bernstein inequality despite the fact that $f'(0) = 0$ for every $f \in \mathcal{A}$.

The next result shows that in most instances the Chebyshev polynomial is close to extremal for Bernstein-type inequalities. This is a theme that will be explored further in later chapters.

Theorem 4.3.2 (A Bernstein-Type Inequality for Chebyshev Spaces). *Let $(1, f_1, \dots, f_n)$ be a Chebyshev system on $[a,b]$ such that each f_i is differentiable at $x_0 \in [a,b]$. Let*

$$T_n := T_n\{1, f_1, \dots, f_n; [a,b]\}$$

be the associated Chebyshev polynomial. Then

$$\frac{|p'(x_0)|}{\|p\|_{[a,b]}} \leq \frac{2}{1 - |T_n(x_0)|}\,|T_n'(x_0)|$$

for every $0 \neq p \in \operatorname{span}\{1, f_1, \dots, f_n\}$, provided $|T_n(x_0)| \neq 1$.

Proof. Let $a = y_0 < y_1 < \cdots < y_n = b$ denote the extreme points of T_n, that is,

$$T_n(y_i) = (-1)^{n-i}, \qquad i = 0,1,\dots,n$$

(see the definition and E.1 a] in Section 3.3). Let $y_k \leq x_0 \leq y_{k+1}$ and

$$0 \neq p \in H_n := \operatorname{span}\{1, f_1, \dots, f_n\}.$$

If $p'(x_0) = 0$, then there is nothing to prove. Assume that $p'(x_0) \neq 0$. Then we may normalize p so that

$$\|p\|_{[a,b]} = 1 \qquad \text{and} \qquad \operatorname{sign}(p'(x_0)) = \operatorname{sign}(T_n(y_{k+1}) - T_n(y_k))\,.$$

Let $\delta := |T_n(x_0)|$. Let $\epsilon \in (0,1)$ be fixed. Then there exists a constant η with $|\eta| \leq \delta + \frac{1}{2}(1-\delta)$ such that

$$\eta + \tfrac{1}{2}(1-\delta)(1-\epsilon)p(x_0) = T_n(x_0)\,.$$

Now let

$$q(x) := \eta + \tfrac{1}{2}(1-\delta)(1-\epsilon)p(x)\,.$$

Then

$$\|q\|_{[a,b]} < 1\,, \qquad q(x_0) = T_n(x_0)\,,$$

and

$$\operatorname{sign}(q'(x_0)) = \operatorname{sign}(T_n(y_{k+1}) - T_n(y_k))\,.$$

If the desired inequality did not hold for p, then for a sufficiently small $\epsilon > 0$

$$|q'(x_0)| > |T_n'(x_0)|\,,$$

so

$$h(x) := q(x) - T_n(x)$$

would have at least three zeros in (y_k, y_{k+1}). But h has at least one zero in each of (y_i, y_{i+1}). Hence $h \in H_n$ has at least $n+2$ zeros in $[a,b]$, which is a contradiction. □

We now state the main result.

Theorem 4.3.3 (Characterization of Denseness by Unbounded Bernstein Inequality). *Suppose $\mathcal{M} := (f_0, f_1, \dots)$ is an infinite Markov system on $[a,b]$ with each $f_i \in C^2[a,b]$, and suppose that $(f_1/f_0)'$ does not vanish on (a,b). Then* span $\mathcal{M}$ *is dense in $C[a,b]$ if and only if* span $\mathcal{M}$ *has an everywhere unbounded Bernstein inequality.*

Proof. The *only if* part of this Theorem is obvious. A good uniform approximation on $[a,b]$ to a function with uniformly large derivative on a subinterval $[\alpha,\beta] \subset [a,b]$ must have large derivative at some points in $[\alpha,\beta]$.

In the other direction we use Theorems 4.3.2 and 4.1.1 in the following way. Without loss of generality we may assume that $f_0 = 1$ (why?). If span $\mathcal{M}$ is not dense in $C[a,b]$, then, by Theorem 4.1.1, there exists a subinterval $[\alpha,\beta] \subset [a,b]$, where all elements of a subsequence of the sequence of associated Chebyshev polynomials,

$$(T_n\{1, f_1, \dots, f_n; [a,b]\}),$$

have no zeros. It remains to show that from this subsequence we can pick another subsequence (T_{n_i}) and a subinterval $[c,d] \subset [\alpha,\beta]$ with

$$\|T_{n_i}\|_{[c,d]} < 1 - \delta \tag{4.3.1}$$

and

$$\|T'_{n_i}\|_{[c,d]} < \gamma \tag{4.3.2}$$

for some absolute constants $\delta > 0$ and $\gamma > 0$. The result will now follow from Theorem 4.3.2. A proof that the above choice of (T_{n_i}) is possible is outlined in E.1. □

Theorem 4.3.3 has the following interesting corollary.

Corollary 4.3.4. *Suppose $(\alpha_k)_{k=1}^{\infty} \subset \mathbb{R} \setminus [-1,1]$ is a sequence of distinct real numbers. Then*

$$\operatorname{span}\left\{1, \frac{1}{x-\alpha_1}, \frac{1}{x-\alpha_2}, \dots\right\}$$

is dense in $C[-1,1]$ if and only if

$$\sum_{k=1}^{\infty} \sqrt{\alpha_k^2 - 1} = \infty.$$

(Here, unlike in Section 3.5, $\sqrt{\alpha_k^2 - 1}$ denotes the principal square root of $\alpha_k^2 - 1$.)

Proof. A combination of Theorem 4.3.3 and Corollary 7.1.3 yields the *only if* part of the corollary.

The Chebyshev polynomials T_n (of the first kind) and U_n (of the second kind) for the Chebyshev space

$$\operatorname{span}\left\{1, \frac{1}{x-\alpha_1}, \dots, \frac{1}{x-\alpha_n}\right\}$$

on $[-1,1]$ were introduced in Section 3.5. The properties of

$$\widetilde{T}_n(\theta) := T_n(\cos\theta) \qquad \text{and} \qquad \widetilde{U}_n(\theta) := U_n(\cos\theta)\sin\theta,$$

established in Section 3.5, include

$$\|\widetilde{T}_n\|_{\mathbb{R}} = 1 \quad \text{and} \quad \|\widetilde{U}_n\|_{\mathbb{R}} = 1, \tag{4.3.3}$$

$$\widetilde{T}_n^2 + \widetilde{U}_n^2 = 1\,, \tag{4.3.4}$$

$$(\widetilde{T}_n')^2 + (\widetilde{U}_n')^2 = \widetilde{B}_n^2\,, \tag{4.3.5}$$

$$\widetilde{T}_n' = -\widetilde{B}_n \widetilde{U}_n\,, \tag{4.3.6}$$

and

$$\widetilde{U}_n' = \widetilde{B}_n \widetilde{T}_n\,, \tag{4.3.7}$$

where

$$\widetilde{B}_n(\theta) := \sum_{k=1}^{n} \frac{\sqrt{\alpha_k^2 - 1}}{|\alpha_k - \cos\theta|}\,, \qquad \theta \in \mathbb{R}$$

