5

Basic Inequalities

Overview

The classical inequalities for algebraic and trigonometric polynomials are treated in the first section. These include the inequalities of Remez, Bernstein, Markov, and Schur. The second section deals with Markov's and Bernstein's inequalities for higher derivatives. The final section is concerned with the size of factors of polynomials.

5.1 Classical Polynomial Inequalities

We start with the classical inequalities of Remez, Bernstein, Markov, and Schur. The most basic and general of these is probably due to Remez. How large can $\|p\|_{[-1,1]}$ be if $p \in \mathcal{P}_n$ and

$$m(\{x \in [-1,1] : |p(x)| \leq 1\}) \geq 2 - s$$

holds? The inequality of Remez [36] answers this question. His inequality and its trigonometric analog can be extended to generalized nonnegative polynomials (discussed in Appendix 4) by a simple density argument. These extensions also play a central role in the proof of various other Bernstein, Markov, Nikolskii, and Schur type inequalities for generalized nonnegative polynomials, where simple density arguments do not work.

Theorem 5.1.1 (Remez Inequality). *The inequality*

$$\|p\|_{[-1,1]} \le T_n\left(\frac{2+s}{2-s}\right)$$

holds for every $p \in \mathcal{P}_n$ *and* $s \in (0,2)$ *satisfying*

$$m\left(\{x \in [-1,1] : |p(x)| \le 1\}\right) \ge 2 - s\,.$$

Here T_n *is the Chebyshev polynomial of degree* n *defined by* (2.1.1). *Equality holds if and only if*

$$p(x) = \pm T_n\left(\frac{\pm 2x + s}{2-s}\right)\,.$$

Proof. The proof is essentially a perturbation argument that establishes that an extremal polynomial is of the required form. Let $\|\cdot\|$ denote the uniform norm on $[-1,1]$, and let

$$\mathcal{P}_n(s) := \{p \in \mathcal{P}_n : m\left(\{x \in [-1,1] : |p(x)| \le 1\}\right) \ge 2 - s\}\,. \tag{5.1.1}$$

The set $\mathcal{P}_n(s)$ is compact, say, in the uniform norm on $[-1,1]$, by E.1, and the function $p \to \|p\|$ is continuous. Hence there is a $p^* \in \mathcal{P}_n(s)$ such that

$$\|p^*\| = \sup_{p \in \mathcal{P}_n(s)} \|p\|\,.$$

First assume that $p^*(1) = \|p^*\|$. We claim that all the zeros of p^* are real and lie in $[-1,1)$. Indeed, if p^* vanishes at a nonreal z, then for sufficiently small $\eta > 0$ and $\epsilon > 0$,

$$q(x) := (1+\eta)p^*(x)\left(1 - \frac{\epsilon(x-1)^2}{(x-z)(x-\overline{z})}\right)$$

is in $\mathcal{P}_n(s)$ and contradicts the maximality of p^*. If p^* has a real zero z outside $[-1,1]$, then, in similar fashion,

$$q(x) := (1+\eta)p^*(x)\left(1 - \epsilon\ \mathrm{sign}(z)\frac{1-x}{z-x}\right)$$

contradicts the maximality of p^*.

Now we show that $|p^*(\zeta)| = \|p^*\|$ cannot occur with $\zeta \in (-1,1)$. To see this, assume, without loss of generality, that $p^*(\zeta) = \|p^*\|$. Then the polynomials

$$q_1(x) := p^*\left(\frac{\zeta-1}{2} + \frac{\zeta+1}{2}x\right) \quad \text{and} \quad q_2(x) := p^*\left(\frac{\zeta+1}{2} + \frac{\zeta-1}{2}x\right)$$

satisfy $q_j(1) = \|q_j\| = \|p^*\|$, $j = 1,2$. Since $p^* \in \mathcal{P}_n(s)$ is extremal, the Lebesgue measure $m(\Omega_j)$ of

$$\Omega_j := \{x \in [-1,1] : |q_j(x)| > 1\}$$

is at least s, otherwise $q_j + \epsilon$ with sufficiently small $\epsilon > 0$ contradicts the maximality of p^*. On the other hand,

$$\frac{1+\zeta}{2}\, m(\Omega_1) + \frac{1-\zeta}{2}\, m(\Omega_2) = m\left(\{x \in [-1,1] : p^*(x) > 1\}\right) \le s\,.$$

Hence $m(\Omega_1) = m(\Omega_2) = s$, which means that $q_j \in \mathcal{P}_n(s)$, $j = 1,2$, are extremal polynomials attaining their uniform norm on $[-1,1]$ at 1. It now follows from the first part of the proof that q_1 and q_2 have all their zeros in $[-1,1]$, which is impossible since the number of zeros of q_j, $j = 1,2$, in $[-1,1]$ is equal to the number of zeros of p^* in $[-1,\zeta)$ and in $(\zeta,1]$, respectively. This contradiction proves that $|p^*(\zeta)| < \|p^*\|$ for every $\zeta \in (-1,1)$. Hence either $p^*(1) = \pm\|p^*\|$ or $p^*(-1) = \pm\|p^*\|$.

Without loss of generality we may assume that $p^*(1) = \|p^*\|$, otherwise we consider $\pm p^*(-x) \in \mathcal{P}_n(s)$. We now have

$$p^*(1) = \|p^*\| > |p^*(x)|\,, \quad -1 < x < 1\,, \tag{5.1.2}$$

and each zero of p^* lies in $[-1,1)$.

Next we prove that

$$|p^*(x)| \le 1\,, \qquad -1 \le x \le 1 - s\,. \tag{5.1.3}$$

Assume to the contrary that for some $-1 \le \zeta_3 < \zeta_2 < \zeta_1 < 1$,

$$\begin{cases} |p^*(x)| > 1\,, & x \in I_1 := (\zeta_1, 1] \\ |p^*(x)| \le 1\,, & x \in I_2 := [\zeta_2, \zeta_1] \\ |p^*(x)| > 1\,, & x \in I_3 := (\zeta_3, \zeta_2)\,. \end{cases}$$

Let $x_1, x_2, \ldots, x_m$ be the zeros of p^* in (ζ_2, ζ_1). Since all zeros of p^* are in $[-1,1)$, we have $m \ge 1$, otherwise $p^{*\prime}$ would vanish at an x larger than the largest zero of p^*, which is a contradiction. The remaining $n - m$ zeros of p^* lie in $[-1,\zeta_3)$. We set

$$p_1(x) := \prod_{j=1}^{m} (x - x_j)\,, \qquad p_2 := \frac{p^*}{p_1}\,.$$

The polynomial $q(x) := p_1(x+h)p_2(x)$ with $0 < h < \zeta_2 - \zeta_3$ has the following properties:

(1) If $|p^*(x)| \le 1$ for some $x \in [-1, \zeta_3]$, then $|q(x)| \le 1$.

(2) For each $x \in I_2$ we have $|q(x-h)| \le |p^*(x)| \le 1$.

(3) $q(1) = p_1(1+h)p_2(1) > p_1(1)p_2(1) = p^*(1) = \|p^*\|$.

These properties show that $q \in \mathcal{P}_n(s)$ contradicts the maximality of p^*.

By E.2, among all polynomials $p \in \mathcal{P}_n$ with $\|p\| \le 1$, the Chebyshev polynomial T_n increases fastest for $x > 1$. Hence, by a linear transformation, we see that the four polynomials

$$p^*(x) = \pm T_n\left(\frac{\pm 2x + s}{2 - s}\right)$$

are the only extremal polynomials. In particular,

$$\|p^*\| = T_n\left(\frac{2+s}{2-s}\right).$$

□

The next theorem establishes a Remez-type inequality for trigonometric polynomials. Throughout this section, as before, $K := \mathbb{R} \pmod{2\pi}$.

Theorem 5.1.2. *The inequality*

$$\|t\|_K \le \exp(4ns)$$

holds for every $t \in \mathcal{T}_n$ and $s \in (0, \pi/2]$ satisfying

$$m(\{\theta \in [-\pi, \pi) : |t(\theta)| \le 1\}) \ge 2\pi - s\,. \tag{5.1.4}$$

The inequality

$$\|t\|_K \le T_n\left(\frac{2+\sigma}{2-\sigma}\right), \quad \sigma = 1 - \cos(s/2), \quad 0 < s < 2\pi,$$

also holds for every even $t \in \mathcal{T}_n$ and $s \in (0, 2\pi)$ satisfying (5.1.4), and equality holds if and only if

$$t(\theta) = \pm T_n\left(\frac{\pm 2\cos\theta + \sigma}{2 - \sigma}\right), \qquad \sigma = 1 - \cos(s/2).$$

Proof. We prove the second part first. Suppose $t \in \mathcal{T}_n$ is even and satisfies $|t(\theta)| \le 1$ on $K \setminus \Omega$ with $m(\Omega) \le s$. Then the polynomial $p \in \mathcal{P}_n$ defined by $t(\theta) := p(\cos\theta)$ satisfies $|p(x)| \le 1$ on $[-1,1] \setminus \Omega'$, where

$$\Omega' := \{x \in [-1,1] : x = \cos\theta\,,\ \theta \in \Omega \cap [0,\pi]\}.$$

It is easy to see that $m(\Omega') \le 1 - \cos(s/2) =: \sigma$, where equality holds if and only if $\Omega := [-s/2, s/2]$. Hence, Theorem 5.1.1 implies that

$$\|t\|_K \le T_n\left(\frac{2+\sigma}{2-\sigma}\right),$$

where equality holds if and only if p is of the form given in the theorem.

The first part of the theorem can be easily obtained from the second part as follows. Let $t \in \mathcal{T}_n$ satisfy (5.1.4). Without loss of generality we may assume that $t(0) = \|t\|_K$. The polynomial

$$\widetilde{t}(\theta) := \tfrac{1}{2}(t(\theta) + t(-\theta)) \in \mathcal{T}_n$$

is even and

$$m(\{\theta \in [-\pi, \pi) : |\widetilde{t}(\theta)| \le 1\}) \ge 2\pi - 2s\,.$$

Hence, the second part of the theorem yields that

$$\|\widetilde{t}\|_{L_\infty(K)} \le T_n\left(\frac{2+\sigma}{2-\sigma}\right),$$

where $\sigma := 1 - \cos s = 2\sin^2(s/2) \le 1$ for every $s \in (0, \pi/2]$. Since

$$T_n(x) \le \left(x + \sqrt{x^2-1}\right)^n, \qquad x \ge 1\,,$$

we have

$$\begin{aligned} T_n\left(\frac{2+\sigma}{2-\sigma}\right) &\le \left(\frac{1+\sqrt{2\sigma}+\sigma/2}{1-\sigma/2}\right)^n \le \left(1+\sqrt{2\sigma}+\tfrac{7}{2}\sigma\right)^n \\ &\le \exp\left(n\left(\sqrt{2\sigma}+\tfrac{7}{4}\sigma\right)\right) \le \exp\left(n\left(s+\tfrac{7}{4}s^2\right)\right) \\ &\le \exp(4ns) \end{aligned}$$

for every $s \in (0, \pi/2]$. Therefore

$$\|t\|_K = t(0) = \widetilde{t}(0) \le T_n\left(\frac{2+\sigma}{2-\sigma}\right) \le \exp\left(n\left(s+\tfrac{7}{4}s^2\right)\right) \le \exp(4ns)$$

for every $s \in (0, \pi/2]$, and the theorem is proved. Note that we have proved slightly more than we claimed in the statement of the theorem. That is,

$$\|t\|_K \le \exp\left(n\left(s+\tfrac{7}{4}s^2\right)\right)$$

for every $t \in \mathcal{T}_n$ satisfying (5.1.4). □

We now prove the basic inequality that bounds the derivative of a trigonometric polynomial in terms of its maximum modulus on the period K.

Theorem 5.1.3 (Bernstein-Szegő Inequality). *The inequality*

$$t'(\theta)^2 + n^2 t^2(\theta) \le n^2 \|t\|_K^2 , \qquad \theta \in K$$

holds for every $t \in \mathcal{T}_n$. Equality holds if and only if $|t(\theta)| = \|t\|_K$ or t is of the form $t(\tau) = \beta \cos(n\tau - \alpha)$ with α, $\beta \in \mathbb{R}$.

Proof. Assume that there are $t \in \mathcal{T}_n$ and $\theta \in K$ such that $\|t\|_K < 1$ and

$$t'(\theta)^2 + n^2 t^2(\theta) > n^2 . \tag{5.1.5}$$

For the sake of brevity let $T_{n,\alpha}(\tau) := \cos(n\tau - \alpha)$. It is easy to see that there exists an $\alpha \in K$ such that

$$T_{n,\alpha}(\theta) = t(\theta) \qquad \text{and} \qquad \operatorname{sign}(T'_{n,\alpha}(\theta)) = \operatorname{sign}(t'(\theta)) . \tag{5.1.6}$$

Since $T'_{n,\alpha}(\theta)^2 + n^2 T^2_{n,\alpha}(\theta) = n^2$, (5.1.5) and (5.1.6) imply that

$$|t'(\theta)| > |T'_{n,\alpha}(\theta)| \qquad \text{and} \qquad \operatorname{sign}(T'_{n,\alpha}(\theta) = \operatorname{sign}(t'(\theta)) .$$

Hence E.4 yields that $0 \not\equiv t - T_{n,\alpha} \in \mathcal{T}_n$ has at least $2n+2$ distinct zeros in K, which is a contradiction. To find all the extremal polynomials, see the hint to E.5. □

As a corollary of Theorem 5.1.3 we have $\|t'\|_K \le n\|t\|_K$ for every $t \in \mathcal{T}_n$, and by induction on m we obtain the following theorem:

Theorem 5.1.4 (Bernstein's Inequality). *The inequality*

$$\|t^{(m)}\|_K \le n^m \|t\|_K$$

holds for every $t \in \mathcal{T}_n$.

Corollary 5.1.5. *The inequality of Theorem 5.1.4 remains true for all $t \in \mathcal{T}_n^c$.*

Proof. Choose $\alpha \in \mathbb{R}$ such that $e^{i\alpha} t^{(m)}$ attains the value $\|t^{(m)}\|_K$, say, at $\theta = \tau$. Now $\widetilde{t}(\theta) := \operatorname{Re}(e^{i\alpha} t(\theta)) \in \mathcal{T}_n$ and $\|\widetilde{t}\|_K \le \|t\|_K$. On applying Theorem 5.1.4 to $\widetilde{t} \in \mathcal{T}_n$, we obtain

$$\|t^{(m)}\|_K = e^{i\alpha} t^{(m)}(\tau) = \widetilde{t}^{(m)}(\tau) \le n^m \|\widetilde{t}\|_K \le n^m \|t\|_K .$$

□

The above corollary implies the following algebraic polynomial version of Bernstein's inequality on the unit disk.

Corollary 5.1.6. *The inequality*

$$\|p'\|_D \le n \|p\|_D$$

holds for every $p \in \mathcal{P}_n^c$, *where* $D := \{z \in \mathbb{C} : |z| < 1\}$.

Proof. If $p \in \mathcal{P}_n^c$ then $t(\theta) := p(e^{i\tau}) \in \mathcal{T}_n^c$ and by Corollary 5.1.5 we have

$$|p'(z)| = |-ie^{-i\tau}t'(\tau)| \le n\|t\|_K = n\|p\|_D\,, \qquad z = e^{i\tau}\,.$$

The maximum principle (see E.1 d] of Section 1.2) finishes the proof. □

From Corollary 5.1.5, by the substitution $x = \cos\tau$, we get the algebraic polynomial case of Bernstein's inequality on $[-1,1]$.

Theorem 5.1.7 (Bernstein's Inequality). *The inequality*

$$|p'(x)| \le \frac{n}{\sqrt{1-x^2}} \|p\|_{[-1,1]}\,, \qquad -1 < x < 1$$

holds for every $p \in \mathcal{P}_n^c$.