($\sqrt{\alpha_k^2 - 1}$ denotes the principal square root of $\alpha_k^2 - 1$) and the identities hold on the real line. Suppose

$$\sum_{k=1}^{\infty} \sqrt{\alpha_k^2 - 1} = \infty\,.$$

Then

$$\lim_{n\to\infty} \min_{\theta\in[\alpha,\beta]} B_n(\theta) = \infty\,, \qquad 0 < \alpha < \beta < \pi\,. \tag{4.3.8}$$

Assume that there is an interval $[a,b] \subset (-1,1)$ such that

$$\sup_{n\in\mathbb{N}} \|T_n'\|_{[a,b]} < \infty\,.$$

Let $\alpha := \arccos b$ and $\beta := \arccos a$. Then

$$\sup_{n\in\mathbb{N}} \|\widetilde{T}_n'\|_{[\alpha,\beta]} < \infty\,.$$

It follows from properties (4.3.6) and (4.3.8) that

$$\lim_{n\to\infty} \|\widetilde{U}_n\|_{[\alpha,\beta]} = 0\,,$$

and hence, by property (4.3.4),

$$\lim_{n\to\infty} \|\widetilde{T}_n^2 - 1\|_{[\alpha,\beta]} = 0\,.$$

Thus, by properties (4.3.7) and (4.3.8),

$$\lim_{n\to\infty} \min_{\theta\in[\alpha,\beta]} |\widetilde{U}_n'(\theta)| = \infty\,,$$

that is,

$$\lim_{n\to\infty} |\widetilde{U}_n(\beta) - \widetilde{U}_n(\alpha)| = \infty\,,$$

which contradicts property (4.3.3). Hence

$$\sup_{n\in\mathbb{N}} \frac{\|T_n'\|_{[a,b]}}{\|T_n\|_{[-1,1]}} = \sup_{n\in\mathbb{N}} \|T_n'\|_{[a,b]} = \infty$$

for every $[a,b] \subset (-1,1)$, which, together with Theorem 4.3.3 shows the *if* part of the corollary. □

Comments, Exercises, and Examples.

We followed Borwein and Erdélyi [95a] in this section. Corollary 4.3.4, to be found in Achiezer [56], is proved by using entirely different methods.

E.1 The Crucial Detail in the Proof of Theorem 4.3.3. Suppose that $\mathcal{M} := (1, f_1, f_2, \dots)$ is an infinite Markov system of C^2 functions on $[a,b]$ and f_1' does not vanish on (a,b). Suppose that the sequence of associated Chebyshev polynomials (T_n) has a subsequence (T_{n_i}) with no zeros on some subinterval $[\alpha,\beta]$ of $[a,b]$. Show that there exists another subinterval $[c,d]$ and another infinite subsequence (T_{n_i}) such that for some $\delta > 0$ and $\gamma > 0$, and for each n_i,

$$\|T_{n_i}\|_{[c,d]} < 1 - \delta \qquad \text{and} \qquad \|T_{n_i}'\|_{[c,d]} < \gamma\,.$$

Outline. For both inequalities first choose a subinterval $[c_1,d_1] \subset [\alpha,\beta]$ and a subsequence $(n_{i,1})$ of (n_i) such that each alternation point of each $T_{n_{i,1}}$ is outside $[c_1,d_1]$. Then choose a subsequence $(n_{i,2})$ of $(n_{i,1})$ so that either each $T_{n_{i,2}}$ is increasing or each $T_{n_{i,2}}$ is decreasing on $[c_1,d_1]$. Study the first case; the second is analogous. Let $[c_2,d_2]$ be the middle third of $[c_1,d_1]$. If the first inequality fails to hold with $[c_2,d_2]$ and $(n_{i,2})$, then there is a subsequence $(n_{i,3})$ of $(n_{i,2})$ such that $\|T_{n_{i,3}}\|_{[c_2,d_2]} \to 1$ as $n_{i,3} \to \infty$. Hence, there is a subsequence $(n_{i,4})$ of $(n_{i,3})$ such that either

$$\max_{c_2 \le x \le d_2} T_{n_{i,4}}(x) \to 1 \qquad \text{or} \qquad \min_{c_2 \le x \le d_2} T_{n_{i,4}}(x) \to -1\,.$$

Once again, study the first case; the second is analogous. Since each $T_{n_{i,3}}$ is increasing on $[c_1,d_1]$,

$$\lim_{n_{i,4} \to \infty} \|1 - T_{n_{i,4}}\|_{[d_2,d_1]} = 0\,.$$

Now choose $g := a_0 + a_1 f_1 + a_2 f_2$ so that g has two distinct zeros α_1 and α_2 in $[d_2,d_1]$, $\|g\|_{[\alpha_1,\alpha_2]} < 1$, and g is positive on (α_1,α_2). Let $\beta := \max_{\alpha_1 \le x \le \alpha_2} g(x)$ and $\widetilde{g} := g + 1 - \beta$. Show that $T_{n_{i,4}} - \widetilde{g}$ has at least $n+1$ distinct zeros in $[a,b]$ if $n_{i,4}$ is large enough, which is a contradiction.

For the second inequality, note that E.4 of Section 3.2 implies that $(f_1', \dots, f_n')$ is a weak Chebyshev system on $[a,b]$, and so is

$$\left(\left(\frac{T_2'}{T_1'} \right)', \left(\frac{T_3'}{T_1'} \right)', \dots, \left(\frac{T_n'}{T_1'} \right)' \right), \qquad n = 2,3,\dots\,.$$

From this deduce that each $(T_{n_{i,2}}'/T_1')'$ has at most one sign change in $[c_2,d_2]$. Choose a subinterval $[c_3,d_3] \subset [c_2,d_2]$ and a subsequence $(n_{i,5})$ of

$(n_{i,2})$ so that none of $(T'_{n_{i,5}}/T'_1)'$ changes sign in $[c_3, d_3]$. Choose a subsequence $(n_{i,6})$ of $(n_{i,5})$ so that either each $T'_{n_{i,6}}/T'_1$ is increasing or each $T'_{n_{i,6}}/T'_1$ is decreasing on $[c_3, d_3]$. Again, consider the first case; the second is analogous. Let $[c_4, d_4]$ be the middle third of $[c_3, d_3]$. If the second inequality fails to hold with $[c_4, d_4]$ and $(n_{i,6})$, then there is a subsequence $(n_{i,7})$ of $(n_{i,6})$ such that either

$$\lim_{n_{i,7}\to\infty} \max_{c_4\le x\le d_4} \frac{T'_{n_{i,7}}(x)}{T'_1(x)} = \infty$$

or

$$\lim_{n_{i,7}\to\infty} \min_{c_4\le x\le d_4} \frac{T'_{n_{i,7}}(x)}{T'_1(x)} = -\infty\,.$$

Once again we just treat the first case; the second is analogous. In this case, for every $K > 0$ there is an $N \in \mathbb{N}$ such that for every $n_{i,7} \ge N$ we have

$$T'_{n_{i,7}}(x) > K\,, \qquad x \in [d_4, d_3]\,.$$

Hence

$$K(d_3 - d_4) \le \int_{d_4}^{d_3} T'_{n_{i,7}}(x)\,dx = T_{n_{i,7}}(d_3) - T_{n_{i,7}}(d_4) \le 2\,,$$

which is a contradiction. □

E.2 On the Uniform Closure of Nondense Markov Spaces. Suppose that $\mathcal{M} = (f_0, f_1, \dots)$ is an infinite Markov system on $[a, b]$ with each $f_i \in C^2[a, b]$, and suppose that $(f_1/f_0)'$ does not vanish on (a, b). Suppose that span $\mathcal{M}$ fails to be dense in $C[a, b]$.

Show that there exists a subinterval $[\alpha, \beta] \subset [a, b]$, $\alpha < \beta$, such that every $g \in C[a, b]$ in the uniform closure of span $\mathcal{M}$ on $[a, b]$ is differentiable on $[\alpha, \beta]$.

Hint: By Theorem 4.3.3 there exists an interval $[\alpha, \beta] \subset [a, b]$ and a constant $\eta \in \mathbb{R}$ so that

$$\|h'\|_{[\alpha,\beta]} \le \eta \|h\|_{[a,b]}$$

for every $h \in$ span $\mathcal{M}$. Suppose $g \in C[a, b]$ and

$$\lim_{n\to\infty} \|h_n - g\|_{[a,b]} = 0.$$

Choose $n_i \in \mathbb{N}$ such that

$$\|g - h_{n_i}\|_{[a,b]} \le 2^{-i}\,, \qquad i = 0, 1, \dots\,.$$

Then

$$g = h_{n_0} + \sum_{i=1}^{\infty} (h_{n_i} - h_{n_{i-1}})\,.$$

Since

$$\|(h_{n_i} - h_{n_{i-1}})'\|_{[\alpha,\beta]} \le \eta 2^{1-i}\,,$$

it follows that g is differentiable on $[\alpha, \beta]$. □

E.3 An Analog of Theorem 4.3.3 with Applications. Suppose that $\mathcal{M} = (f_0, f_1, \dots)$ is an extended complete Chebyshev (ECT) system of C^∞ functions on $[a,b]$, as in E.3 of Section 3.1. Note that E.3 a] of Section 3.1 implies that f_0 does not vanish on $[a,b]$.

a] Show that the differential operator D defined by

$$D(f) := \left(\frac{f}{f_0}\right)', \qquad f \in C^1[a,b]$$

maps $\mathcal{M}$ to $\mathcal{M}_D$, where

$$\mathcal{M}_D := \left(\left(\frac{f_1}{f_0}\right)', \left(\frac{f_2}{f_0}\right)', \dots\right)$$

and $\mathcal{M}_D$ is once again an ECT system of C^∞ functions on $[a,b]$.

Hint: Use E.8 b] of Section 3.2. □

b] The differential operators $D^{(n)}(f)$ are defined for every $f \in C^n[a,b]$ as follows. Let

$$F_{i,0} := f_i, \qquad i = 0,1,2,\dots,$$
$$F_{i,n} := \left(\frac{F_{i+1,n-1}}{F_{0,n-1}}\right)', \qquad i = 0,1,2,\dots, \quad n = 1,2,\dots,$$
$$D^{(0)}(f) := f, \quad D^{(n)}(f) := \left(\frac{D^{(n-1)}(f)}{F_{0,n-1}}\right)', \quad n = 1,2,\dots.$$

Let

$$\mathcal{M}_{D^{(1)}} := \mathcal{M}_D \qquad \text{and} \qquad \mathcal{M}_{D^{(n)}} := \left(\mathcal{M}_{D^{(n-1)}}\right)_D, \qquad n = 2,3,\dots.$$

Show that if span $\mathcal{M}_{D^{(n)}}$ is dense in $C[a,b]$, then so is span $\mathcal{M}$.

c] Suppose that span $\mathcal{M}$ fails to be dense in $C[c,d]$ for every subinterval $[c,d] \subset [a,b]$, $c < d$. Show that for each $n \in \mathbb{N}$, there exists an interval $[\alpha_n, \beta_n] \subset [a,b]$, $\alpha_n < \beta_n$, such that

$$\sup\left\{\frac{\|D^{(n)}(f)\|_{[\alpha_n,\beta_n]}}{\|f\|_{[a,b]}} : \ 0 \neq f \in \text{span } \mathcal{M}\right\} < \infty.$$

Hint: Use Theorem 4.3.1 and induction on n. □

d] Suppose that span $\mathcal{M}$ fails to be dense in $C[c,d]$ for every subinterval $[c,d] \subset [a,b]$ $c < d$. Show that for each $n \in \mathbb{N}$, there exists an interval $[\alpha_n, \beta_n] \subset [a,b]$, $\alpha_n < \beta_n$, where every $g \in C[a,b]$ in the uniform closure of span $\mathcal{M}$ on $[a,b]$ is n times continuously differentiable.

Hint: Use part c]. The argument is similar to the one given in the hint to E.2. □

e] Suppose that span $\mathcal{M}$ fails to be dense in $C[c,d]$ for every subinterval $[c,d] \subset [a,b]$, $c < d$.

Show that every function in the uniform closure of span $\mathcal{M}$ on $[a,b]$ is C^∞ on a dense subset of $[a,b]$.

(This is the case for Müntz systems

$$\mathcal{M} := (x^{\lambda_0}, x^{\lambda_1}, \dots), \qquad \lambda_i \in \mathbb{R}, \qquad \sum_{\substack{i=1 \\ \lambda_i \neq 0}}^{\infty} \frac{1}{|\lambda_i|} < \infty$$

on $[a,b]$, $0 \leq a < b$; see E.7 of Section 4.2.)

E.4 Bounded Bernstein-Type Inequality for Nondense Müntz Spaces. Suppose $(\lambda_i)_{i=1}^{\infty}$ is a sequence of distinct positive numbers satisfying

$$\sum_{i=1}^{\infty} \frac{\lambda_i}{\lambda_i^2 + 1} < \infty .$$

Then for every $\epsilon > 0$, there is a constant c_ϵ such that

$$|p'(x)| \leq \frac{c_\epsilon}{x} \|p\|_{[0,1]}, \qquad x \in (0, 1-\epsilon]$$

for every

$$p \in \operatorname{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}.$$

To prove this proceed as follows. Let $\lambda_0 := 0$, and let

$$T_n := T_n\{\lambda_0, \lambda_1, \dots, \lambda_n; [0,1]\}$$

be the Chebyshev polynomial for $\operatorname{span}\{1, x^{\lambda_1}, \dots, x^{\lambda_n}\}$ on $[0,1]$. Let

$$M(\Lambda) := \operatorname{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}.$$

a] Observe that for every $\epsilon > 0$ there exists a $k_\epsilon \in \mathbb{N}$ depending only on $(\lambda_i)_{i=1}^{\infty}$ and ϵ (and not on n) such that T_n has at most k_ϵ zeros in $[\epsilon, 1-\epsilon]$.