The next theorem improves the previous result if x is close to ± 1.

Theorem 5.1.8 (Markov's Inequality). *The inequality*

$$\|p'\|_{[-1,1]} \le n^2 \|p\|_{[-1,1]}$$

holds for every $p \in \mathcal{P}_n$.

A proof can be given as a simple combination of Theorem 5.1.7 and the next theorem.

Theorem 5.1.9 (Schur's Inequality). *The inequality*

$$\|p\|_{[-1,1]} \le n \left\|p(x)\sqrt{1-x^2}\right\|_{[-1,1]}$$

holds for every $p \in \mathcal{P}_{n-1}$.

Proof. Let

$$x_k := \cos\tfrac{(2k-1)\pi}{2n}\,, \qquad k = 1, 2, \dots, n\,,$$

so the numbers x_k are the zeros of the Chebyshev polynomial

$$T_n(x) = \cos(n \arccos x)\,.$$

Assume that $p \in \mathcal{P}_{n-1}$ and $\|p(x)\sqrt{1-x^2}\|_{[-1,1]} \le 1$. If $|y| \le x_n$, then

$$\begin{aligned} |p(y)| &\le (1-y^2)^{-1/2} \le (1-x_n^2)^{-1/2} \\ &= \left(\sin\frac{\pi}{2n}\right)^{-1} \le \left(\frac{2}{\pi}\frac{\pi}{2n}\right)^{-1} = n\,. \end{aligned}$$

Now let $|y| \in (x_n, 1]$. Without loss of generality we may assume that $y \in (x_n, 1]$. The Lagrange interpolation polynomial of a $p \in \mathcal{P}_{n-1}$ with nodes $x_1, x_2, \dots, x_n$ is just p itself, hence E.6 of Section 1.1 yields

$$\begin{aligned} |p(y)| &= \left|\sum_{k=1}^{n} p(x_k)\frac{T_n(y)}{T_n'(x_k)(y-x_k)}\right| \\ &= \frac{1}{n}\left|\sum_{k=1}^{n} p(x_k)\sqrt{1-x_k^2}\,\frac{T_n(y)}{y-x_k}\right| \\ &\le \frac{1}{n}\sum_{k=1}^{n}\frac{T_n(y)}{y-x_k} \le \frac{1}{n}\,T_n'(y) \le \frac{1}{n}\,T_n'(1) = n\,, \end{aligned}$$

where we use the facts that

$$\frac{T_n(y)}{y-x_k} > 0\,, \qquad k = 1, 2, \dots, n\,,$$

T_n' is increasing on (x_n, ∞), and $T_n'(1) = n^2$. □

Proof of Theorem 5.1.8. Let $p \in \mathcal{P}_n$. Then $p' \in \mathcal{P}_{n-1}$ and Theorems 5.1.4 and 5.1.9 yield

$$\|p'\|_{[-1,1]} \le n\left\|p'(x)\sqrt{1-x^2}\right\|_{[-1,1]} \le n^2\|p\|_{[-1,1]}\,,$$

and the theorem is proved. □

Comments, Exercises, and Examples.

Every result of this section was proved by the person it is named after; see Remez [36], Bernstein [12], Szegő [28] or [82], A. A. Markov [1889], Schur [19], and M. Riesz [14]. However, earlier less complete versions of these basic polynomial inequalities also appear in the literature. Some of the proofs were simplified later. For example, in the proof of Theorem 5.1.1 we followed the method given in Erdélyi [89b], while in the proof of Theorem 5.1.8 a method of Pólya and Szegő [76] is used. Theorem 5.1.3 is also obtained in Corput and Schaake [35], however, it follows from an earlier result of Szegő [28]. Theorem 5.1.2 was established in Erdélyi [92a] in a slightly weaker form. Various extensions of the inequalities of this section are discussed in Sections 5.2 to 7.2 and in the appendices. Rahman and Schmeisser [83] also

offers a collection of Markov- and Bernstein-type inequalities. Some further classical inequalities are in E.5 of Appendix 3.

An interesting extension of Markov's inequality is due to Bojanov [82a]. It states that

$$\|p'\|_{L_q[-1,1]} \le \|T_n'\|_{L_q[-1,1]} \|p\|_{[-1,1]}$$

for every $p \in \mathcal{P}_n$ and $q \in [1,\infty]$, where T_n is the Chebyshev polynomial of degree n as in (2.1.1).

The following result is due to Szegő [25]. The inequality

$$|p'(0)| \le cn^{2\alpha} \|p\|_{D_\alpha}$$

holds for every $p \in \mathcal{P}_n^c$, where c is an absolute constant and

$$D_\alpha := \{z \in \mathbb{C} : |z| \le 1, \ |\arg(z)| \le \pi(1-\alpha)\}, \qquad \alpha \in (0,1].$$

Throughout the exercises T_n denotes the Chebyshev polynomial of degree n as defined by (2.1.1).

E.1 A Detail in the Proof of Theorem 5.1.1. Show that the sets $\mathcal{P}_n(s)$ defined by (5.1.1) are compact in the uniform norm on $[-1,1]$ for every fixed $n \in \mathbb{N}$ and $s \in (0,2)$.

E.2 Chebyshev's Inequality. Prove that

$$|p(y)| \le |T_n(y)| \cdot \|p\|_{[-1,1]}, \qquad y \in \mathbb{R} \setminus [-1,1]$$

for every $p \in \mathcal{P}_n$, and equality holds if and only if $p = cT_n$ for some $c \in \mathbb{R}$.

Extend the above inequality to every $p \in \mathcal{P}_n^c$ and find all $p \in \mathcal{P}_n^c$ for which equality holds.

Hint: If $p \in \mathcal{P}_n$, $\|p\|_{[-1,1]} = 1$, and $|p(y)| > |T_n(y)|$ for some $y \in \mathbb{R} \setminus [-1,1]$, then with $\lambda := T_n(y)p(y)^{-1} \in [-1,1]$, the polynomial $\lambda p - T_n \in \mathcal{P}_n$ has at least $n+1$ zeros (counting multiplicities). □

E.3 Trigonometric Chebyshev Polynomials on Subintervals of K.

a] For $n \in \mathbb{N}$ and $\omega \in (0,\pi)$, let

$$Q_{n,\omega}(\theta) := T_{2n}\left(\frac{\sin(\theta/2)}{\sin(\omega/2)}\right).$$

Show that $Q_{n,\omega} \in \mathcal{T}_n$ attains the values $\pm\|Q_{n,\omega}\|_{[-\omega,\omega]} = \pm 1$ with alternating sign $2n+1$ times on $[-\omega,\omega]$.

b] Prove that

$$|t(\theta)| \le Q_{n,\omega}(\theta)\|t\|_{[-\omega,\omega]}, \qquad \theta \in K \setminus [-\omega,\omega]$$

for every $t \in \mathcal{T}_n$, and for every fixed $\theta \in K \setminus [-\omega,\omega]$. Equality holds if and only if $p = cQ_{n,\omega}$ for some $c \in \mathbb{R}$.

c] Show that there exist absolute constants $c_1 > 0$ and $c_2 > 0$ such that

$$\tfrac{1}{2}\exp(c_1 n(\pi-\omega)) \le \|Q_{n,\omega}\|_K = Q_{n,\omega}(\pi) \le \exp(c_2 n(\pi-\omega)).$$

E.4 A Zero Counting Lemma. *Let $\theta \in K$ be fixed. Suppose f and g are continuous functions on $K := \mathbb{R} \pmod{2\pi}$, differentiable at a fixed $\theta \in K$, and suppose f and g have the properties*

(1) $\|f\|_K = 1, \quad \|g\|_K < 1,$

(2) *there are* $-\pi \le x_1 < x_2 < \cdots < x_{2n} < \pi$ *so that* $f(x_j) = (-1)^j$,

(3) $f(\theta)) = g(\theta), \quad |g'(\theta)| > |f'(\theta)|, \quad$ *and* $\quad \operatorname{sign}(g'(\theta) = \operatorname{sign}(f'(\theta))$.

Then $f - g$ has at least $2n + 2$ distinct zeros on K.

E.5 Sharpness of Theorem 5.1.3. Let $\theta \in K$ be fixed. Suppose $t \in \mathcal{T}_n$ and

$$t'(\theta)^2 + n^2 t^2(\theta) = n^2 \|t\|_K^2 .$$

Show that either $|t(\theta)| = \|t\|_K$ or t is of the form

$$t(\tau) = \beta \cos(n\tau - \alpha) , \qquad \alpha, \beta \in \mathbb{R} .$$

Hint: Suppose $\|t\|_K = 1$ and $|t(\theta)| < 1$. Choose $\alpha \in K$ such that (5.1.6) is satisfied and show that $t - T_{n,\alpha} \in \mathcal{T}_n$ has at least $2n + 2$ zeros on K (counting multiplicities). □

E.6 Sharpness of Theorem 5.1.4. Show that Theorem 5.1.4 is sharp and equality holds if and only if t is of the form

$$t(\tau) = \beta \cos(n\tau - \alpha) , \qquad \alpha, \beta \in \mathbb{R} .$$

E.7 Sharpness of Corollary 5.1.6. Show that Corollary 5.1.5 is sharp and equality holds if and only if p is of the form $p(z) = cz^n, \ \ c \in \mathbb{C}$.

E.8 Sharpness of Theorem 5.1.7. Show that for a fixed integer $n \ge 1$, Theorem 5.1.7 is sharp if and only if x is a zero of the Chebyshev polynomial T_n, that is,

$$x = \cos \tfrac{(2k-1)\pi}{2n} , \qquad k = 1, 2, \ldots , n ,$$

and $p = cT_n$ for some $c \in \mathbb{R}$.

E.9 Sharpness of Theorem 5.1.9. Show that Theorem 5.1.9 is sharp and equality holds if and only if $p = cU_n$ for some $c \in \mathbb{R}$, where U_n is the Chebyshev polynomial of the second kind defined in E.10 of Section 2.1.

E.10 Sharpness of Theorem 5.1.8. Show that Theorem 5.1.8 is sharp and equality holds if and only if $p = cT_n$ for some $c \in \mathbb{R}$.

E.11 A Property of the Zeros of $t \in \mathcal{T}_n$. Let $\theta \in K$ be fixed. Show that every $t \in \mathcal{T}_n$ has at most

$$M := enr|t(\theta)|^{-1} \|t\|_K$$

zeros (counting multiplicities) in the interval $[\theta - r, \theta + r], \ \ r > 0$.

Proof. Assume that $t \in \mathcal{T}_n$ has $m > M$ zeros in $[\theta - r, \theta + r]$. Interpolate t at these m zeros by a Hermite interpolation polynomial of degree at most $m-1$ (see E.7 of Section 1.1). This gives the identically zero polynomial. The formula for the remainder term of the Hermite interpolation polynomial and Theorem 5.1.4 (Bernstein's inequality) yield that there exists a $\zeta \in (\theta - r, \theta + r)$ such that

$$\begin{aligned} |t(\theta)| &= \frac{1}{m!} r^m |t^{(m)}(\zeta)| < \left(\frac{e}{m}\right)^m r^m n^m \|t\|_K \\ &< \left(\frac{enr}{M}\right)^m \|t\|_K \le |t(\theta)|^m \|t\|_K^{1-m} \le |t(\theta)|\,, \end{aligned}$$

which is impossible. □

E.12 A Property of the Zeros of a $p \in \mathcal{P}_n$. Show that every $p \in \mathcal{P}_n$ has at most

$$M := \frac{e}{\sqrt{2}}\, n\sqrt{r}\, |p(1)|^{-1} \|p\|_{[-1,1]}$$

zeros (counting multiplicities) in $[1 - r, 1]$, $r > 0$.

Hint: Use the substitution $x = \cos\tau$, E.11, and the inequality

$$\cos r < 1 - \tfrac{1}{4} r^2\,, \qquad 0 < r \le 2\,.$$

□

E.13 Riesz's Lemma.

a] *Suppose $t \in \mathcal{T}_n$ and $t(\alpha) = \|t\|_K = 1$ for some $\alpha \in K$. Then*

$$t(\theta) \ge \cos(n(\theta - \alpha))\,, \qquad \theta \in \left[\alpha - \tfrac{\pi}{2n}, \alpha + \tfrac{\pi}{2n}\right]\,,$$

and equality holds for a fixed $\theta \in \left[\alpha - \frac{\pi}{2n}, \alpha + \frac{\pi}{2n}\right]$ if and only if t is of the form $t(\tau) = \cos(n(\tau - \alpha))$. In particular, t does not vanish in $\left(\alpha - \frac{\pi}{2n}, \alpha + \frac{\pi}{2n}\right)$.

Hint: If this were false, then

$$q(\tau) := t(\tau) - \cos(n(\tau - \alpha))$$

would have more than $2n$ zeros on K (counting multiplicities). □

b] *Suppose $p \in \mathcal{P}_n$ and $p(1) = \|p\|_{[-1,1]} = 1$. Then that*

$$p(x) \ge T_n(x)\,, \qquad x \in \left[\cos\tfrac{\pi}{2n}, 1\right]\,,$$

and equality holds for a fixed $x \in \left[\cos\frac{\pi}{2n}, 1\right]$ if and only if $p = T_n$, where T_n is the Chebyshev polynomial of degree n as defined by (2.1.1). In particular, p does not vanish in $\left(\cos\frac{\pi}{2n}, 1\right]$.

The next two exercises follows Erdélyi [88] and Erdélyi and Szabados [89b].

E.14 A Markov-Type Inequality for Trigonometric Polynomials on $[-\omega,\omega]$. Show that there exists a constant $0 < c \le 16\pi$ such that

$$\|s'\|_{[-\omega,\omega]} \le \left(n + c\,\frac{\pi-\omega}{\omega}n^2\right)\|s\|_{[-\omega,\omega]}$$

for every $s \in \mathcal{T}_n$ and $\omega \in (0,\pi]$.

Proof. If $\pi - \omega > (2n)^{-1}$, then

$$(3\tan^2(\omega/2)+1)^{1/2} < 8n\,,$$

and E.19 c] gives the result. If $\pi - \omega \le (2n)^{-1}$, then Theorem 5.1.4 (Bernstein's inequality) combined with the Mean Value Theorem yields

$$\|s\|_{[-\pi,\pi]} \le \|s\|_{[-\omega,\omega]} + (\pi-\omega)n\,\|s\|_{[-\pi,\pi]}\,,$$

and hence

$$\|s\|_{[-\omega,\omega]} \ge (1 - n(\pi-\omega))\,\|s\|_{[-\pi,\pi]}$$

for every $s \in \mathcal{T}_n$. Therefore, using Theorem 5.1.4 (Bernstein's inequality) and $\pi - \omega \le (2n)^{-1}$, we get

$$\begin{aligned}\|s'\|_{[-\omega,\omega]} &\le \|s'\|_{[-\pi,\pi]} \le n\,\|s\|_{[-\pi,\pi]}\\ &\le \frac{n}{1-n(\pi-\omega)}\,\|s\|_{[-\omega,\omega]} \le (n + 2(\pi-\omega)n^2)\|s\|_{[-\omega,\omega]}\end{aligned}$$

for every $s \in \mathcal{T}_n$. □

E.15 Schur-Type Inequality for $\mathcal{T}_n$ on $[-\omega,\omega]$. Let $\omega \in (0,2\pi]$. Show that

$$\|s\|_{[-\omega,\omega]} \le \frac{2n+1}{\sin(\omega/2)}\left\|s(\tau)\left(\tfrac{1}{2}(\cos\tau - \cos\omega)\right)^{1/2}\right\|_{[-\omega,\omega]}$$

for every $s \in \mathcal{T}_n$, and equality holds if and only if s is of the form

$$s(\tau) = c\,\frac{\sin\left[(2n+1)\arccos\frac{\sin(\tau/2)}{\sin(\omega/2)}\right]}{(\cos\tau - \cos\omega)^{1/2}}\,,\qquad c \in \mathbb{R}\,.$$

Note that the right-hand side of the above is an element of $\mathcal{T}_n$.