Hint: This is proved in the outline of the proof of E.18 d] of Section 4.2. □

b] Show that every nonempty $(a,b) \subset (0,1)$ contains a nonempty subinterval (α, β) for which there are integers $0 < n_1 < n_2 < \cdots$ such that none of the Chebyshev polynomials T_{n_i} vanishes on (α, β).

Hint: Use part a]. □

c] Show that every nonempty $(a,b) \subset (0,1)$ contains a nonempty subinterval (α,β) such that

$$\sup\left\{\frac{\|p'\|_{[\alpha,\beta]}}{\|p\|_{[0,1]}}: \ \ 0 \neq p \in M(\Lambda)\right\} < \infty\,.$$

Hint: Modify the proof of Theorem 4.3.3. □

d] Finish the proof of the initial statement of the exercise.

Hint: Use part c] and a linear scaling. □

E.5 The Closure of Nondense Müntz Spaces in $C[0,1]$. Suppose $(\lambda_i)_{i=1}^{\infty}$ is a sequence of distinct positive numbers satisfying

$$\sum_{i=1}^{\infty}\frac{\lambda_i}{\lambda_i^2+1} < \infty\,.$$

Show that

$$\overline{\text{span}}\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\} \subset C^{\infty}(0,1)\,,$$

that is, if f is the uniform limit of a sequence from $\text{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}$, then f is infinitely many times differentiable on $(0,1)$.

Hint: Use E.4 with the substitution $x = e^{-t}$. □

E.6 A Nondense Markov Space with Unbounded Bernstein Inequality on a Subinterval. One may incorrectly suspect that nondense Markov spaces on $[a,b]$ can be characterized by an *everywhere bounded* Bernstein inequality on (a,b), at least under the assumptions of Theorem 4.3.3. The purpose of this exercise is to show that this is far from true, and in a sense, Theorem 4.3.3 is the best possible result.

The same construction can be used to give a nondense Markov space on $[a,b]$ such that the set

$$Z := \{x \in [a,b] : T_n(x) = 0 \ \text{ for some } \ n \in \mathbb{N}\}$$

is neither dense nor nowhere dense in $[a,b]$. Here $(T_n)_{n=0}^{\infty}$ is the sequence of associated Chebyshev polynomials on $[a,b]$.

We construct an infinite Markov system on $(-\infty,\infty)$ as follows. Suppose $\Lambda := (\lambda_i)_{i=0}^{\infty}$ is a sequence of even integers satisfying

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots\,, \qquad \sum_{i=1}^{\infty}\frac{1}{\lambda_i} < \infty\,.$$

Suppose $m > 0$. Let $\varphi_k \in C(-\infty,\infty)$, $k = 0,1,\dots$, be defined by

$$\varphi_{2k}(x) := \begin{cases} x^{2k+m}\,, & x \geq 0 \\ c_{2k}x^{\lambda_{2k}}\,, & x \leq 0 \end{cases}$$

and

$$\varphi_{2k+1}(x) := \begin{cases} x^{2k+m+1}\,, & x \geq 0 \\ -c_{2k+1}x^{\lambda_{2k+1}}\,, & x \leq 0\,, \end{cases}$$

where $(c_i)_{i=0}^{\infty}$ is a sequence of positive numbers associated with a fixed sequence of integers $0 := n_0 < n_1 < n_2 < \cdots$ and a constant $\delta > 0$, and it is chosen as follows. Let

$$m_j := n_{j+1} - n_j - 1\,, \qquad j = 0, 1, \ldots\,.$$

Let

$$T_n(x) := \cos(n \arccos(2x-1)) =: \sum_{i=0}^{n} \alpha_{i,n} x^i$$

be the nth Chebyshev polynomial on $[0,2]$. Now choose the constants $c_i > 0$ such that

$$\sum_{j=0}^{\infty} \sum_{i=n_j}^{n_{j+1}-1} c_i \delta^{\lambda_i} |\alpha_{i-n_j,m_j}| \leq 1\,.$$

a] Show that $(\varphi_0, \varphi_1, \ldots)$ is a Markov system on $(-\infty, \infty)$.

Hint: If

$$p(x) = \sum_{i=0}^{n} a_i \varphi_i(x)\,, \qquad a_i \in \mathbb{R}\,,$$

then

$$p(x) = \sum_{i=0}^{n} a_i x^{\lambda_i}\,, \qquad x \in [0, \infty)\,,$$

while

$$p(x) = \sum_{i=0}^{n} (-1)^i a_i x^{\lambda_i}\,, \qquad x \in (-\infty, 0]\,.$$

Now apply Theorem 3.2.4 (Descartes' rule of signs). □

b] Show that $\operatorname{span}\{\varphi_0, \varphi_1, \ldots\}$ is not dense in $C[-\delta, 2]$.

c] Show that there is a sequence of integers $0 := n_0 < n_1 < n_2 < \cdots$ such that

$$\sup \left\{ \frac{\|p'\|_{[\alpha,\beta]}}{\|p\|_{[-\delta,2]}} : \quad 0 \not\equiv p \in \operatorname{span}\{\varphi_0, \varphi_1, \ldots\} \right\} = \infty$$

for every nonempty interval $[\alpha, \beta] \subset [0, 2]$, while

$$\sup \left\{ \frac{|p'(x)|}{\|p\|_{[-\delta,2]}} : \quad 0 \not\equiv p \in \operatorname{span}\{\varphi_0, \varphi_1, \ldots\} \right\} < \infty$$

for every $x \in (-\delta, 0)$.

d] Suppose the sequence $(\varphi_i)_{i=0}^{\infty}$ is defined associated with a fixed sequence of integers $0 := n_0 < n_1 < n_2 < \cdots$ and $\delta := -2$. Let

$$T_n := T_n\{\varphi_0, \varphi_1, \dots, \varphi_n; [-2,2]\}$$

be the Chebyshev polynomial for $\text{span}\{\varphi_0, \varphi_1, \dots, \varphi_n\}$ on $[-2,2]$. Let

$$Z := \{x \in [-2,2] : T_n(x) = 0 \text{ for some } n \in \mathbb{N}\}$$

and let Z' be the set of all limit points of Z, and let Z'' be the set of all limit points of Z'. Show that the sequence of integers $0 := n_0 < n_1 < n_2 < \cdots$ in the definition of $(\varphi_i)_{i=0}^{\infty}$ can be chosen so that

$$Z'' \cap (-2,0) = \emptyset \qquad \text{and} \qquad [0,2] \cup \{-2\} \subset Z''\,.$$

E.7 Nikolskii-Type Inequalities for Nondense Müntz Spaces. Suppose that $p \in [1,\infty]$. Suppose $(\lambda_i)_{i=0}^{\infty}$ is a sequence of distinct real numbers greater than $-1/p$ satisfying

$$\sum_{i=0}^{\infty} \frac{\lambda_i + \frac{1}{p}}{\left(\lambda_i + \frac{1}{p}\right)^2 + 1} < \infty\,.$$

Show that for every $\epsilon > 0$, there exists a constant $c_\epsilon > 0$ depending only on ϵ and p so that

$$|q(x)| \le c_\epsilon x^{-1/p} \|q\|_{L_p[0,1]}$$

for every $q \in \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ and for every $x \in [0, 1-\epsilon]$.

In particular, for every $\epsilon > 0$, there exists a constant $c_\epsilon > 0$ depending only on ϵ so that

$$\|q\|_{[\epsilon,1-\epsilon]} \le c_\epsilon \|q\|_{L_p[0,1]}$$

for every $q \in \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$.

Thus, $\text{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ is not dense in $L_p[0,1]$.

Hint: Use Hölder's inequality to show, as in Operstein [to appear], that the operator

$$J : L_p[0,1] \mapsto L_\infty[0,1]$$

defined by

$$J(\varphi)(0) := 0\,, \qquad J(\varphi)(x) := x^{1/p-1} \int_0^x \varphi(t)\,dt\,, \quad x > 0$$

is a bounded linear operator. That is, there is a constant $c \ge 0$ such that

$$\|J(\varphi)\|_{L_\infty[0,1]} \le c\,\|\varphi\|_{L_p[0,1]}$$

for every $\varphi \in L_p[0,1]$. Now observe that $q \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ implies that $J(q) \in \operatorname{span}\{x^{\mu_0}, x^{\mu_1}, \dots\}$, where $\mu_i := \lambda_i + \frac{1}{p}$, and so $(\mu_i)_{i=0}^{\infty}$ is a sequence of positive numbers satisfying

$$\sum_{i=0}^{\infty} \frac{\mu_i}{\mu_i^2 + 1} < \infty \,.$$

Therefore, by E.4, for every $\epsilon > 0$, there exists a constant $c_\epsilon > 0$ such that

$$|(J(q))'(x)| \le \frac{c_\epsilon}{x} \, \|J(q)\|_{[0,1]} \le \frac{c_\epsilon}{x} \, c \, \|q\|_{L_p[0,1]}$$

for every $q \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ and for every $x \in (0, 1-\epsilon]$. Note that for $x \in (0,1)$,

$$(J(q))'(x) = x^{1/p-1} q(x) - \left(1 - \tfrac{1}{p}\right) x^{1/p-2} \int_0^x q(t)\, dt \,,$$

where

$$\left| \int_0^x q(t)\, dt \right| = x^{1-1/p} |J(q)(x)| \le c x^{1-1/p} \|q\|_{L_p[0,1]} \,.$$

Therefore

$$x^{1/p} |q(x)| - \left(1 - \tfrac{1}{p}\right) \|q\|_{L_p[0,1]} \le c_\epsilon c \, \|q\|_{L_p[0,1]}$$

for every $q \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ and for every $x \in (0, 1-\epsilon]$. □

E.8 The Closure of Nondense Müntz Spaces in $L_p[0,1]$. Let $p \in [1, \infty]$. Suppose $(\lambda_i)_{i=0}^{\infty}$ is a sequence of distinct real numbers greater than $-1/p$ satisfying

$$\sum_{i=0}^{\infty} \frac{\lambda_i + \frac{1}{p}}{\left(\lambda_i + \frac{1}{p}\right)^2 + 1} < \infty \,.$$

Show that if f is a function in the $L_p[0,1]$ closure of

$$\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\} \,,$$

then $f \in C^\infty(0,1)$, that is, f is infinitely many times differentiable on $(0,1)$.

Hint: Combine E.5 and E.7. □

4.4 Müntz Rationals

A surprising and beautiful theorem, conjectured by Newman and proved by Somorjai [76], states that rational functions derived from any infinite Müntz system are always dense in $C[a,b]$, $a \geq 0$. More specifically, we have the following result.

Theorem 4.4.1 (Denseness of Müntz Rationals). *Let $(\lambda_i)_{i=0}^{\infty}$ be any sequence of distinct real numbers. Suppose $a > 0$. Then*

$$\left\{ \frac{\sum_{i=0}^{n} a_i x^{\lambda_i}}{\sum_{i=0}^{n} b_i x^{\lambda_i}} : \ a_i, b_i \in \mathbb{R}, \ n \in \mathbb{N} \right\}$$

is dense in $C[a,b]$.

The same result holds when $a = 0$, however, the proof in this case requires a few more technical details; see E.1 b].

The proof of this theorem, primarily due to Somorjai, rests on the next theorem. We introduce the following notation. A function Z defined on (a,b) is called an *ϵ-zoomer* ($\epsilon > 0$) at $\zeta \in (a,b)$ if

$$\begin{aligned} &Z(x) > 0\,, && x \in [a,b]\,, \\ &Z(x) \leq \epsilon\,, && x < \zeta - \epsilon\,, \\ &Z(x) \geq \epsilon^{-1}, && x > \zeta + \epsilon\,. \end{aligned} \tag{4.4.1}$$

While (approximate) δ-functions are the building blocks for polynomial approximations, the existence of ϵ-zoomers is all that is needed for rational approximations. More precisely, we have the following result.

Theorem 4.4.2 (Existence of Zoomers and Denseness). *Let S be a linear subspace of $C[a,b]$. Suppose that S contains an ϵ-zoomer for every $\epsilon > 0$ at every $\zeta \in (a,b)$. Then*

$$R(S) := \left\{ \frac{p}{q} : p, q \in S \right\}$$

is dense in $C[a,b]$.

Proof. It suffices to consider the case $[a,b] = [0,1]$. Let $n \in \mathbb{N}$ and $\epsilon > 1/n$ be fixed. We construct a partition of unity inductively as follows. Let Z_0 be any positive function in S and choose functions $Z_1, Z_2, \ldots, Z_n \in S$ positive on $[0,1]$ so that

$$Z_k(x) < \epsilon Z_{k-1}(x)\,, \qquad x < \frac{k}{n} - \epsilon$$

and

$$Z_k(x) > \epsilon^{-1} Z_{k-1}(x)\,, \qquad x > \frac{k-1}{n} + \epsilon$$

(which the existence of ϵ-zoomers allows for). Let

$$\Delta_k(x) := \frac{Z_k(x)}{\sum_{i=0}^n Z_i(x)}\,, \qquad k = 0, 1, \ldots, n\,. \tag{4.4.2}$$

So

$$\Delta_k(x) > 0\,, \qquad x \in [0,1]$$

and

$$\sum_{k=0}^n \Delta_k(x) = 1\,.$$

Since for every $x \in [0,1]$,

$$\sum_{\substack{k=0 \\ \frac{k}{n}-\epsilon > x}}^n Z_k(x) + \sum_{\substack{k=0 \\ \frac{k}{n}+\epsilon < x}}^n Z_k(x) < 2\epsilon \sum_{k=0}^n Z_k(x)\,,$$

we also know that

$$\sum_{\substack{k=0 \\ \left|\frac{k}{n}-x\right| > \epsilon}}^n \Delta_k(x) \le 2\epsilon\,, \qquad x \in [0,1]\,.$$