Hint: Define $2n+1$ distinct points in $(-\omega,\omega)$ by

$$\tau_k := 2\arcsin\left(\sin\frac{\omega}{2}\sin\frac{k\pi}{2n+1}\right)\,,\qquad k = 0,\pm1,\dots,\pm n\,.$$

Distinguish two cases in estimating $|s_n(\theta)|$ for $\theta \in [-\omega,\omega]$.

Case 1: $|\theta| \le |\tau_n|$. Use the inequality

$$\frac{1}{2}(\cos\theta - \cos\omega) > \frac{\sin^2(\omega/2)}{(2n+1)^2}$$

to get the desired result.

Case 2: $\tau_n < |\theta| \le \omega$. Let

$$M_n(s) := \max_{|k|\le n}\{|s(\tau_k)|((\cos\tau_k - \cos\omega)/2)^{1/2}\}.$$

Let $n \in \mathbb{N}$, $\omega \in (0,\pi]$, and $\theta \in [-\omega,-\tau_n)\cup(\tau_n,\omega]$ be fixed. Show that there is an $\widetilde{s}_n \in \mathcal{T}_n$ such that

$$\frac{|\widetilde{s}_n(\theta)|}{M_n(\widetilde{s}_n)} = \max_{s\in\mathcal{T}_n}\frac{|s(\theta)|}{M_n(s)}\,.$$

Show by a variational method that $\widetilde{s}_n$ is of the form

$$\widetilde{s}_n(\tau) = c\,\frac{\sin\left[(2n+1)\arccos\frac{\sin(\tau/2)}{\sin(\omega/2)}\right]}{(\cos\tau-\cos\omega)^{1/2}}\,, \qquad c\in\mathbb{R}\,.$$

□

E.16 Another Proof of Schur's Inequality.

a] Prove Theorem 5.1.9 (Schur's inequality) by using the method given in the hint to E.15.

b] Prove the result of E.15 by using interpolation, as in the proof of Theorem 5.1.9.

Proof. See Erdélyi and Szabados [89b]. □

E.17 Growth of Polynomials in the Complex Plane. Let

$$D := \{z\in\mathbb{C} : |z|<1\} \qquad \text{and} \qquad D_\varrho := \{z\in\mathbb{C} : |z|<\varrho\}.$$

a] Show that

$$|p(z)| \le |z|^n \,\|p\|_D$$

for every $p \in \mathcal{P}_n^c$ and $z \in \mathbb{C}\setminus\overline{D}$.

Find all $p \in \mathcal{P}_n^c$ for which equality holds.

Hint: Apply the maximum principle (see E.1 d] of Section 1.2) with D and

$$q(z) := z^n p(z^{-1}) \in \mathcal{P}_n^c\,.$$

□

b] Show that the transformation $z = M(w) := \frac{1}{2}(w + w^{-1})$ can be written as

$$x = \tfrac{1}{2}(\varrho + \varrho^{-1})\cos\theta\,, \qquad y = \tfrac{1}{2}(\varrho - \varrho^{-1})\sin\theta\,,$$

where $z = x + iy$, $x, y \in \mathbb{R}$ and $w = \varrho e^{i\theta}$, $\varrho > 0$, $\theta \in K$.

c] For $\varrho > 0$, let E_ϱ be the image of the circle ∂D_ϱ under M. Show that E_ϱ is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{with semiaxes} \quad a := \tfrac{1}{2}(\varrho + \varrho^{-1}) \quad \text{and} \quad b := \tfrac{1}{2}|\varrho - \varrho^{-1}|.$$

Furthermore, $E_\varrho = E_{\varrho^{-1}}$, and E_1 is the interval $[-1, 1]$ covered twice.

d] Show that

$$|p(z)| \le \varrho^n \|p\|_{[-1,1]}$$

for every $p \in \mathcal{P}_n^c$, $z \in E_\varrho$, $\varrho > 1$.

Proof. Applying the maximum principle (see E.1 d] of Section 1.2) with $D = D_1$ and

$$Q(w) := w^n p\left(\tfrac{1}{2}(w + w^{-1})\right) \in \mathcal{P}_{2n}^c\,,$$

we obtain

$$|Q(w)| \le \|Q\|_{D_1} = \|p\|_{[-1,1]}\,, \qquad w \in \partial D_{\varrho^{-1}}\,.$$

This, together with part c], yields

$$|p(z)| \le \varrho^n \|p\|_{[-1,1]}\,, \qquad z \in E_\varrho\,.$$

□

e] Show that there exists an absolute constant c such that

$$|p(z)| \le c\,\|p\|_{-1,1]}\,, \qquad p \in \mathcal{P}_n^c$$

whenever

$$z = x + iy\,, \quad x, y \in \mathbb{R}\,, \quad |x| \le 1 + \frac{1}{n^2}\,, \quad |y| \le \frac{\sqrt{|1 - x^2|_+}}{n} + \frac{1}{n^2}$$

($|1 - x^2|_+ := \max\{1 - x^2, 0\}$).

Hint: Use part d]. □

f] Prove the following Markov-Bernstein inequality. There is a constant $c(m)$ depending only on m such that

$$|p^{(m)}(x)| \le c(m)\left(\min\left\{n^2, \frac{n}{\sqrt{1 - x^2}}\right\}\right)^m \|p\|_{[-1,1]}$$

for every $p \in \mathcal{P}_n^c$ and $x \in [-1, 1]$.

Hint: Use part e] and Cauchy's integral formula (see E.1 a] of Section 1.2). □

Bernstein established Theorem 5.1.4 in order to prove inverse theorems of approximation. Bernstein's method is presented in the proof of the next exercise, which is one of the simplest cases. However, several other inverse theorems of approximation can be proved by straightforward modifications of the proof of this exercise. That is why Bernstein- and Markov-type inequalities play a significant role in approximation theory. Direct and inverse theorems of approximation and related matters may be found in many books on approximation theory, including Cheney [66], Lorentz [86a], and DeVore and Lorentz [93].

E.18 An Inverse Theorem of Approximation. Let Lip_α, $\alpha \in (0,1]$, denote the family of all real-valued functions g defined on K satisfying

$$\sup\left\{\frac{|g(x)-g(y)|}{|x-y|^\alpha} : x \neq y \in K\right\} < \infty.$$

For $f \in C(K)$, let

$$E_n(f) := \inf\{\|t-f\|_K : t \in \mathcal{T}_n\}.$$

An example for a direct theorem of approximation is stated in part a]. Part b] deals with its inverse result.

a] *Suppose f is m times differentiable on K and $f^{(m)} \in \mathrm{Lip}_\alpha$ for some $\alpha \in (0,1]$. Then there is a constant C depending only on f so that*

$$E_n(f) \le Cn^{-(m+\alpha)}, \qquad n = 1,2,\dots .$$

Proof. See, for example, Lorentz [86a]. □

b] *Suppose m is a nonnegative integer, $\alpha \in (0,1)$, and $f \in C(K)$. Suppose there is a constant $C > 0$ depending only on f such that*

$$E_n(f) \le Cn^{-(m+\alpha)}, \qquad n = 1,2,\dots .$$

Then f is m times continuously differentiable on K and $f^{(m)} \in \mathrm{Lip}_\alpha$.

Outline. We show only that f is m times continuously differentiable on K. The rest can be proved similarly, but its proof requires more technical details. See, for example, Lorentz [86a].

For each $k \in \mathbb{N}$, let $Q_{2^k} \in \mathcal{T}_{2^k}$ be chosen so that

$$\|Q_{2^k} - f\|_K \le C\,2^{-k(m+\alpha)}.$$

Then

$$\|Q_{2^{k+1}} - Q_{2^k}\|_K \le 2C\,2^{-k(m+\alpha)}\,.$$

Now

$$f(\theta) = Q_{2^0}(\theta) + \sum_{k=1}^{\infty}(Q_{2^{k+1}} - Q_{2^k})(\theta)\,, \qquad \theta \in K\,,$$

and by Theorem 5.1.4 (Bernstein's inequality)

$$\begin{aligned}
|Q_{2^0}^{(j)}| + \left|\sum_{k=1}^{\infty}(Q_{2^{k+1}} - Q_{2^k})^{(j)}(\theta)\right| & \\
&\le \|Q_1\|_K + \sum_{k=1}^{\infty}\bigl(2^{k+1}\bigr)^j\|Q_{2^{k+1}} - Q_{2^k}\|_K \\
&\le \|Q_1\|_K + \sum_{k=1}^{\infty}\bigl(2^{k+1}\bigr)^j 2C\,2^{-k(m+\alpha)} \\
&\le \|Q_1\|_K + 2^{j+1}C\sum_{k=1}^{\infty}(2^{j-m-\alpha})^k < \infty
\end{aligned}$$

for every $\theta \in K$ and $j = 0, 1, \dots, m$, since $\alpha > 0$. Now we can conclude that $f^{(j)}(\theta)$ exists and

$$f^{(j)}(\theta) = Q_1^{(j)}(\theta) + \sum_{k=1}^{\infty}(Q_{2^{k+1}} - Q_{2^k})^{(j)}(\theta)$$

for every $\theta \in K$ and $j = 0, 1, \dots, m$. The fact that $f^{(m)} \in C(K)$ can be seen by the Weierstrass M-test. □

The next exercise follows Videnskii [60].

E.19 Videnskii's Inequalities. The main results of this exercise are the Bernstein- (part b]) and Markov-type (part c]) inequalities for trigonometric polynomials on an interval shorter than the period.

Let $\omega \in (0, \pi)$,

$$t_n(\theta) := Q_{n,\omega}(\theta) = \cos\left(2n \arccos\left(\frac{\sin(\theta/2)}{\sin(\omega/2)}\right)\right)\,,$$

and

$$u_n(\theta) := \sin\left(2n \arccos\left(\frac{\sin(\theta/2)}{\sin(\omega/2)}\right)\right)\,.$$

a] Recall that $t_n \in \mathcal{T}_n$ by E.3 a].

b] Show that

$$\begin{aligned} |s_n'(\theta)| &\le |t_n'(\theta) + iu_n'(\theta)|\, \|s_n\|_{[-\omega,\omega]} \\ &= n\left(1 - \frac{\cos^2(\omega/2)}{\cos^2(\theta/2)}\right)^{-1/2} \|s_n\|_{[-\omega,\omega]} \end{aligned}$$

for every $s_n \in \mathcal{T}_n$ and $\theta \in (-\omega, \omega)$. Equality holds if and only if $s_n = ct_n$ for some $c \in \mathbb{R}$ and $s_n(\theta) = 0$.

Hint: First show that for every $n \in \mathbb{N}$, $\omega \in (0, \pi]$, and $\theta \in [-\omega, \omega]$, there exists an $\widetilde{s} \in \mathcal{T}_n$ such that

$$\frac{|\widetilde{s}\,'(\theta)|}{\|\widetilde{s}\|_{[-\omega,\omega]}} = \max_{s_n \in \mathcal{T}_n} \frac{|s_n'(\theta)|}{\|s_n\|_{[-\omega,\omega]}}\,.$$

Use a variational method to show that either

$$\|\widetilde{s}_n\|_{[-\omega,\omega]} = \|\widetilde{s}_n\|_{[-\pi,\pi]}$$

or there exist $\alpha \in (-\pi, -\omega]$ and $\beta \in [\omega, \pi)$ such that $\widetilde{s}_n = cT_n$ for some $c \in \mathbb{R}$, where, as in Section 3.3,

$$T_n := T_n\{1, \cos\tau, \sin\tau, \ldots, \cos n\tau, \sin n\tau; [\alpha, \beta]\}$$

is the Chebyshev polynomial for $\mathcal{T}_n$ on $[\alpha, \beta]$. In the first case use Theorem 5.1.4 (Bernstein's inequality). In the second case observe that

$$T_n(\tau) = t_n\left(\frac{\sin((\tau - \gamma)/2)}{\sin(\widetilde{\omega}/2)}\right)$$

with

$$\gamma := \tfrac{1}{2}(\alpha + \beta) \qquad \text{and} \qquad \widetilde{\omega} := \tfrac{1}{2}(\beta - \alpha)\,.$$

□

c] Show that if $2n > (3\tan^2(\omega/2) + 1)^{1/2}$, then

$$\|s_n'\|_{[-\omega,\omega]} \le t_n'(\omega)\|s_n\|_{[-\omega,\omega]} = 2n^2\cot(\omega/2)\|s_n\|_{[-\omega,\omega]}$$

for every $s_n \in \mathcal{T}_n$, and equality holds if and only if $s_n = ct_n$, $c \in \mathbb{R}$.

Outline. Let $-\omega = \xi_0 < \xi_1 < \cdots < \xi_{2n} = \omega$ be the points where

$$t_n(\xi_j) = (-1)^j\,, \qquad j = 0, 1, \ldots, 2n$$

and

$$u_n(\xi_j) = 0\,, \qquad j = 1, 2, \ldots, 2n-1\,,$$

and let $\theta_1 < \theta_2 < \cdots < \theta_{2n}$ be the zeros of t_n (which lie in $(-\omega, \omega)$). Note that

$$u_n'(\theta_j) = 0\,, \qquad j = 1, 2, \ldots, 2n\,.$$

Step 1. Show that $2n > (3\tan^2(\omega/2) + 1)^{1/2}$ implies $t_n''(\theta) > 0$ for every $\theta \in [\xi_{2n-1}, \omega]$, so t_n' is increasing on $[\xi_{2n-1}, \omega]$.

Step 2. Use part b] to show that if $\theta \in [\theta_1, \theta_{2n}]$, then

$$|t_n'(\theta)| \le |t_n'(\theta) + iu_n'(\theta)| \le |t_n'(\theta_{2n}) + iu_n'(\theta_{2n})| = |t_n'(\theta_{2n})|\,.$$

Step 3. Deduce from Steps 1 and 2 that

$$|t_n'(\theta)| < |t_n'(\pm\omega)|\,, \qquad \theta \in (-\omega, \omega)\,.$$

Step 4. Show that there is an $\widetilde{s} \in \mathcal{T}_n$ such that

$$\frac{\|\widetilde{s}_n'\|_{[-\omega,\omega]}}{\|\widetilde{s}_n\|_{[-\omega,\omega]}} = \max_{s \in \mathcal{T}_n} \frac{\|s'\|_{[-\omega,\omega]}}{\|s\|_{[-\omega,\omega]}}\,.$$

For the rest of the proof let $\widetilde{s}_n$ be normalized by $\|\widetilde{s}\|_{[-\omega,\omega]} = 1$ and let $\theta^* \in [-\omega, \omega]$ be chosen so that

$$|\widetilde{s}_n'(\theta^*)| = \|\widetilde{s}_n'(\theta)\|_{[-\omega,\omega]}.$$

Let $(a_1 < a_2 < \cdots < a_m)$ be an alternation sequence of maximal length for $\widetilde{s}_n \in C[-\omega, \omega]$ on $[-\omega, \omega]$. We would like to show that $m = 2n + 1$. Clearly $m < 2n + 2$.

Step 5. Use a variational method to show that $2n \le m$.

Step 6. Use a variational method to show that $\theta^* = \pm\omega$ implies $m = 2n+1$, so $m = 2n$ implies $\theta^* \in (-\omega, \omega)$.

Step 7. Show by a variational method that

$$|s_n'(\pm\omega)| \le |t_n'(\pm\omega)|\,\|s_n\|_{[-\omega,\omega]}$$

for every $s_n \in \mathcal{T}_n$, and equality holds if and only if $s_n = ct_n$, $c \in \mathbb{R}$. In particular, if $m = 2n$, then

$$|\widetilde{s}_n'(\pm\omega)| < |t_n'(\pm\omega)|\,.$$

Step 8. Use part b] and Step 3 to show that

$$|\widetilde{s}_n'(\theta)| \le t_n'(\theta_{2n}) < t_n'(\omega)\,, \qquad \theta \in [\theta_1, \theta_{2n}]\,.$$

Step 9. Use part b] to show that $m = 2n$ implies

$$|\widetilde{s}_n'(\theta_j)| < |t_n'(\theta_j)| = (-1)^j t_n'(\theta_j)\,, \qquad j = 1, 2, \ldots, 2n\,.$$

Step 10. Suppose $m = 2n$. Use Steps 6 and 8 to show that

$$\theta^* \in (-\omega, \theta_1) \cup (\theta_{2n}, \omega)\,.$$

Step 11. Suppose $m = 2n$. Use the defining property of $\widetilde{s}_n$ and Steps 1 and 10 to show that

$$|\widetilde{s}_n'(\theta^*)| = \|\widetilde{s}_n'\|_{[-\omega,\omega]} \geq |t_n'(\pm\omega)| > |t_n'(\theta^*)|\,.$$

Step 12. Suppose $m = 2n$ and $s_n'(\theta^*) \geq 0$. Use Steps 7 to 11 to show that $t_n' - \widetilde{s}\,'_n$ has at least $2n+1$ distinct zeros in $(-\omega, \omega)$, a contradiction.

Step 13. Show that $m = 2n+1$ and $\widetilde{s}_n = \pm t_n$. □

E.20 Inequalities for Entire Functions of Exponential Type. Entire functions of (exponential) type τ are defined in E.17 of Section 4.2. Denote by E_τ the set of all entire functions of exponential type at most τ. This exercise collects some of the interesting inequalities known for E_τ. Since a trigonometric polynomial of degree n belongs to E_n, these results can be viewed as extensions of the corresponding inequalities for trigonometric polynomials. More on various inequalities for entire functions of exponential type may be found in Rahman and Schmeisser [83].

a] Bernstein's Inequality. The inequality

$$\|f'\|_{\mathbb{R}} \leq \tau\,\|f\|_{\mathbb{R}}\,, \qquad x \in \mathbb{R}$$

holds for every $f \in E_\tau$.

Proof. See Bernstein [23] or Rahman and Schmeisser [83]. □

b] Extension of the Bernstein-Szegő Inequality. The inequality

$$(f'(x))^2 + (\tau f(x))^2 \leq \tau^2 \|f\|_{\mathbb{R}}^2\,, \qquad x \in \mathbb{R}$$

holds for every $f \in E_\tau$ taking real values on the real line.

Proof. See Duffin and Schaeffer [37]. □

c] The Growth of $f \in E_\tau$. The inequality

$$|f(x+iy)| \leq e^{\tau|y|}\|f\|_{\mathbb{R}}\,, \qquad x, y \in \mathbb{R}$$

holds for every $f \in E_\tau$.

Proof. See Rahman and Schmeisser [83]. □

d] The Growth of $f \in E_\tau$ Taking Real Values on $\mathbb{R}$. The inequality

$$|f(x+iy)| \le (\cosh \tau y)\, \|f\|_{\mathbb{R}}, \qquad x, y \in \mathbb{R}$$

holds for every $f \in E_\tau$ taking real values on the real line.

Proof. See Schaeffer and Duffin [38]. □

e] Bernstein-Type Inequality in L_p. Let $p \in (0, \infty)$. The inequality

$$\|f'\|_{L_p(\mathbb{R})} \le \tau \, \|f\|_{L_p(\mathbb{R})}$$

holds for every $f \in E_\tau$.

Proof. See Rahman and Schmeisser [90]. □

E.21 Markov-Type Inequality on Connected Subsets of the Complex Plane. Let E be a connected compact set of the complex plane. Then

$$\|p'\|_E \le \frac{e}{2} \frac{n^2}{\operatorname{cap}(E)} \|p\|_E$$

for every $p \in \mathcal{P}_n^c$.

Proof. See Pommerenke [59c]. □

Erdős conjectured that the constant $\frac{e}{2}$ in the above inequality can be replaced by $\frac{1}{2}$. This result would contain Theorem 5.1.8 (Markov's inequality) as a special case. However, Rassias, Rassias, and Rassias [77] disproved the conjecture. Erdős still speculates that $\frac{e}{2}$ in Pommerenke's inequality may be replaced by $\frac{1}{2}(1 + o(1))$.

The result of the next exercise is formulated so that its proof is elementary at the expense of precision and generality.

E.22 The Interval where the Sup Norm of a Weighted Polynomial Lives. Suppose $w = \exp(-Q)$, where

(1) $Q : \mathbb{R} \to \mathbb{R}$ is continuous and even,

(2) Q' is continuous and positive in $(0, \infty)$,

(3) $tQ'(t)$ is increasing in $(0, \infty)$, and

(4) $\lim_{t \to 0+} tQ'(t) = 0$ and $\lim_{t \to \infty} tQ'(t) = \infty$.

Let $a_n > 0$ be chosen so that

$$a_n^n w(a_n) = \max_{x \in \mathbb{R}} |x^n w(x)|.$$

a] Show that

$$n = a_n Q'(a_n) \qquad \text{and} \qquad 0 < a_1 < a_2 < a_3 < \cdots.$$

b] Show that

$$\|pw\|_{\mathbb{R}} \le \|pw\|_{[-2a_{2n},2a_{2n}]}$$

for every $p \in \mathcal{P}_n^c$.

Proof. Using Chebyshev's inequality (see E.2 of Section 5.1) and the explicit form (2.1.1) of the Chebyshev polynomial T_n, we can deduce that

$$|p(x)| \le \left| \frac{x}{a} + \sqrt{\left(\frac{x}{a}\right)^2 - 1} \right|^n \|p\|_{[-a,a]} \le \left(\frac{2|x|}{a}\right)^n \|p\|_{[-a,a]}$$

for every $p \in \mathcal{P}_n^c$, $x \in \mathbb{R} \setminus [-a,a]$, and $a > 0$. Choosing $a := a_n$, and using the fact that w is decreasing on $[0,\infty)$, we obtain

$$|(pw)(x)| \le \left(\frac{2|x|}{a_n}\right)^n \frac{w(x)}{w(a_n)} \|pw\|_{[-a_n,a_n]}$$

for every $p \in \mathcal{P}_n^c$ and $x \in \mathbb{R} \setminus [-a_n,a_n]$. Now if $x \in \mathbb{R} \setminus [-2a_{2n},2a_{2n}]$, then

$$\begin{aligned} \frac{|x|^n w(x)}{a_n^n w(a_n)} &\le \exp\left(\int_{a_n}^{|x|} \frac{d}{dt} \log(t^n w(t)) \right) dt \\ &= \exp\left(\int_{a_n}^{|x|} \frac{n - tQ'(t)}{t}\, dt \right) \\ &\le \exp\left(\int_{a_{2n}}^{2a_{2n}} \frac{n-2n}{t}\, dt \right) = 2^{-n}\,, \end{aligned}$$

where we used $a_{2n} \ge a_n$, $tQ'(t) \ge 2n$ for $t \ge a_{2n}$, and $tQ'(t) \ge n$ for $t \ge a_n$. Combining the above inequality with the previous one, we obtain that

$$|(pw)(x)| \le \|pw\|_{[-a_n,a_n]}\,, \qquad x \in \mathbb{R} \setminus [-2a_{2n},2a_{2n}]$$

for every $p \in \mathcal{P}_n^c$, from which the result follows. □

c] Let $Q(x) := |x|^\alpha$, $\alpha > 0$. Show that Q satisfies the assumptions of the exercise, and

$$a_n = \left(\frac{n}{\alpha}\right)^{1/\alpha}.$$

The idea of infinite-finite range inequalities, of which E.22 b] is an example, goes back to Freud (a_n is the *Freud number*); see Nevai's survey paper [86]. The sharp form of these is due to Mhaskar and Saff [85]; see also Lubinsky and Saff [88] and Saff and Totik [to appear]. They are not that difficult to prove, but need the maximum principle for subharmonic functions.

E.23 A Theorem of Markov. Let T_n be the Chebyshev polynomial of degree n defined by (2.1.1). Then

$$T_n(x) = \sum_{k=0}^{n} c_{k,n} x^k \,.$$

The *Markov numbers* $M_{k,n}$, $0 \le k \le n$, are defined by

$$M_{k,n} := \begin{cases} |c_{k,n}| & \text{if } k \equiv n \pmod 2 \\ |c_{k,n-1}| & \text{if } k \equiv n-1 \pmod 2 \,. \end{cases}$$

These can be explicitly computed from (2.1.1).

a] The inequalities

$$|a_{k,n}| \le M_{k,n} \, \|p\|_{[-1,1]}$$

hold for every $p \in \mathcal{P}_n$ of the form

$$p(x) = \sum_{k=0}^{n} a_{k,n} x^k, \qquad a_{k,n} \in \mathbb{R} \,.$$

Proof. See, for example, Natanson [64]. □

b] Show that

$$|p'(0)| \le (2n-1) \, \|p\|_{[-1,1]}$$

for every $p \in \mathcal{P}_{2n}$.

Hint: Use part a]. □

c] Show that

$$|p'(x)| \le \frac{2n-1}{1-|x|} \, \|p\|_{[-1,1]}$$

for every $p \in \mathcal{P}_{2n}$ and $x \in (-1,1)$.

Hint: Use part b] and a linear transformation. □

Of course, part c] gives a better result than Theorem 5.1.7 (Bernstein's Inequality) only if x is very close to 0. This is exactly the case we need in our application of part c] in E.4 c] of Section 6.1.

5.2 Markov's Inequality for Higher Derivatives

From Theorem 5.1.8 (Markov's Inequality), by induction on m it follows that

$$\|p^{(m)}\|_{[-1,1]} \le (n(n-1)\cdots(n-m+1))^2 \|p\|_{[-1,1]}$$

for every $p \in \mathcal{P}_n$.

However, this is not the best possible result. The main result of this section is the following inequality of Duffin and Schaeffer [41], which gives a sharp improvement of the above. E.2 d] extends the following result to polynomials with complex coefficients:

Theorem 5.2.1. *If $p \in \mathcal{P}_n$ satisfies*

$$\left|p\left(\cos\tfrac{j\pi}{n}\right)\right| \le 1\,, \qquad j = 0, 1, \dots, n\,,$$

then for every $m = 1, 2, \dots, n$,

$$|p^{(m)}(x+iy)| \le |T_n^{(m)}(1+iy)|\,, \qquad x \in [-1,1]\,, \quad y \in \mathbb{R}\,,$$

where T_n is the Chebyshev polynomial of degree n defined by (2.1.1). Equality can occur only if $p = \pm T_n$.

To prove this inequality we need three lemmas, which are of some interest in their own right.

Lemma 5.2.2. *Suppose $\lambda_1, \lambda_2, \dots, \lambda_n$ are distinct real numbers,*

$$q(z) := c\prod_{j=1}^{n}(z-\lambda_j)\,, \qquad 0 \ne c \in \mathbb{R}\,,$$

and $p \in \mathcal{P}_n^c$ satisfies

$$|p'(\lambda_j)| \le |q'(\lambda_j)|\,, \qquad j = 1, 2, \dots, n\,.$$

Then, for every $m \in \mathbb{N}$,

$$|p^{(m)}(x)| \le |q^{(m)}(x)| \tag{5.2.1}$$

whenever x is a zero of $q^{(m-1)}$.

Proof. For $m = 1$, inequality (5.2.1) is simply a restatement of the assumption of the lemma, so consider the case $m = 2$. The Lagrange interpolation formula (see E.6 of Section 1.1) gives

$$\frac{p'(z)}{q(z)} = \sum_{j=1}^{n}\frac{p'(\lambda_j)}{q'(\lambda_j)}\frac{1}{z-\lambda_j} = \sum_{j=1}^{n}\frac{\delta_j}{z-\lambda_j}\,, \tag{5.2.2}$$

where, by the hypotheses of the lemma, $|\delta_j| \le 1$. There is a similar expression for $q'(z)/q(z)$ in which each δ_j is equal to 1. On differentiating (5.2.2), we obtain

$$\frac{p''(z)q(z) - p'(z)q'(z)}{q(z)^2} = -\sum_{j=1}^{n}\frac{\delta_j}{(z-\lambda_j)^2}\,.$$

Thus at the points x where $q'(x) = 0$ we have

$$\left|\frac{p''(x)}{q(x)}\right| = \left|\sum_{j=1}^{n} \frac{\delta_j}{(x-\lambda_j)^2}\right| \le \sum_{j=1}^{n} \frac{1}{(x-\lambda_j)^2} = \left|\frac{q''(x)}{q(x)}\right|,$$

and it follows that the lemma is true for $m = 2$.

The proof for $m > 2$ is by induction. Let $|p^{(m)}(x)| \le |q^{(m)}(x)|$ at the zeros of $q^{(m-1)}$ (which are real and distinct). Applying the previous argument to $p^{(m)}$ and $q^{(m)}$ instead of p' and q', we obtain

$$|p^{(m+1)}(x)| \le |q^{(m+1)}(x)|$$

at the zeros of $q^{(m)}$. This completes the induction. □

Lemma 5.2.3. *Let $q \in \mathcal{P}_n$ have n distinct zeros in $(-\infty, b)$, and suppose that in a strip of the complex plane it satisfies the inequality*

$$|q(x+iy)| \le |q(b+iy)|, \qquad x \in [a,b], \quad y \in \mathbb{R}. \tag{5.2.3}$$

Suppose also that $p' \in \mathcal{P}_{n-1}$ satisfies

$$|p'(x)| \le |q'(x)| \tag{5.2.4}$$

whenever x is a zero of q. Then the derivatives of p and q satisfy

$$|p^{(m)}(x+iy)| \le |q^{(m)}(b+iy)|, \qquad x \in [a,b], \quad y \in \mathbb{R}. \tag{5.2.5}$$

Proof. First we show that at every point $x_0 + iy_0$ in the strip

$$|p'(x_0+iy_0)| \le |q'(b+iy_0)|.$$

Let

$$q(z) := c\prod_{j=1}^{n}(z-\lambda_j), \qquad 0 \ne c \in \mathbb{R}, \quad \lambda_j \in (-\infty, b).$$

Let $h(z)$ be another polynomial with the same leading coefficient as q and whose zeros are obtained by reflecting about x_0 those zeros of q that lie to the right of x_0. Thus

$$h(z) := c\prod_{j=1}^{n}(z-\beta_j),$$

where

$$\beta_j := \begin{cases} 2x_0 - \lambda_j & \text{if } \lambda_j > x_0 \\ \lambda_j & \text{if } \lambda_j \le x_0. \end{cases} \tag{5.2.6}$$

Then on the line $z = x_0 + iy,\ \ y \in \mathbb{R}$, we have $|z - \beta_j| = |z - \lambda_j|$, so

$$|h(x_0 + iy)| = |q(x_0 + iy)|\,. \tag{5.2.7}$$

We now show that

$$|p'(x_0 + iy_0)| \le |h'(x_0 + iy_0)|\,. \tag{5.2.8}$$

Note that

$$\frac{h'(z)}{h(z)} = \sum_{j=1}^{n} \frac{1}{z - \beta_j}\,, \tag{5.2.9}$$

and recalling (5.2.2), we have

$$\frac{p'(z)}{q(z)} = \sum_{j=1}^{n} \frac{\delta_j}{z - \lambda_j} \tag{5.2.10}$$

with $\delta_j \in [-1, 1]$ for each j. Comparing the right-hand sides of (5.2.9) and (5.2.10), respectively, at $z = x_0 + iy_0$, we obtain

$$\begin{aligned}
\left| \sum_{j=1}^{n} \frac{\delta_j}{x_0 + iy_0 - \lambda_j} \right| &= \left| \sum_{j=1}^{n} \frac{\delta_j (x_0 - \lambda_j)}{(x_0 - \lambda_j)^2 + y_0^2} - i \sum_{j=1}^{n} \frac{y_0 \delta_j}{(x_0 - \lambda_j)^2 + y_0^2} \right| \\
&\le \left| \sum_{j=1}^{n} \frac{x_0 - \beta_j}{(x_0 - \beta_j)^2 + y_0^2} - i \sum_{j=1}^{n} \frac{y_0}{(x_0 - \beta_j)^2 + y_0^2} \right| \\
&= \left| \sum_{j=1}^{n} \frac{1}{x_0 - \beta_j + iy_0} \right|
\end{aligned}$$

since by construction $|x_0 - \lambda_j| = x_0 - \beta_j$. Therefore

$$\left| \frac{p'(x_0 + iy_0)}{q(x_0 + iy_0)} \right| \le \left| \frac{h'(x_0 + iy_0)}{h(x_0 + iy_0)} \right|\,,$$

and (5.2.7) yields (5.2.8).