Now let $f \in C[0,1]$, and consider the approximation

$$\sum_{k=0}^n f\left(\tfrac{k}{n}\right) \Delta_k(x) \in R(S)\,.$$

Then

$$\begin{aligned}
&\left| f(x) - \sum_{k=0}^n f\left(\tfrac{k}{n}\right) \Delta_k(x) \right| \\
&\le \left| \sum_{\substack{k=0 \\ \left|\frac{k}{n}-x\right| \le \epsilon}}^n \left(f(x) - f\left(\tfrac{k}{n}\right)\right) \Delta_k(x) \right| + \left| \sum_{\substack{k=0 \\ \left|\frac{k}{n}-x\right| > \epsilon}}^n \left(f(x) - f\left(\tfrac{k}{n}\right)\right) \Delta_k(x) \right| \\
&\le\ \omega_f(\epsilon) + 2\epsilon\, \omega_f(1)\,,
\end{aligned}$$

which finishes the proof. □

We can now finish the proof of Theorem 4.4.1:

Proof of Theorem 4.4.1. We may suppose, on passing to a subsequence if necessary, that $(\lambda_i)_{i=0}^\infty$ is a convergent sequence (possibly converging to infinity). We may also assume that $a := 1/b$ with $b > 1$. Since $C[1/b, b]$ is

invariant under $x \to 1/x$, we may assume that $(\lambda_i)_{i=0}^{\infty}$ has a nonnegative (possibly infinite) limit.

Case 1: The sequence $(\lambda_i)_{i=0}^{\infty}$ has a finite nonnegative limit. Then Müntz's theorem on $[a,b]$, $a > 0$ (see E.7 of Section 4.2), yields that the Müntz polynomials themselves associated with $(\lambda_i)_{i=0}^{\infty}$ are already dense in $C[1/b, b]$.

Case 2: The sequence $(\lambda_i)_{i=0}^{\infty}$ tends to infinity. In this case $(x/\zeta)^{\lambda_i}$ is an ϵ-zoomer at $\zeta \in (a,b)$ for sufficiently large λ_i, and the result follows from Theorem 4.4.2. □

Comments, Exercises, and Examples.

A comparison between Müntz's theorem and the main result of this section shows the power of a single division in approximation. In what other contexts does allowing a division create a spectacularly different result? Newman [78, p. 12] conjectures that if $\mathcal{M}$ is any infinite Markov system on $[0,1]$, then the set

$$\left\{ \frac{p}{q} : p, q \in \text{span } \mathcal{M} \right\}$$

of rational functions is dense in $C[0,1]$.

Newman calls this a "wild conjecture in search of a counterexample." It does, however, hold for both

$$\mathcal{M} = (x^{\lambda_0}, x^{\lambda_1}, \dots), \qquad \lambda_i \geq 0 \text{ are distinct}$$

(see E.1 b]) and

$$\mathcal{M} = \left(\frac{1}{x-\alpha_1}, \frac{1}{x-\alpha_2}, \dots \right), \qquad \alpha_i \in \mathbb{R} \setminus [0,1] \text{ are distinct}$$

(see E.2). A counterexample to the full generality of this conjecture is presented in E.6. However, the characterization of the class of Markov systems for which it holds remains as an interesting question. In particular, it is open if Newman's conjecture holds for Descartes systems.

The reader is referred to Newman [78] for an extensive treatment of these matters; see also Zhou [92a]. In [78, p. 50] Newman asks about the denseness of the products $(\sum a_i x^{i^2})(\sum b_i x^{i^2})$ in $C[0,1]$ (see E.3). He speculates that this "extra" multiplication of Müntz polynomials should not carry the utility of the "extra" division. This is proved in Section 6.2, where it is shown that products pq of Müntz polynomials from nondense Müntz spaces never form a dense set in $C[0,1]$.

E.1 Denseness of Müntz Rationals on $[0,1]$. A function C defined on $[a,b]$ is called an *ϵ-crasher*, $\epsilon > 0$, at $\zeta \in (a,b)$ if

$$\begin{aligned} C(x) &> 0, && x \in [a,b], \\ C(x) &\leq \epsilon, && x > \zeta + \epsilon, \\ C(x) &\geq \epsilon^{-1}, && x \leq \zeta - \epsilon. \end{aligned}$$

a] Let S be a linear subspace of $C[a,b]$. Suppose that S contains an ϵ-crasher for every $\epsilon > 0$ at every $\zeta \in (a,b)$. Show that

$$R(S) := \left\{ \frac{p}{q} : p, q \in S \right\}$$

is dense in $C[a,b]$.

Hint: The argument is a trivial modification of the proof of Theorem 4.4.2. □

Let $(\lambda_i)_{i=0}^{\infty}$ be a sequence of distinct real numbers. Let $R(\Lambda)$ be the space of functions $f \in C[0,1]$ of the form

$$f(x) = \frac{\sum_{i=0}^{n} a_i x^{\lambda_i}}{\sum_{i=0}^{n} b_i x^{\lambda_i}}, \quad a_i, b_i \in \mathbb{R}, \quad n \in \mathbb{N}, \quad x \in (0,1].$$

b] Show that if $(\lambda_i)_{i=0}^{\infty}$ is a sequence of distinct nonnegative real numbers with $\lambda_0 := 0$, then $R(\Lambda)$ is dense in $C[0,1]$.

Hint: As in the proof of Theorem 4.4.1, we may suppose, on passing to a subsequence if necessary, that $(\lambda_i)_{i=1}^{\infty}$ is a convergent sequence (possibly converging to ∞). Distinguish the following two cases.

Case 1: $\lim_{i\to\infty} \lambda_i = \infty$. Given $\epsilon > 0$ and $\zeta \in (0,1)$, show that

$$Z(x) := \frac{\epsilon}{2} + \left(\frac{x}{\zeta}\right)^{\lambda_i}$$

is an ϵ-zoomer at ζ on $[0,1]$ if λ_i is large enough. Use Theorem 4.4.2 to finish the proof.

Case 2: $(\lambda_i)_{i=1}^{\infty}$ is a sequence with finite limit. Given $\epsilon > 0$ and $\zeta \in (0,1)$, show that

$$C(x) := \frac{\epsilon}{2} + (-1)^n \frac{\epsilon}{4} T_n\{\lambda_0, \lambda_1, \ldots, \lambda_n; [\zeta, 1]\}(x)$$

is an ϵ-crasher at ζ on $[0,1]$ if n is large enough. This can be proved by using Müntz's theorem (E.7 of Section 4.2) on $[\zeta - \epsilon, 1]$ with $\epsilon \in (0, \zeta)$, E.3 c] of Section 3.3, and the monotonicity of C on $[0, \zeta]$. Finish the proof by part a]. □

The result of E.1 b] is due to Bak and Newman [78]. Zhou [92b] extends this result to sequences of arbitrary distinct real numbers.