Let $\alpha \in \mathbb{C},\ \ |\alpha| < 1$, be an arbitrary constant and let

$$\varphi(z) := q(z) - \alpha h(z + x_0 - b)\,.$$

Let Γ be the simple closed curve consisting of a segment of the line $z = b + iy,\ \ y \in \mathbb{R}$, and the portion of a circle with center at b and radius ϱ that lies to the right of this line. Relations (5.2.3) and (5.2.7) show that on the line segment of Γ,

$$|q(z)| > |\alpha h(z + x_0 - b)|\,.$$

If ϱ is sufficiently large, the same inequality is true for the circular portion of Γ since q and h have the same leading coefficient. Thus Rouché's theorem gives that φ and h have the same number of zeros inside Γ. We conclude that φ has no zeros on or to the right of the line $z = b + iy,\ \ y \in \mathbb{R}$. The last statement, together with Theorem 1.3.1, implies that φ' has no zeros on the line $z = b + iy,\ \ y \in \mathbb{R}$. Thus for $|\alpha| < 1$,

$$q'(b + iy_0) - \alpha h'(x_0 + iy_0) \neq 0\,,$$

which, together with (5.2.8), yields

$$|p'(x_0 + iy_0)| \leq |h'(x_0 + iy_0)| \leq |q'(b + iy_0)|\,.$$

This proves the lemma for $m = 1$.

We turn now to the case $m > 1$. Applying the lemma with $m = 1$ when $p = q$, we have

$$|q'(x + iy)| \leq |q'(b + iy)|\,, \qquad x \in [a, b]\,, \quad y \in \mathbb{R}\,.$$

Thus q' satisfies all the requirements that are imposed on the interpolating polynomial q (with n replaced by $n - 1$) in the lemma, and by Lemma 5.2.2, $|p''(x)| \leq |q''(x)|$ at the $n-1$ zeros of q'. Applying the lemma to $p'(x)$ instead of p in the case $m = 1$, for which it has already been proved, we have

$$|p''(x + iy)| \leq |q''(b + iy)|\,, \qquad x \in [a, b]\,, \quad y \in \mathbb{R}\,,$$

which proves that inequality (5.2.5) is true for $m = 2$. Repetition of this argument completes the proof for larger values of m. □

Lemma 5.2.4. *Suppose $p \in P_n^c$ and $|p(\lambda_j)| \leq 1$ for*

$$\lambda_j := \cos \tfrac{j\pi}{n}\,, \qquad j = 0, 1, \ldots, n\,.$$

Then

$$|p'(x_k)| \leq \frac{n}{\sqrt{1 - x_k^2}} \qquad \text{for} \quad x_k := \cos \tfrac{(2k-1)\pi}{2n}\,, \quad k = 0, 1, \ldots, n\,.$$

For every fixed k, equality holds if and only if $p = cT_n$ for some $c \in \mathbb{C}$ with $|c| = 1$.

Proof. Let

$$f(x) = (1 - x^2)T_n'(x) = a \prod_{j=0}^{n} \left(x - \cos \tfrac{j\pi}{n}\right).$$

Then by the differential equation for T_n (see E.3 e] of Section 2.1), we have

$$f'(x) = -xT_n'(x) - n^2 T_n(x)\,.$$

Differentiating the Lagrange interpolation formula for p (see E.6 of Section 1.1) gives

$$p'(x) = \sum_{j=0}^{n} \frac{p(\lambda_j)}{f'(\lambda_j)} \frac{(x-\lambda_j)f'(x) - f(x)}{(x-\lambda_j)^2}\,.$$

For the zeros of T_n, this reduces to

$$p'(x) = -T_n'(x) \sum_{j=0}^{n} \frac{p(\lambda_j)}{f'(\lambda_j)} \frac{(1-x\lambda_j)}{(x-\lambda_j)^2} \tag{5.2.11}$$

since at these points

$$\begin{aligned}(x-\lambda_j)f'(x) - f(x) &= -x(x-\lambda_j)T_n'(x) - (1-x^2)T_n'(x)\\ &= -(1-x\lambda_j)T_n'(x)\,.\end{aligned}$$

In the same way, we obtain for the zeros of T_n,

$$T_n'(x) = -T_n'(x) \sum_{j=0}^{n} \frac{T_n(\lambda_j)}{f'(\lambda_j)} \frac{(1-x\lambda_j)}{(x-\lambda_j)^2}$$

and since $T_n(\lambda_j)$ and $f'(\lambda_j)$ are of opposite sign ($f'(\lambda_j) = -n^2 T_n(\lambda_j)$), this gives

$$T_n'(x) = T_n'(x) \sum_{j=0}^{n} \left|\frac{1}{f'(\lambda_j)}\right| \frac{(1-x\lambda_j)}{(x-\lambda_j)^2}\,. \tag{5.2.12}$$

Since $|p(\lambda_j)| \le 1$ in (5.2.11), on comparing (5.2.11) and (5.2.12), we obtain for every zero x of T_n that

$$|p'(x)| \le |T_n'(x)| = \frac{n}{\sqrt{1-x^2}}\,.$$

For a fixed zero of T_n, the equality occurs if and only if

$$p(\lambda_j) = cT_n(\lambda_j)\,, \qquad j = 0, 1, \ldots, n$$

for some $c \in \mathbb{C}$, $|c| = 1$, that is, if and only if $p = cT_n$ for some $c \in \mathbb{C}$ with $|c| = 1$. □

Proof of Theorem 5.2.1. Suppose $p \ne T_n$ satisfies the assumption of the theorem. Then by Lemma 5.2.4 there exists a constant $\alpha > 1$ such that $|\alpha p'(x)| \le |T_n'(x)|$ at the zeros of T_n. Applying Lemma 5.2.3 with p and q replaced by αp and T_n (assumption (5.2.3) is satisfied with $[a,b] := [-1,1]$ by E.1 b]), we obtain

$$|p^{(m)}(x+iy)| \le \alpha^{-1} \left|T_n^{(m)}(1+iy)\right|\,, \qquad x \in [-1,1]\,, \quad y \in \mathbb{R}$$

for every $m = 1, 2, \ldots, n$. If $p = T_n$, then we have the same inequality with α^{-1} replaced by 1. □

Comments, Exercises, and Examples.

The inequality

$$\|p^{(m)}\|_{[-1,1]} \le T_n^{(m)}(1) \cdot \|p\|_{[-1,1]}\,, \quad p \in \mathcal{P}_n$$

was first proved by V. A. Markov [16]. He was the brother of the more famous A. A. Markov who proved the above inequality for $m = 1$ in [1889] (see Theorem 5.1.8). However, their ingenious proofs are rather complicated. Bernstein presented a shorter variational proof of V. A. Markov's inequality in 1938 (see Bernstein [58], which includes a complete list of Bernstein's publications). Our discussion in this section follows Duffin and Schaeffer [41].

E.1 A Property of Chebyshev Polynomials.

a] Let $(\alpha_i)_{i=1}^{2n}$ be a sequence of $2n$ nonnegative numbers, and let (α_i') be a rearrangement of this sequence according to magnitude,

$$\alpha_1' \ge \alpha_2' \ge \cdots \ge \alpha_{2n}' \ge 0\,.$$

Show that for every $y \ge 0$,

$$(\alpha_1\alpha_2 + y)(\alpha_3\alpha_4 + y)\cdots(\alpha_{2n-1}\alpha_{2n} + y) \tag{5.2.13}$$

is not greater than

$$(\alpha_1'\alpha_2' + y)(\alpha_3'\alpha_4' + y)\cdots(\alpha_{2n-1}'\alpha_{2n}' + y)\,.$$

Proof. If α_1 and α_3 are at least as large as any of the remaining numbers α_i, then

$$(\alpha_1\alpha_3 + y)(\alpha_2\alpha_4 + y) - (\alpha_1\alpha_2 + y)(\alpha_3\alpha_4 + y) = y(\alpha_1 - \alpha_4)(\alpha_3 - \alpha_2) \ge 0\,.$$

This shows that the numbers α_i in (5.2.13) can be rearranged so that the two largest occur in the same factor without decreasing (5.2.13). Then the two largest of the remaining numbers α_i may be brought into the same factor without decreasing (5.2.13), and so on. □

b] Show that the Chebyshev polynomials T_n defined by (2.1.1) satisfy the inequality

$$|T_n(x + iy)| \le |T_n(1 + iy)|\,, \qquad x \in [-1,1]\,, \quad y \in \mathbb{R}\,.$$

Proof. We have

$$|T_n(x + iy)|^2 = c^2 \prod_{j=1}^{n} \left((x - \cos\theta_j)^2 + y^2\right),$$

where $\theta_j = \frac{(2j-1)\pi}{2n}$ and $c = 2^{n-1}$. With $x = \cos\theta$ we write

$$|T_n(x+iy)|^2 = c^2 \prod_{j=1}^{n} \left(\tfrac{1}{4} \left| e^{i\theta} - e^{-i\theta_j} \right|^2 \left| e^{i\theta} - e^{i\theta_j} \right|^2 + y^2 \right).$$

Geometrically, $e^{\pm i\theta_j}$, $j = 1, 2, \ldots, n$, represent $2n$ points equally distributed on the unit circle. Connect these points by chords to the point $e^{i\theta}$. Then the lengths of these $2n$ chords are given by $|e^{i\theta} - e^{\pm i\theta_j}|$. If θ is increased or decreased by any multiple of $\frac{\pi}{n}$, we obtain a new set of chords, but the aggregate of their lengths is unchanged. Choose φ such that

$$\varphi \equiv \theta \pmod{\tfrac{\pi}{n}}, \qquad -\tfrac{\pi}{2n} \le \varphi \le \tfrac{\pi}{2n}.$$

If $x^* = \cos\varphi$, then

$$T_n(x^* + iy) = c^2 \prod_{j=1}^{n} \left(\tfrac{1}{4} \left| e^{i\varphi} - e^{-i\theta_j} \right|^2 \left| e^{i\varphi} - e^{i\theta_j} \right|^2 + y^2 \right),$$

where the numbers $|e^{i\varphi} - e^{\pm i\theta_j}|^2$ are simply a rearrangement of the numbers $|e^{i\theta} - e^{\pm i\theta_j}|^2$. Use part a] to show that

$$|T_n(x+iy)|^2 \le |T_n(x^* + iy)|^2.$$

Note that $\cos\theta_1 \le x^* \le 1$, where $\cos\theta_1$ is the right most zero of T_n. Hence

$$|T_n(x^* + iy)|^2 \le |T_n(1+iy)|^2,$$

and the proof is finished. □

E.2 Markov's Inequality for Higher Derivatives for $\mathcal{P}_n^c$.

a] Show that if p and q satisfy the conditions of Lemma 5.2.3 with $p \in \mathcal{P}_n$ replaced by $p \in \mathcal{P}_n^c$, then

$$|p^{(m)}(x)| \le |q^{(m)}(b)|, \qquad x \in [a,b]$$

for every $m \in \mathbb{N}$.

Hint: After differentiating (5.2.2) $m-1$ times, we obtain

$$p^{(m)}(x) = \sum_{j=1}^{n} \delta_j \frac{d^{m-1}}{dx^{m-1}} \left(\frac{q(x)}{x - \lambda_j} \right),$$

where $|\delta_j| = |p'(\lambda_j)/q'(\lambda_j)| \le 1$. It is evident that if $x \in (a,b)$ is fixed, then $|p^{(m)}(x)|$ attains its maximum when $\delta_j = \pm 1$ for each j, in which case p' has real coefficients. Now use Lemma 5.2.3. □

b] Show that if p satisfies the assumption of Theorem 5.2.1 with $p \in \mathcal{P}_n$ replaced by $p \in \mathcal{P}_n^c$, then

$$\|p^{(m)}\|_{[-1,1]} \le |T_n^{(m)}(1)|$$

for every $m \in \mathbb{N}$. The equality can hold only if $p = cT_n$ for some $c \in \mathbb{C}$, $|c| = 1$.

Hint: Modify the proof of Theorem 5.2.1 by using part a]. □

c] Show that

$$T_n^{(m)}(1) = \frac{n^2(n^2-1)(n^2-2^2)\cdots(n^2-(m-1)^2)}{1\cdot 3\cdot 5\cdots(2m-1)}\,.$$

Hint: Differentiating the second-order differential equation for T_n (see E.3 e] of Section 2.1) $m-1$ times gives

$$(1-x^2)T_n^{(m+2)}(x) - (2m+1)xT_n^{(m+1)}(x) + (n^2-m^2)T_n^{(m)}(x) = 0$$

from which

$$(2m+1)\,T_n^{(m+1)}(1) = (n^2-m^2)\,T_n^{(m)}(1)$$

follows. Use induction and $T_n(1) = 1$ to finish the proof. □

d] The Main Inequality. Suppose $p \in \mathcal{P}_n^c$ satisfies

$$\left|p\left(\cos\tfrac{j\pi}{n}\right)\right| \le 1\,, \quad j = 1, 2, \ldots, n\,.$$

Show that for $m = 1, 2, \ldots, n$,

$$\|p^{(m)}\|_{[-1,1]} \le \frac{n^2(n^2-1)(n^2-2^2)\cdots(n^2-(m-1)^2)}{1\cdot 3\cdot 5\cdots(2m-1)}\,,$$

and the equality can occur only if $p = cT_n$ for some $c \in \mathbb{C}$, $|c| = 1$.

Hint: Combine parts c] and b]. □

A slightly weaker version of Markov's inequality for higher derivatives is much easier to prove.

E.3 A Weaker Version of Markov's Inequality. Show that

$$\|p^{(m)}\|_{[-1,1]} \le 2^m n^{2m} \|p\|_{[-1,1]}$$

for every $p \in \mathcal{P}_n$.

Hint: First show by a variational method that the extremal problem

$$\max_{0 \not\equiv p \in \mathcal{P}_n} \frac{|p'(\pm 1)|}{\|p\|_{[-1,1]}}$$

is solved by the Chebyshev polynomial T_n, hence

$$|p'(\pm 1)| \le n^2 \|p\|_{[-1,1]} .$$

Then, by using a linear transformation, show that

$$|p'(y)| \le \frac{2}{1-y} n^2 \|p\|_{[y,1]} \le 2n^2 \|p\|_{[-1,1]} , \qquad -1 \le y \le 0$$

and

$$|p'(y)| \le \frac{2}{1+y} n^2 \|p\|_{[-1,y]} \le 2n^2 \|p\|_{[-1,1]} , \qquad 0 \le y \le 1 .$$

Hence the inequality of the exercise is proved when $m = 1$. For larger values of m use induction. □

E.4 Weighted Bernstein and Markov Inequalities. Let $w \in C[-1,1]$ be strictly positive on $[-1,1]$.

a] Show that for every $\epsilon > 0$ there exists an n_0 depending on ϵ and w such that

$$\left\| p'(x) w(x) \sqrt{1-x^2} \right\|_{[-1,1]} \le n(1+\epsilon) \|pw\|_{[-1,1]}$$

for every $p \in \mathcal{P}_n$, $n \ge n_0$.

Proof. By the Weierstrass approximation theorem, for every $\eta > 0$ there is a $q \in \mathcal{P}_k$ such that

$$w(x) \le q(x) \le (1+\eta) w(x) , \qquad x \in [-1,1] .$$

Let $m := \min\{w(x) : x \in [-1,1]\}$. Applying Theorem 5.1.7 (Bernstein's inequality) to $pq \in \mathcal{P}_{n+k}$ and then to $q \in \mathcal{P}_k$, we obtain

$$\begin{aligned}
\left| p'(x) w(x) \sqrt{1-x^2} \right| &\le \left| p'(x) q(x) \sqrt{1-x^2} \right| \\
&\le \left| (pq)'(x) \sqrt{1-x^2} \right| + \left| p(x) q'(x) \sqrt{1-x^2} \right| \\
&\le (n+k) \|pq\|_{[-1,1]} + \|p\|_{[-1,1]} k \|q\|_{[-1,1]} \\
&\le (n+k)(1+\eta) \|pw\|_{[-1,1]} + \frac{1}{m} \|pw\|_{[-1,1]} k (1+\eta) \|w\|_{[-1,1]} \\
&\le n(1+\epsilon) \|pw\|_{[-1,1]}
\end{aligned}$$

for every $p \in \mathcal{P}_n$, provided $\eta > 0$ is sufficiently small and $n \ge n_0$. □

b] Show that for every $\epsilon > 0$ there exists an n_0 depending on ϵ and w such that

$$\|p'w\|_{[-1,1]} \le n^2(1+\epsilon)\|pw\|_{[-1,1]}$$

for every $p \in \mathcal{P}_n$, $n \ge n_0$.

Hint: Use the idea given in the previous proof. □

Schaeffer and Duffin [38] prove an extension of Theorem 5.1.7 (Bernstein's inequality) to higher derivatives. They show that

$$\left|\frac{d^m}{dx^m}\,p(x)\right| \le \left|\frac{d^m}{dx^m}\,\exp(in\arccos x)\right|, \qquad x \in (-1,1)$$

for every $p \in \mathcal{P}_n$. The following exercise gives a slightly weaker version of this, which is much simpler to prove. Some of this follows Lachance [84].

E.5 Bernstein's Inequality for Higher Derivatives.

a] Show that there exists a constant $c(m)$ depending only on m such that

$$|p^{(m)}(x)| \le c(m)\left(\frac{n}{\sqrt{1-x^2}}\right)^m \|p\|_{[-1,1]}, \qquad x \in (-1,1)$$

for every $p \in \mathcal{P}_n^c$. (That we can choose $c(m) \le 2^m$ is shown in parts c], d], and e].)

Hint: For $j = 1, 2, \ldots, m$, let

$$a_j := x - \frac{(m-j)(1+x)}{m} \qquad \text{and} \qquad b_j := x + \frac{(m-j)(1-x)}{m}.$$

Use Corollary 5.1.5 to show that there are constants $c_j(m)$ depending only on m such that

$$\|p^{(j)}\|_{[a_j,b_j]} \le c_j(m)\frac{n}{\sqrt{1-x^2}}\|p^{(j-1)}\|_{[a_{j-1},b_{j-1}]}$$

for every $p \in \mathcal{P}_n^c$ and $j = 1, 2, \ldots, m$. □

b] Show that there exists a constant $c(m) > 0$ depending only on m such that

$$\sup_{0 \ne p \in \mathcal{P}_n} \frac{|p^{(m)}(x)|}{\|p\|_{[-1,1]}} \ge c(m)\left(\min\left\{n^2, \frac{n}{\sqrt{1-x^2}}\right\}\right)^m$$

for every $x \in [-1,1]$ and $m = 1, 2, \ldots, n$.

Hint: First show that

$$\|T_n^{(m)}\|_{I(x)} \ge c(m)\left(\min\left\{n^2, \frac{n}{\sqrt{1-x^2}}\right\}\right)^m,$$

where $I(x) := [x - \frac{1}{2}(1-|x|),\, x + \frac{1}{2}(1-|x|)]$, then use a shift and a scaling. □

In the rest of the exercise we show that $c(m) = 2^m$ is a suitable choice for the constant in part a].

c] Let k be a positive integer. Then

$$\|p(x)(1-x^2)^{(k-1)/2}\|_{[-1,1]} \le \frac{n+k}{k}\|p(x)(1-x^2)^{k/2}\|_{[-1,1]}$$

for every $p \in \mathcal{P}_n$.

Hint: Let $p \in \mathcal{P}_n$ be normalized so that $\|p(x)(1-x^2)^{k/2}\|_{[-1,1]} = 1$. Apply Theorem 5.1.3 (Bernstein-Szegő inequality) with

$$t(\theta) := p(\cos\theta)\sin^k\theta \in \mathcal{T}_{n+k}\,.$$

If $x_0 = \cos\theta_0$ denotes a relative extreme point for p on $(-1,1)$, then $p'(\cos\theta_0) = 0$, and after simplification we obtain that

$$(p(\cos\theta_0)\sin^{k-1}\theta_0)^2 \le \frac{(n+k)^2}{((n+k)^2-k^2)\sin^2\theta_0 + k^2} \le \left(\frac{n+k}{k}\right)^2\,.$$

□

d] Let k be a positive integer. Then

$$\|p'(x)(1-x^2)^{(k+1)/2}\|_{[-1,1]} \le 2(n+k)\|p(x)(1-x^2)^{k/2}\|_{[-1,1]}$$

for every $p \in \mathcal{P}_n$.

Proof. Let $p \in \mathcal{P}_n$ be normalized so that $\|p(x)(1-x^2)^{k/2}\|_{[-1,1]} = 1$. Applying Theorem 5.1.4 (Bernstein's inequality) with $m = 1$ and

$$t(\theta) := p(\cos\theta)\sin^k\theta \in \mathcal{T}_{n+k}\,,$$

we obtain

$$|p'(\cos\theta)\sin^{k+1}\theta + p(\cos\theta)k\sin^{k-1}\theta\cos\theta| \le n+k\,.$$

Now the triangle inequality and part c] yield

$$\begin{aligned}|p'(\cos\theta)\sin^{k+1}\theta| &\le (n+k) + k\left|p(\cos\theta)\sin^{k-1}\theta\right| \\ &\le (n+k) + k\,\frac{n+k}{k} \le 2(n+k)\,.\end{aligned}$$

□

e] Show that

$$|p^{(m)}(x)| \le \left(\frac{2n}{\sqrt{1-x^2}}\right)^m \|p\|_{[-1,1]}$$

for every $p \in \mathcal{P}_n$ and $x \in (-1,1)$.

Hint: Use induction on m, Theorem 5.1.7, and part d]. □

5.3 Inequalities for Norms of Factors

A typical result of this section is the following inequality due to Kneser [34].

Theorem 5.3.1. *Suppose $p = qr$, where $q \in \mathcal{P}_m^c$ and $r \in \mathcal{P}_{n-m}^c$. Then*

$$\|q\|_{[-1,1]}\|r\|_{[-1,1]} \le \tfrac{1}{2} C_{n,m} C_{n,n-m} \|p\|_{[-1,1]},$$

where

$$C_{n,m} := 2^m \prod_{k=1}^{m} \left(1 + \cos\tfrac{(2k-1)\pi}{2n}\right).$$

Furthermore, for any n and $m \le n$ the inequality is sharp in the case that p is the Chebyshev polynomial T_n of degree n defined by (2.1.1), and the factor $q \in \mathcal{P}_m^c$ is chosen so that q vanishes at the m zeros of p closest to -1.

Before proving the above theorem, we establish an asymptotic formula for $C_{n,m}$ and formulate a corollary.

If $f \in C^2[a,b]$, then by the midpoint rule of numerical integration

$$\int_a^b f(x)\,dx = (b-a) f\left(\tfrac{1}{2}(a+b)\right) + \frac{(b-a)^3}{24} f''(\xi)$$

for some $\xi \in [a,b]$. Let $f(x) := \log(2 + 2\cos \pi x)$. Then

$$f''(x) = \frac{-\pi^2}{(1+\cos \pi x)^2}.$$

On applying the midpoint rule to the above f, we obtain

$$\int_0^{m/n} \log(2+2\cos \pi x)\,dx = \frac{1}{n}\sum_{k=1}^{m} \log\left(2 + 2\cos\tfrac{(2k-1)\pi}{2n}\right) + O\left(m\,\frac{1}{24}\,\frac{1}{n^3}\,\frac{n^4}{(n-m)^4}\right)$$

for all integers $0 \le m \le n$. Thus

$$C_{n,m} = \exp\left(\log \prod_{k=1}^{m} \left(2 + 2\cos\tfrac{(2k-1)\pi}{2n}\right)\right) \tag{5.3.1}$$
$$= \left(\exp\left(\frac{1}{n}\sum_{k=1}^{m} \log\left(2 + 2\cos\tfrac{(2k-1)\pi}{2n}\right)\right)\right)^n$$
$$= \exp\left(O\left(\frac{mn^2}{(n-m)^4}\right)\right)\left(\exp(I(n,m))\right)^n,$$

where

$$I(n,m) := \int_0^{m/n} \log(2 + 2\cos\pi x)\, dx\,.$$

So

$$\left(C_{n,\lfloor n/2\rfloor}\right)^{1/n} \sim \exp\left(\int_0^{1/2} \log(2 + 2\cos\pi x)\, dx\right) = 1.7916\ldots \tag{5.3.2}$$

and

$$\left(C_{n,\lfloor 2n/3\rfloor}\right)^{1/n} \sim \exp\left(\int_0^{2/3} \log(2 + 2\cos\pi x)\, dx\right) = 1.9081\ldots\,. \tag{5.3.3}$$

We use the notation $a_n \sim b_n$ and $a_n \lesssim b_n$ to mean $\lim\limits_{n\to\infty} a_n/b_n = 1$ and $\limsup\limits_{n\to\infty} a_n/b_n \le 1$, respectively.

On estimating $\frac{1}{2}C_{n,m}C_{n,n-m}$ in Theorem 5.3.1 and using (5.3.2), we obtain the following:

Corollary 5.3.2. *Let $p \in \mathcal{P}_n^c$ and suppose $p = qr$ for some polynomials q and r. Then*

$$\|q\|_{[-1,1]}\|r\|_{[-1,1]} \le 2^{n-1} \prod_{k=1}^{\lfloor n/2\rfloor} \left(1 + \cos\tfrac{(2k-1)\pi}{2n}\right)^2 \|p\|_{[-1,1]}$$
$$\le \tfrac{1}{2}C_{n,\lfloor n/2\rfloor}^2 \|p\|_{[-1,1]}$$

and equality holds when p is the Chebyshev polynomial T_n of degree n, and the factor $q \in \mathcal{P}_m^c$ is chosen so that $m := \lfloor n/2\rfloor$ and q vanishes at the m zeros of T_n closest to -1. Here $C_{n,\lfloor n/2\rfloor}^{2/n} \sim 3.20991\ldots$, hence

$$\left(\frac{\|q\|_{[-1,1]}\|r\|_{[-1,1]}}{\|p\|_{[-1,1]}}\right)^{1/n} \lesssim 3.20991\ldots\,.$$

The proof of Theorem 5.3.1 proceeds through a number of lemmas. For the remainder of the proof we assume that $0 < m < n$ are fixed. Now

$$\sup\{\|q\|_{[-1,1]}\|r\|_{[-1,1]} : \|qr\|_{[-1,1]} = 1\,,\ \ q \in \mathcal{P}_m^c\,,\ \ r \in \mathcal{P}_{n-m}^c\} \tag{5.3.4}$$

is attained for some $q \in \mathcal{P}_m^c$ and $r \in \mathcal{P}_{n-m}^c$. We proceed to show that there are extremal polynomials $q \in \mathcal{P}_m^c$ and $r \in \mathcal{P}_{n-m}^c$ such that $p := qr$ is the Chebyshev polynomial T_n of degree of n, and that the factors q and r are as advertised, that is,

$$p(x) = (qr)(x) = T_n(x) = \frac{1}{2}\prod_{k=1}^{n} 2\left(x - \cos\tfrac{(2k-1)\pi}{2n}\right)\,,$$

and the extremal factors, q and r, are given by

$$q(x) := \frac{1}{\sqrt{2}}\prod_{k=1}^{m} 2\left(x - \cos\tfrac{(2k-1)\pi}{2n}\right)$$

and

$$r(x) := \frac{1}{\sqrt{2}}\prod_{k=m+1}^{n} 2\left(x - \cos\tfrac{(2k-1)\pi}{2n}\right)\,,$$

respectively. Note that for the above q and r we have

$$\|q\|_{[-1,1]} = |q(-1)| = \frac{1}{\sqrt{2}}\,C_{n,m}$$

and

$$\|r\|_{[-1,1]} = |r(1)| = \frac{1}{\sqrt{2}}\,C_{n,n-m}\,.$$

First we show that there exist extremal polynomials $q \in \mathcal{P}_m^c$ and $r \in \mathcal{P}_{n-m}^c$ such that

$$|q(-1)| = \|q\|_{[-1,1]} \qquad \text{and} \qquad |r(1)| = \|r\|_{[-1,1]}\,. \tag{5.3.5}$$

To see this, choose $\alpha,\ \beta \in [-1,1]$ such that

$$|q(\alpha)| = \|q\|_{[-1,1]} \qquad \text{and} \qquad |r(\beta)| = \|r\|_{[-1,1]}\,,$$

where, considering $q(-z)$ and $r(-z)$ if necessary, we may assume that $\alpha \le \beta$. Note that $\alpha = \beta$ cannot happen, so $\alpha < \beta$. We have

$$\frac{\|q\|_{[\alpha,\beta]}\|r\|_{[\alpha,\beta]}}{\|qr\|_{[\alpha,\beta]}} \ge \frac{\|q\|_{[-1,1]}\|r\|_{[-1,1]}}{\|qr\|_{[-1,1]}}$$

since the numerators are equal and

$$\|qr\|_{[\alpha,\beta]} \le \|qr\|_{[-1,1]}\,.$$

Let $\widetilde{q} \in \mathcal{P}_m$ be defined by shifting q from $[\alpha,\beta]$ to $[-1,1]$ linearly so that $\alpha \to -1$. Let $\widetilde{r} \in \mathcal{P}_{n-m}^c$ be defined by shifting r from $[\alpha,\beta]$ to $[-1,1]$ linearly so that $\beta \to 1$. Then $\widetilde{q} \in \mathcal{P}_m^c$ and $\widetilde{r} \in \mathcal{P}_{n-m}^c$ are extremal polynomials for which (5.3.5) holds.