E.2 Another Markov-System with Dense Rationals. Let $(\beta_i)_{i=1}^{\infty}$ be any sequence of distinct numbers in $\mathbb{R} \setminus [0,1]$. Show that

$$\left\{ \frac{\sum_{i=1}^{n} \frac{a_i}{x-\beta_i}}{\sum_{i=1}^{n} \frac{b_i}{x-\beta_i}} : \ a_i, b_i \in \mathbb{R}, \ n \in \mathbb{N} \right\}$$

is dense in $C[0,1]$.

E.3 Products of Müntz Spaces. Associated with a sequence $\Lambda := (\lambda_i)_{i=0}^{\infty}$ of real numbers, let

$$M^2(\Lambda) := \left\{ \left(\sum_{i=0}^{n} a_i x^{\lambda_i} \right) \left(\sum_{i=0}^{n} b_i x^{\lambda_i} \right) : \ a_i, b_i \in \mathbb{R}, \ n \in \mathbb{N} \right\} .$$

a] Show that if $(\lambda_k)_{k=0}^{\infty} = (k^{\alpha})_{k=0}^{\infty}$, $\alpha > 2$, then $M^2(\Lambda)$ is not dense in $C[0,1]$.

b] Show that with $(\lambda_k)_{k=0}^{\infty} = (k^2)_{k=0}^{\infty}$ the nondenseness of $M^2(\Lambda)$ does not follow from Müntz's theorem since $\sum_{k \in \Gamma} 1/k = \infty$, where Γ is the set of natural numbers k of the form $k = n^2 + m^2$ with nonnegative integers n and m.

It is shown in Section 6.2 that $M^2(\Lambda)$ is not dense in $C[0,1]$ whenever Λ is a sequence of nonnegative real numbers satisfying $\sum_{k=1}^{\infty} 1/\lambda_k < \infty$, so the "extra multiplication" is of no spectacular utility.

It is not always possible to extend Theorem 4.4.2 to the case when the numerators and the denominators are coming from different infinite Müntz spaces. Somorjai [76] shows that

$$\left\{ \frac{\sum_{i=0}^{n} a_i x^{\lambda_i}}{\sum_{i=0}^{n} b_i x^{\delta_i}} : \ a_i, b_i \in \mathbb{R}, \ n \in \mathbb{N} \right\}$$

is not dense in $C[0,1]$ when, for example,

$$\eta^2 \lambda_k < \eta \delta_{k+1} < \lambda_{k+1} , \qquad k = 0, 1, \ldots$$

for some $\eta > 1$.

E.4 Nondense Ratios of Müntz Spaces. Suppose $0 \le \lambda_0 < \lambda_1 < \cdots$. Let $a > 0$. Show that

$$\left\{ \frac{\sum_{i=0}^{n} a_i x^{\lambda_i}}{\sum_{i=0}^{n} b_i x^{-\lambda_i}} : \ a_i, b_i \in \mathbb{R}, \ n \in \mathbb{N} \right\}$$

is dense in $C[a,b]$ if and only if $\sum_{i=1}^{\infty} 1/\lambda_i = \infty$.

Hint: For one direction use Müntz's theorem. For the other direction use E.5 a] and b] and E.8 b] and c] of Section 4.2. □

E.5 On the Rate of Approximation by Müntz Rationals. Let $(\lambda_i)_{i=0}^{\infty}$ be a fixed sequence of nonnegative real numbers. We wish to estimate the error of the best uniform approximation to $f \in C[0,1]$ on $[0,1]$ from $\operatorname{span}\{x^{\lambda_0}, \ldots, x^{\lambda_n}\}$. We let

$$R_n^*(f) := \inf \left\{ \left\| f(x) - \frac{\sum_{i=0}^{n} a_i x^{\lambda_i}}{\sum_{i=0}^{n} b_i x^{\lambda_i}} \right\|_{[0,1]} : \ a_i, b_i \in \mathbb{R} \right\} ,$$

where the infimum is taken for all $a_i, b_i \in \mathbb{R}$, $i = 0, 1, \ldots, n$.

In the case when $\lambda_{n+1} - \lambda_n \geq 1,\ n = 0, 1, \ldots$, Newman [78] claims (without proof) that there exists a constant C independent of n such that

$$R_n^*(f) = C\omega_f\left(\tfrac{1}{n}\right)$$

and that this is the best possible. He conjectures, in Newman [78], that this estimate holds for every sequence $(\lambda_i)_{i=0}^{\infty}$ of distinct nonnegative real numbers.

a] Observe in the last line of the proof of Theorem 4.4.2, with $\epsilon = \frac{2}{n}$, we have

$$\left| f(x) - \sum_{k=0}^{n} f\left(\tfrac{k}{n}\right) \Delta_k(x) \right| \leq 6\,\omega_f\left(\tfrac{1}{n}\right)$$

since $\omega_f(1) \leq n\omega_f\left(\frac{1}{n}\right)$.

b] What growth conditions on the sequence $(\lambda_i)_{i=0}^{\infty}$ guarantees that

$$R_n^*(f) = O\left(\omega_f\left(\tfrac{1}{n}\right)\right)?$$

Hint: Estimate the "degree" of the zoomers Z_k defined in Theorem 4.4.2. Use the zoomers defined in Case 1 of the hint for E.1 b]. □

c] Let

$$f(x) := \begin{cases} x\sin(1/x), & x \in \mathbb{R} \setminus \{0\} \\ 0, & x = 0. \end{cases}$$

Show that

$$R_n^*(f) \geq c_1 n^{-1} \geq c_2\,\omega_f\left(\tfrac{1}{n}\right)$$

for every sequence $(\lambda_i)_{i=0}^{\infty}$ of nonnegative real numbers, where c_1 and c_2 are positive constants independent of $(\lambda_i)_{i=0}^{\infty}$.

Hint: $\left(\sum_{i=0}^{n} a_i x^{\lambda_i}\right) / \left(\sum_{i=0}^{n} b_i x^{\lambda_i}\right)$ has at most n zeros on $[0,1]$. □

E.6 A Markov System with Nondense Rationals. This example outlines a construction of Markov systems on $[-1,1]$ whose rationals are not dense in $C[-1,1]$. It follows and corrects Borwein and Shekhtman [93].

We construct an infinite Markov system on $(-\infty, \infty)$ as follows. Suppose that $\Lambda := (\lambda_i)_{i=0}^{\infty}$, where $2\lambda_i + 1 = 7^{2i}$. Then Λ is a sequence of even integers satisfying

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots, \qquad \sum_{i=1}^{\infty} \frac{1}{\lambda_i} < \infty.$$

Let $\varphi_k \in C[-1,1],\ k = 0, 1, \ldots$, be defined by

$$\varphi_{2k}(x) := x^{\lambda_{2k}}$$

and

$$\varphi_{2k+1}(x) := \begin{cases} x^{\lambda_{2k+1}}, & x \geq 0 \\ -x^{\lambda_{2k+1}}, & x \leq 0. \end{cases}$$

a] Show that $(\varphi_0, \varphi_1, \dots)$ is a Markov system on $(-\infty, \infty)$.