Lemma 5.3.3. *Suppose $q \in \mathcal{P}_m^c$ and $r \in \mathcal{P}_{n-m}^c$ are extremal polynomials for which (5.3.5) holds. Then there are extremal polynomials $\widetilde{q} \in \mathcal{P}_m^c$ and $\widetilde{r} \in \mathcal{P}_{n-m}^c$ having only real zeros for which (5.3.5) holds.*

Proof. Let $\widetilde{q}(z)$ be defined by replacing every factor $z-\alpha$ with nonreal α by $z-(|\alpha+1|-1)$ in the factorization of q. Let $\widetilde{r}(z)$ be defined by replacing every factor $z-\alpha$ with nonreal α by $z-(1-|\alpha-1|)$ in the factorization of r. Now it is elementary geometry to show that $\widetilde{q} \in \mathcal{P}_m^c$ and $\widetilde{r} \in \mathcal{P}_{n-m}^c$ are extremal polynomials for which (5.3.5) holds, and all the zeros of both $\widetilde{q}$ and $\widetilde{r}$ are real. □

Lemma 5.3.4. *Suppose $q \in \mathcal{P}_m^c$ and $r \in \mathcal{P}_{n-m}^c$ are extremal polynomials having only real zeros for which (5.3.5) holds. Then there are extremal polynomials $\widetilde{q} \in \mathcal{P}_m^c$ and $\widetilde{r} \in \mathcal{P}_{n-m}^c$ having all their zeros in $[-1,1]$ for which (5.3.5) holds.*

Proof. Let $\widetilde{q}(z)$ be defined by replacing every factor $z-\alpha$ by $z-1$ if $\alpha > 1$, and by 1 if $\alpha < -1$, in the factorization of q. Let $\widetilde{r}(z)$ be defined by replacing every factor $z-\alpha$ by $z+1$ if $\alpha < -1$, and by 1 if $\alpha > 1$. Now it is again elementary geometry to show that $\widetilde{q} \in \mathcal{P}_m^c$ and $\widetilde{q} \in \mathcal{P}_{n-m}^c$ are extremal polynomials having all their zeros in $[-1,1]$ for which (5.3.5) holds. □

So we now assume that $q \in \mathcal{P}_m^c$ and $r \in \mathcal{P}_{n-m}^c$ are extremal polynomials having all their zeros in $[-1,1]$ for which (5.3.5) holds. We may also assume that $\deg(q) = m$ and $\deg(r) = n-m$, otherwise we would study

$$\widetilde{q}(z) := z^{m-\deg(q)} q(z) \in \mathcal{P}_m^c$$

and

$$\widetilde{r}(z) := z^{n-m-\deg(r)} r(z) \in \mathcal{P}_{n-m}^c ,$$

which are also extremal polynomials having all their zeros in $[-1,1]$ for which (5.3.5) holds.

It is now clear that if q and r are extremal polynomials with the above properties, then the smallest zero of q is not less than the largest zero of r. Indeed, if there were numbers $-1 \le \alpha < \beta \le 1$ so that $q(\alpha) = r(\beta) = 0$, then the polynomials

$$\widetilde{q}(z) := q(z)\frac{z-\beta}{z-\alpha} \in \mathcal{P}_m^c$$

and

$$\widetilde{r}(z) := r(z)\frac{z-\alpha}{z-\beta} \in \mathcal{P}_{n-m}^c$$

would contradict the extremality of q and r since

$$\|\widetilde{q}\|_{[-1,1]} \geq |\widetilde{q}(-1)| > |q(-1)| = \|q\|_{[-1,1]}\,,$$

$$\|\widetilde{r}\|_{[-1,1]} \geq |\widetilde{r}(1)| > |r(1)| = \|r\|_{[-1,1]}\,,$$

and

$$\|\widetilde{q}\widetilde{r}\|_{[-1,1]} = \|qr\|_{[-1,1]}\,.$$

So now we have extremal polynomials $q \in \mathcal{P}_m^c$ and $r \in \mathcal{P}_{n-m}^c$ of the form

$$q(z) = \sqrt{a}\prod_{k=1}^{m}(z-\beta_k) \qquad \text{and} \qquad r(z) = \sqrt{a}\prod_{k=1}^{n-m}(z-\alpha_k)$$

satisfying

$$|q(-1)| = \|q\|_{[-1,1]} \qquad \text{and} \qquad |r(1)| = \|r\|_{[-1,1]}\,,$$

where

$$-1 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{n-m} \leq \beta_1 \leq \beta_2 \leq \cdots \leq \beta_m \leq 1$$

and the constant $a > 0$ is chosen so that for $p := qr$ we have $\|p\|_{[-1,1]} = 1$. Now we are ready to prove Theorem 5.3.1.

Proof of Theorem 5.3.1. We show three properties of $p = qr$:

(1) $|p(-1)| = 1$ and $\|p(1)| = 1$.

(2) $\|p(x)\|_{[\alpha_i,\alpha_{i+1}]} = 1, \quad i = 1,2,\ldots,n-m-1,$
$\|p(x)\|_{[\beta_i,\beta_{i+1}]} = 1, \quad i = 1,2,\ldots,m-1.$

(3) $\|p\|_{[\alpha_{n-m},\beta_1]} = 1.$

These three facts show that p is indeed the Chebyshev polynomial $\pm T_n$ defined by (2.1.1) since $\pm T_n$ are the only polynomials of degree at most n that equioscillate $n+1$ times on $[-1,1]$ with uniform norm 1.

To prove (1), assume to the contrary that $|p(-1)| < 1$. Then there is a $\delta < -1$ such that

$$\|p\|_{[\delta,1]} = \|p\|_{[-1,1]}\,.$$

Since $|q|$ is strictly decreasing on $[\delta,-1]$,

$$\|q\|_{[\delta,1]} \geq |q(\delta)| > |q(-1)| = \|q\|_{[-1,1]}$$

and, of course,

$$\|r\|_{[\delta,1]} \geq \|r\|_{[-1,1]}\,.$$

Let $\widetilde{q} \in \mathcal{P}_m^c$ and $\widetilde{r} \in \mathcal{P}_{n-m}^c$ be the polynomials q and r shifted linearly from $[\delta,1]$ to $[-1,1]$ so that $1 \mapsto 1$. By the previous observations, these $\widetilde{q}$

and $\widetilde{r}$ contradict the extremality of q and r, hence $|p(-1)| = 1$, and we are finished. A similar argument shows that $|p(1)| = 1$.

To prove (2) let

$$s(z) := (z - \alpha_i)(z - \alpha_{i+1}) .$$

Given $\epsilon > 0$, we can find $0 < \delta_1, \delta_2 < \epsilon$ such that

$$t(z) := (z - (\alpha_i - \delta_1))(z - (\alpha_{i+1} + \delta_2))$$

satisfies

$$t(1) = s(1) , \qquad \|t - s\|_{[-1,1]} < \epsilon ,$$

and

$$|t(x)| < |s(x)| , \qquad x \in [-1, \alpha_i - \delta_1] \cup [\alpha_{i+1} + \delta_2, 1) .$$

Suppose $\|p\|_{[\alpha_i, \alpha_{i+1}]} < 1$. Let

$$\widehat{q}(z) := q(z) \in \mathcal{P}_m^c$$

and

$$\widehat{r}(z) := r(z)\frac{t(z)}{s(z)} \in \mathcal{P}_{n-m}^c .$$

If $\epsilon > 0$ is sufficiently small, then

$$\|\widehat{q}\widehat{r}\|_{[-1,1]} \le \|qr\|_{[-1,1]} \qquad \text{and} \qquad |(\widehat{q}\widehat{r})(-1)| < 1 .$$

The second inequality guarantees that there exists a $\delta < -1$ such that

$$\|\widehat{q}\widehat{r}\|_{[\delta,1]} = \|\widehat{q}\widehat{r}\|_{[-1,1]} .$$

Since $|\widehat{q}|$ is (strictly) decreasing on $(-\infty, 1]$,

$$\|\widehat{q}\|_{\delta,1]} \ge |\widehat{q}(\delta)| > |\widehat{q}(-1)| = |q(-1)| = \|q\|_{[-1,1]} .$$

Also

$$\|\widehat{r}\|_{[\delta,1]} \ge \|\widehat{r}\|_{[-1,1]} \ge |\widehat{r}(1)| = |r(1)| = \|r\|_{[-1,1]} .$$

Now let $\widetilde{q} \in \mathcal{P}_m^c$ and $\widetilde{r} \in \mathcal{P}_{n-m}^c$ be the polynomials $\widehat{p}$ and $\widehat{q}$ shifted linearly from $[\delta, 1]$ to $[-1, 1]$ so that $-1 \to -1$. By the previous observations, these $\widetilde{q}$ and $\widetilde{r}$ contradict the extremality of q and r. Hence $\|p\|_{[\alpha_i, \alpha_{i+1}]} = 1$, and the proof is finished. The proof of $\|p\|_{[\beta_i, \beta_{i+1}]} = 1$ is identical.

To prove (3) assume that $\|p\|_{[\alpha_{n-m}, \beta_1]} < 1$. Let

$$\widetilde{q}(z) := \sqrt{a}(z - (\beta_1 + \epsilon)) \prod_{k=2}^{m} (z - \beta_k)$$

and

$$\widetilde{r}(z) := \sqrt{a}(z - (\alpha_{n-m} - \epsilon)) \prod_{k=1}^{n-m-1} (z - \alpha_k)\,.$$

If $\epsilon > 0$ is sufficiently small, then

$$\|\widetilde{q}\|_{[-1,1]} \geq |\widetilde{q}(-1)| > |q(-1)| = \|q\|_{[-1,1]}\,,$$

$$\|\widetilde{r}\|_{[-1,1]} \geq |\widetilde{r}(1)| > |r(1)| = \|r\|_{[-1,1]}\,,$$

and

$$\|\widetilde{q}\widetilde{r}\|_{[-1,1]} \leq \|qr\|_{[-1,1]}\,,$$

which contradicts the extremality of q and r. Hence $\|p\|_{[\alpha_{n-m},\beta_1]} = 1$, indeed. □

Theorem 5.3.5. *Suppose $p \in \mathcal{P}_n^c$ is monic and $q \in \mathcal{P}_m^c$ is a monic factor of p. Then*

$$|q(-\beta)| \leq \beta^{m-n} 2^{n-1} \prod_{k=1}^{m} \left(1 + \cos\tfrac{(2k-1)\pi}{2n}\right) \|p\|_{[-\beta,\beta]}$$

for every $\beta > 0$. Equality holds if p is the Chebyshev polynomial $T_{n,\beta}$ of degree n on $[-\beta,\beta]$ (normalized to be monic), and the monic factor $q \in \mathcal{P}_m^c$ is chosen so that q vanishes at the m zeros of $T_{n,\beta}$ closest to β. Note that $T_{n,\beta}(x) = \beta^n T_n(x/\beta)$, where T_n is defined by (2.1.1).

The proof of Theorem 5.3.5 is outlined in E.1.

Corollary 5.3.6. *Suppose $p \in \mathcal{P}_n^c$ is monic and $q \in \mathcal{P}_m^c$ is a monic factor of p. Then*

$$|q(-2)| \leq 2^{m-1} \prod_{k=1}^{m} \left(1 + \cos\tfrac{(2k-1)\pi}{2n}\right) \|p\|_{[-2,2]} = \frac{1}{2} C_{n,m} \|p\|_{[-2,2]}$$

and the inequality is sharp for all $m \leq n$. Here, for all $m \leq n$,

$$C_{n,m}^{1/n} \leq C_{n,\lfloor 2n/3 \rfloor}^{1/n} \sim 1.9081\ldots\,,$$

and hence

$$\left(\frac{|q(-2)|}{\|p\|_{[-2,2]}}\right)^{1/n} \lesssim 1.9081\ldots\,.$$

Proof. Take $\beta = 2$ in Theorem 5.3.5. Note that

$$2^m \prod_{k=1}^{m} \left(1 + \cos \tfrac{(2k-1)\pi}{2n}\right) \le 2^{\gamma(n)} \prod_{k=1}^{\gamma(n)} \left(1 + \cos \tfrac{(2k-1)\pi}{2n}\right),$$

where $\gamma(n)+1 := \lfloor \frac{2}{3}n + \frac{3}{2} \rfloor$ is the smallest $k \in \mathbb{N}$ for which $\cos \frac{(2k-1)\pi}{2n} \le -\frac{1}{2}$. So, for every $m = 0, 1, \ldots, n$,

$$C_{n,m} \le C_{n,\gamma(n)}$$

and by (5.3.3)

$$C_{n,m}^{1/n} \lesssim C_{n,\lfloor 2n/3 \rfloor} \lesssim 1.9081\ldots.$$

□

Theorem 5.3.7. *Suppose $p \in \mathcal{P}_n^c$ is monic and has a monic factor of the form qr, where $q \in \mathcal{P}_{m_1}^c$ and $r \in \mathcal{P}_{m_2}^c$. Then*

$$|q(-\beta)||r(\beta)| \le \beta^{m_1+m_2-n} 2^{n-1} \times \prod_{k=1}^{m_1} \left(1 + \cos \tfrac{(2k-1)\pi}{2n}\right) \prod_{k=1}^{m_2} \left(1 + \cos \tfrac{(2k-1)\pi}{2n}\right) \|p\|_{[-\beta,\beta]}.$$

Equality holds if p is the Chebyshev polynomial $T_{n,\beta}$ of degree n on $[-\beta, \beta]$ normalized to be monic, and the factors $q \in \mathcal{P}_{m_1}^c$ and $r \in \mathcal{P}_{m_2}^c$ are chosen so that q vanishes at the m_1 zeros of $T_{n,\beta}$ closest to β, while r vanishes at the m_2 zeros of $T_{n,\beta}$ closest to $-\beta$.

The proof of Theorem 5.3.7 is analogous to the proof of Theorem 5.3.1 and is left as an exercise (see E.2).

Theorem 5.3.8. *Suppose $p \in \mathcal{P}_n^c$ is monic and has a monic factor of the form $q_1 q_2 \cdots q_j$, where $q_i \in \mathcal{P}_{m_i}^c$ and $m := m_1 + m_2 + \cdots + m_j \le n$. Then*

$$\prod_{i=1}^{j} \|q_i\|_{[-\beta,\beta]} \le \beta^{m-n} 2^{n-1} \times \prod_{k=1}^{\lfloor m/2 \rfloor} \left(1 + \cos \tfrac{(2k-1)\pi}{2n}\right) \prod_{k=1}^{\lceil m/2 \rceil} \left(1 + \cos \tfrac{(2k-1)\pi}{2n}\right) \|p\|_{[-\beta,\beta]}$$

for every $\beta > 0$. Equality holds if p is the Chebyshev polynomial $T_{n,\beta}$ of degree n on $[-\beta, \beta]$ normalized to be monic, $j = 2$, and the factors $q_1 \in \mathcal{P}_{\lfloor m/2 \rfloor}^c$ and $q_2 \in \mathcal{P}_{\lceil m/2 \rceil}^c$ are chosen so that q_1 vanishes at the $\lfloor m/2 \rfloor$ zeros of $T_{n,\beta}$ closest to β, while q_2 vanishes at the $\lceil m/2 \rceil$ zeros of $T_{n,\beta}$ closest to $-\beta$.

Proof. For each q_i, write

$$q_i = r_i s_i \,,$$

where r_i is the monic factor of q_i composed of the roots of q_i with negative real part, while s_i is the monic factor of q_i composed of the roots of q_i with nonnegative real part. So

$$\|r_i\|_{[-\beta,\beta]} = |r_i(\beta)| \qquad \text{and} \qquad \|s_i\|_{[-\beta,\beta]} = |s_i(-\beta)| \,.$$

Thus

$$\prod_{i=1}^{j} \|q_i\|_{[-\beta,\beta]} \le \left|\prod_{i=1}^{j} r_i(\beta)\right| \cdot \left|\prod_{i=1}^{j} s_i(-\beta)\right| .$$

We now apply Theorem 5.3.7 to the two factors $\prod_{i=1}^{j} r_i$ and $\prod_{i=1}^{j} s_i$ to finish the proof. □

As before, let $D := \{z \in \mathbb{C} : |z| < 1\}$. We now derive inequalities on the disk from those on the interval. A continuous function on $\overline{D}$ has the same uniform norm on both D and $\overline{D}$ and it is notationally convenient to state the remaining theorems over D. Suppose $t \in \mathcal{P}_n^c$, $s \in \mathcal{P}_m^c$, and $v \in \mathcal{P}_{n-m}^c$ are monic, and $t = sv$. By the maximum principle, t, s, and v achieve their maximum on $\overline{D}$ somewhere on ∂D. Now consider

$$p(x) := t(z)t(z^{-1}) \,,$$

$$q(x) := s(z)s(z^{-1}) \,, \qquad \text{and} \qquad r(x) := v(z)v(z^{-1})$$

with

$$x := z + z^{-1} \,.$$

The effect of this transformation on linear factors is

$$(z-\alpha)(z^{-1}-\alpha) = -\alpha x + 1 + \alpha^2 \,,$$

so $p \in \mathcal{P}_n^c$, $q \in \mathcal{P}_m^c$, $r \in \mathcal{P}_{n-m}^c$, and $p = qr$. Also

$$\|p\|_{[-2,2]} \le \|t\|_D^2 \,.$$

If $t(0) \neq 0$, then the modulus of the leading coefficient of p is $|t(0)|$, while the modulus of the leading coefficient of q is $|s(0)|$, and the modulus of the leading coefficient of r is $|v(0)|$.

From these transformations and the interval inequalities we can deduce the next three theorems.

Theorem 5.3.9. *Let $t \in \mathcal{P}_n^c$ be monic and suppose $t = sv$, where $s \in \mathcal{P}_m^c$ and $v \in \mathcal{P}_{n-m}^c$. Then*

$$|v(0)|^{1/2}\|s\|_D \le \left(\tfrac{1}{2}C_{n,m}\right)^{1/2}\|t\|_D\,,$$

where $C_{n,m}$ is the same as in Theorem 5.3.1. This bound is attained when $m \le n$ are even, $t(z) = z^n + 1$, and $s \in \mathcal{P}_m^c$ vanishes at m adjacent zeros of t on the unit circle.

Proof. We may assume, by performing an initial rotation if necessary, that

$$\|s\|_D = |s(-1)|\,.$$

So from Corollary 5.3.6 we deduce that

$$\begin{aligned}(5.3.6)\qquad \|s\|_D^2 = |s(-1)|^2 &= |q(-2)| \\ &\le |s(0)/t(0)|2^{m-1}\prod_{k=1}^{m}\left(1+\cos\tfrac{(2k-1)\pi}{2n}\right)\|p\|_{[-2,2]} \\ &\le |s(0)/t(0)|2^{m-1}\prod_{k=1}^{m}\left(1+\cos\tfrac{(2k-1)\pi}{2n}\right)\|t\|_D^2\,,\end{aligned}$$

where $s(0)/t(0) = 1/v(0)$. □

Theorem 5.3.10. *Suppose $t = sv$, where $s \in \mathcal{P}_m$ and $r \in \mathcal{P}_{n-m}$. Then*

$$\|s\|_D\|v\|_D \le \left(\tfrac{1}{2}C_{n,m}C_{n,n-m}\right)^{1/2}\|t\|_D\,,$$

where $C_{n,m}$ is the same as in Theorem 5.3.1 and

$$(C_{n,m}C_{n,n-m})^{1/(2n)} \le C_{n,\lfloor n/2\rfloor}^{1/n} \sim 1.7916\ldots\,.$$

This bound is attained when $m \le n$ are even, $t(z) = z^n + 1$, and $s \in \mathcal{P}_m^c$ vanishes at the m zeros of t closest to 1 and $v \in \mathcal{P}_{n-m}$ vanishes at the $n - m$ zeros of t closest to -1.

Proof. From Theorem 5.3.1 we can deduce that if a, $b \in \partial D$, then

$$\begin{aligned}|s(a)|^2|v(b)|^2 &= |s(a)s(a^{-1})||v(b)v(b^{-1})| \\ &= |q(a+a^{-1})||r(b+b^{-1})| \\ &\le \tfrac{1}{2}C_{n,m}C_{n,n-m}\|p\|_{[a+a^{-1},b+b^{-1}]} \\ &\le \tfrac{1}{2}C_{n,m}C_{n,n-m}\|p\|_{[-2,2]} \\ &\le \tfrac{1}{2}C_{n,m}C_{n,n-m}\|t\|_D^2\,,\end{aligned}$$

where, without loss of generality, we may assume that $a + a^{-1} \le b + b^{-1}$. The result now follows on choosing a and b to be points on ∂D where s and v, respectively, achieve their uniform norm on D. □

In the multifactor case we have the following theorem:

Theorem 5.3.11. *Suppose $t \in \mathcal{P}_n$ is of the form $t = vs_1s_2\cdots s_j$, where $s_i \in \mathcal{P}_{m_i}$ and $v \in \mathcal{P}_{n-m}$ with $m := m_1 + m_2 + \cdots + m_j \le n$. Then*

$$|v(0)|^{1/2}\prod_{i=1}^{j}\|s_i\|_D \le 2^{(m-1)/2} \times \left(\prod_{k=1}^{\lfloor m/2\rfloor}\left(1+\cos\tfrac{(2k-1)\pi}{2n}\right)\prod_{k=1}^{\lceil m/2\rceil}\left(1+\cos\tfrac{(2k-1)\pi}{2n}\right)\right)^{1/2}\|t\|_D\,.$$

Equality holds if $t(z) = z^n + 1$, $j = 2$, $m_1 = m_2 := m/2$ and n are even, and the factors $q_1 \in \mathcal{P}_{m/2}$ and $q_2 \in \mathcal{P}_{m/2}$ are chosen so that q_1 vanishes at the $m/2$ zeros of t closest to 1 and q_2 vanishes at the $m/2$ zeros of t closest to -1.

Proof. This follows from Theorem 5.3.8 in exactly the same way as Theorem 5.3.10 follows from Theorem 5.3.1. □

Comments, Exercises, and Examples.

The first result of this section is due to Kneser [34] and in part to Aumann [33]. The proof follows Borwein [94], as does most of the section. There are many variations and generalizations. See Boyd [92], [93a], [93b], [94a], and [94b]; Beauzamy and Enflo [85]; Beauzamy, Bombieri, Enflo, and Montgomery [90]; Gel'fond [60]; Glesser [90]; Granville [90]; Mahler [60], [62], and [64]; and Mignotte [82]. Some of these are presented in the exercises. In particular, E.6 reproduces a very pretty proof of Boyd [92] that

$$\|q\|_D\|r\|_D \le (1.7916\ldots)^n\|p\|_D\,,$$

where $p \in \mathcal{P}_n^c$ and $p = qr$ with some $q \in \mathcal{P}_m^c$ and $r \in \mathcal{P}_{n-m}^c$. (Note that we have not assumed real coefficients unlike in Theorem 5.3.10, and we have $\le$ instead of $\lesssim$.)

E.1 Proof of Theorem 5.3.5.

Outline. Let $m < n$ and $\beta > 0$ be fixed. The value

$$\sup\left\{\frac{|q(-\beta)|}{\|p\|_{[-\beta,\beta]}}:\ q \in \mathcal{P}_m^c \text{ and } p \in \mathcal{P}_n^c \text{ are monic and } q \text{ divides } p\right\}$$

is attained for some monic $q \in \mathcal{P}_m^c$ and $p \in \mathcal{P}_n^c$. We can now argue, exactly as in the proof of Lemma 5.3.3, that there are extremal polynomials $p \in \mathcal{P}_n^c$ and $q \in \mathcal{P}_m^c$ such that all the zeros of p are real and lie in $[-\beta,\infty)$. Arguing as in Lemma 5.3.4 gives that p has all its roots in $[-\beta,\beta]$. Thus q must be composed of the m roots of p closest to β.

The argument of the proof of Theorem 5.3.1 now applies essentially verbatim and proves that an extremal $p \in \mathcal{P}_n^c$ can be chosen to be the Chebyshev polynomial on $[-\beta, \beta]$ normalized to be monic. Thus on $[-1, 1]$

$$p(x) = \prod_{k=1}^{n} \left(x - \cos \tfrac{(2k-1)\pi}{2n} \right) \qquad \text{and} \qquad q(x) = \prod_{k=1}^{m} \left(x - \cos \tfrac{(2k-1)\pi}{2n} \right)$$

from which the result follows (on considering $\beta^n p(x/\beta)$ and $\beta^m q(x/\beta)$ on $[-\beta, \beta]$). □

E.2 Proof of Theorem 5.3.7.

Hint: Proceed as in the proof of Theorem 5.3.1 (or E.1). □

E.3 A Version of Theorem 5.3.10 for Complex Polynomials. Suppose $t = sv$, where $s \in \mathcal{P}_m^c$ and $v \in \mathcal{P}_{n-m}^c$. Then

$$|s(-1)||v(1)| \le \left(\tfrac{1}{2} C_{n,m} C_{n,n-m}\right)^{1/2} \|t\|_D$$

and if t is monic

$$|t(0)|^{1/2} \|s\|_D \|v\|_D \le \tfrac{1}{2} (C_{n,m} C_{n,n-m})^{1/2} \|t\|_D^2 .$$

Hint: The first inequality follows as in the proof of Theorem 5.3.10 with $a := -1$ and $b := 1$. The second part is immediate from Theorem 5.3.9. □

E.4 Mahler's Measure. Let $F : \mathbb{C}^k \to \mathbb{C}$, and let the *Mahler measure* of F be defined by

$$M_k(F) := \exp \left\{ \int_0^1 \cdots \int_0^1 \log |F(e^{2\pi i t_1}, \ldots, e^{2\pi i t_k})| \, dt_1 \cdots dt_k \right\}$$

if the integral exist.

a] Show that if

$$p(z) = c \prod_{i=1}^{n} (z - \alpha_i) , \qquad c \, , \, \alpha_i \in \mathbb{C} ,$$

then

$$M_1(p) = |c| \prod_{i=1}^{n} \max\{1, |\alpha_i|\} .$$

Hint: Use Jensen's formula (see E.10 c] of Section 4.2). □

b] Show that if $F := F(z_1, \ldots, z_k)$ and $G := G(z_1, \ldots, z_k)$, then

$$M_k(FG) = M_k(F) M_k(G) .$$

c] Show that

$$\begin{aligned} M_1(ax+b) &= \max(|a|,|b|)\,, \\ M_2(1+x+y) &= M_1(\max\{1,|1+x|\})\,, \\ M_2(1+x+y-xy) &= M_1(\max\{|1-x|,|1+x|\})\,. \end{aligned}$$

d] One can numerically check that

$$M_2(1+x+y) = 1.381356\ldots$$

and

$$M_2(1+x+y-xy) = 1.791622\ldots\,.$$

E.5 The Norm of a Factor of a $p \in \mathcal{P}_n^c$ on the Unit Disk. Suppose $p \in \mathcal{P}_n^c$ is monic and has a monic factor $q \in \mathcal{P}_m^c$. Then

$$\|q\|_D \le \beta^n \|p\|_D\,,$$

where $\beta := M_2(1+x+y) = 1.3813\ldots$.

Outline. Let

$$p(x) := \prod_{i=1}^{n}(x-\alpha_i) \qquad \text{and} \qquad q(x) := \prod_{i=1}^{m}(x-\alpha_i)\,, \qquad \alpha_i \in \mathbb{C}.$$

Suppose $\|q\|_D = |q(u)|$, where $u \in \partial D$. Then

$$\begin{aligned} \|q\|_D = |q(u)| &= \prod_{i=1}^{m}|u-\alpha_i| \le \prod_{i=1}^{n}\max\{|u-\alpha_i|,1\} \\ &= M_1(p(x+u)) \le M_1(\max\{1,|x+u|^n\}\|p\|_D)\,, \end{aligned}$$

where the last equality holds by E.4 a], and the last inequality follows because

$$|p(z)| \le \max\{1,|z|^n\}\cdot\|p\|_D \tag{5.3.7}$$

holds for every $p \in \mathcal{P}_n^c$ and $z \in \mathbb{C}$ by E.18 a] of Section 5.1. Now using E.4 b] and

$$M_1(\max\{1,|x+u|\}) = M_1(\max\{1,|x+1|\})\,,$$

we obtain

$$\begin{aligned} \|q\|_D &\le M_1(\max\{1,|x+u|^n\})\,\|p\|_D \\ &= M_1((\max\{1,|x+u|\})^n)\,\|p\|_D \\ &= M_1((\max\{1,|x+1|\})^n)\,\|p\|_D \\ &= (M_1(\max\{1,|x+1|\}))^n\,\|p\|_D \\ &= (M_2(1+x+y))^n\|p\|_D = \beta^n\|p\|_D\,. \end{aligned}$$

□

E.6 Another Inequality for the Factors of a $p \in \mathcal{P}_n^c$ on the Unit Disk. Let $p = qr$, where $q \in \mathcal{P}_m^c$ and $r \in \mathcal{P}_{n-m}^c$. Then

$$\|q\|_D \|r\|_D \le \delta^n \|p\|_D$$

where $\delta := M_2(1 + x + y - xy) = 1.7916\ldots$.

Outline. Without loss of generality we may assume that q and r are monic. Let

$$q(x) := \prod_{i=1}^{m}(x - \alpha_i) \qquad \text{and} \qquad r(x) := \prod_{i=m+1}^{n}(x - \alpha_i)\,, \qquad \alpha_i \in \mathbb{C}.$$

Choose $u \in \partial D$ and $v \in \partial D$ such that $|q(u)| = \|q\|_D$ and $|r(v)| = \|r\|_D$. Then, using E.4 b] and c], we obtain

$$\begin{aligned}
\|q\|_D \|r\|_D = |q(u)||r(v)| &= \prod_{i=1}^{m} |u - \alpha_i| \prod_{i=m+1}^{n} |v - \alpha_i| \\
&\le \prod_{i=1}^{n} \max\{|u - \alpha_i|, |v - \alpha_i|\} \\
&= \prod_{i=1}^{n} |u - \alpha_i| \max\left\{1, \left|\frac{v - \alpha_i}{u - \alpha_i}\right|\right\} \\
&= M_1\left((x-1)^n p\left(\frac{ux - v}{x - 1}\right)\right)\,.
\end{aligned}$$

Now, by (5.3.7),

$$\left|(x-1)^n p\left(\frac{xu - v}{x - 1}\right)\right| \le \left(\max\left\{|x - 1|, \left|x - \frac{v}{u}\right|\right\}\right)^n \|p\|_D\,,$$

hence

$$\begin{aligned}
\|q\|_D \|r\|_D &\le M_1((\max\{|x-1|, |x - v/u|\})^n)\|p\|_D \\
&= (M_1(\max\{|x-1|, |x - v/u|\}))^n \|p\|_D \\
&\le (M_1(\max\{|1-x|, |1+x|\}))^n \|p\|_D \\
&= (M_2(1 + x + y - xy))^n \|p\|_D \\
&= (1.7916\ldots)^n \|p\|_D\,.
\end{aligned}$$

□

E.7 Bombieri's Norm. For $Q(z) := \sum_{k=0}^{n} a_k z^k$ the *Bombieri p norm* is defined by

$$[Q]_p := \left(\sum_{k=0}^{n} \binom{n}{k}^{1-p} |a_k|^p \right)^{1/p}.$$

Note that this is a norm on $\mathcal{P}_n^c$ for every $p \in [1, \infty)$, but it varies with varying n. The following remarkable inequality holds (see Beauzamy et al. [90]). If $Q = RS$ with $Q \in \mathcal{P}_n^c$, $R \in \mathcal{P}_m^c$, and $S \in \mathcal{P}_{n-m}^c$, then

$$[R]_2 [S]_2 \leq \binom{n}{m}^{1/2} [Q]_2$$

and this is sharp.

One feature of this inequality is that it extends naturally to the multivariate case. See Beauzamy, Enflo, and Wang [94] and Reznick [93] for further discussion.