Hint: If

$$p(x) = \sum_{i=0}^{n} a_i \varphi_i(x)\,, \qquad a_i \in \mathbb{R}\,,$$

then

$$p(x) = \sum_{i=0}^{n} a_i x^{\lambda_i}\,, \qquad x \in [0, \infty)\,,$$

while

$$p(x) = \sum_{i=0}^{n} (-1)^i a_i x^{\lambda_i}\,, \qquad x \in (-\infty, 0]\,.$$

Now apply Theorem 3.2.4 (Descartes' rule of signs). □

b] There exists an absolute constant $c > 0$ (independent of n) such that

$$\left\| \sum_{i=0}^{n} a_i x^{\lambda_i} \right\|_{L_2[0,1]} \le c \left\| \sum_{i=0}^{n} a_i x^{\lambda_i} \right\|_{L_2[1/2,1]}$$

for all choices of $a_i \in \mathbb{R}$.

Hint: Use E.8 a] of Section 4.2 and Hölder's inequality. □

c] The inequalities

$$\frac{1}{3} \sum_{i=0}^{n} |a_i|^2 \le \int_0^1 \left(\sum_{i=0}^{n} a_i \sqrt{2\lambda_i + 1}\, x^{\lambda_i} \right)^2 dx \le \frac{5}{3} \sum_{i=0}^{n} |a_i|^2$$

hold for all choices of $a_i \in \mathbb{R}$.

Proof. We have

$$\int_0^1 \left(\sum_{i=0}^{n} a_i \sqrt{2\lambda_i + 1}\, x^{\lambda_i} \right)^2 dx = \sum_{i=0}^{n} |a_i|^2 + 2 \sum_{k=1}^{n} \sum_{j=0}^{n-k} a_j a_{j+k} \frac{\sqrt{2\lambda_j + 1}\sqrt{2\lambda_{j+k} + 1}}{\lambda_j + \lambda_{j+k} + 1}\,.$$

Here

$$\frac{\sqrt{2\lambda_j + 1}\sqrt{2\lambda_{j+k} + 1}}{\lambda_j + \lambda_{j+k} + 1} = 2\, \frac{7^j 7^{j+k}}{7^{2j} + 7^{2j+2k}} < 2\, \frac{7^{2j+k}}{7^{2j+2k}} = 2 \cdot 7^{-k}\,,$$

so on applying the Cauchy-Schwarz inequality n times, we get

$$2\left|\sum_{k=1}^{n}\sum_{j=0}^{n-k} a_j a_{j+k}\frac{\sqrt{2\lambda_j+1}\sqrt{2\lambda_{j+k}+1}}{\lambda_j+\lambda_{j+k}+1}\right|$$
$$\leq 4\left(\sum_{k=1}^{n} 7^{-k}\right)\sum_{i=0}^{n}|a_i|^2 \leq \frac{2}{3}\sum_{i=0}^{n}|a_i|^2\,,$$

and the result follows. □

d] The inequality

$$\left\|\sum_{i=0}^{n} a_i x^{\lambda_i}\right\|_{L_2[0,1]} \leq \sqrt{5}\left\|\sum_{i=0}^{n}(-1)^i a_i x^{\lambda_i}\right\|_{L_2[0,1]}$$

holds for all choices of $a_i \in \mathbb{R}$.

Hint: Use part c]. □

e] The rational functions of the form

$$\frac{\sum_{i=0}^{n} a_i\varphi_i}{\sum_{i=0}^{n} b_i\varphi_i}\,, \quad a_i, b_i \in \mathbb{R}\,, \quad n \in \mathbb{N}$$

are not dense in $C[-1,1]$.

Proof. Consider $f \in C[-1,1]$ defined by

$$f(x) := \begin{cases} 1 & \text{if} \quad x \in [-1,-1/2] \\ 0 & \text{if} \quad x \in [0,1] \\ -2x & \text{if} \quad x \in [-1/2,0]\,. \end{cases}$$

We show that f is not uniformly approximable on $[-1,1]$ by the above rational functions. Suppose that

$$\left\|\frac{\sum_{i=0}^{n} a_i\varphi_i}{\sum_{i=0}^{n} b_i\varphi_i} - f\right\|_{[-1,1]} < \epsilon < 1\,, \qquad a_i, b_i \in \mathbb{R}\,.$$

This implies

$$\left\|\frac{\sum_{i=0}^{n} a_i x^{\lambda_i}}{\sum_{i=0}^{n} b_i x^{\lambda_i}}\right\|_{[0,1]} < \epsilon \tag{4.4.3}$$

and

$$\left\|\frac{\sum_{i=0}^{n}(-1)^i a_i x^{\lambda_i}}{\sum_{i=0}^{n}(-1)^i b_i x^{\lambda_i}} - 1\right\|_{[1/2,1]} < \epsilon\,. \tag{4.4.4}$$

Without loss of generality we may assume that

$$\left\| \sum_{i=0}^{n} b_i x^{\lambda_i} \right\|_{L_2[0,1]} = 1 \,. \tag{4.4.5}$$

From (4.4.3) and (4.4.5) it follows that

$$\left\| \sum_{i=0}^{n} a_i x^{\lambda_i} \right\|_{L_2[0,1]} < \epsilon \,. \tag{4.4.6}$$

Part d], together with (4.4.3) and (4.4.5), implies that

$$\left\| \sum_{i=0}^{n} (-1)^i a_i x^{\lambda_i} \right\|_{L_2[0,1]} < \sqrt{5}\,\epsilon \,. \tag{4.4.7}$$

Part d], together with (4.4.5), also implies that

$$\left\| \sum_{i=0}^{n} (-1)^i b_i x^{\lambda_i} \right\|_{L_2[0,1]} \geq \frac{1}{\sqrt{5}} \,. \tag{4.4.8}$$

Combining part b] and (4.4.8), we obtain that

$$\left\| \sum_{i=0}^{n} (-1)^i b_i x^{\lambda_i} \right\|_{L_2[1/2,1]} \geq \frac{1}{\sqrt{5}\,c} \,. \tag{4.4.9}$$

Now (4.4.7) and (4.4.9) yield the right-hand side of

$$1 - \epsilon \leq \frac{\left\| \sum_{i=0}^{n} (-1)^i a_i x^{\lambda_i} \right\|_{L_2[1/2,1]}}{\left\| \sum_{i=0}^{n} (-1)^i b_i x^{\lambda_i} \right\|_{L_2[1/2,1]}} \leq 5c\epsilon$$

while (4.4.4) yields the left-hand side of it. This shows that $\epsilon > 0$ cannot be arbitrarily small. □