6

Inequalities in Müntz Spaces

Overview

Versions of Markov's inequality for Müntz spaces, both in $C[a,b]$ and $L_p[0,1]$, are given in the first section of this chapter. Bernstein- and Nikolskii-type inequalities are treated in the exercises, as are various other inequalities for Müntz polynomials and exponential sums. The second section provides inequalities, including most significantly a Remez-type inequality, for nondense Müntz spaces.

6.1 Inequalities in Müntz Spaces

We first present a simplified version of Newman's beautiful proof of an essentially sharp Markov-type inequality for Müntz polynomials. This simplification allows us to prove the L_p analogs of Newman's inequality. Then, using the results of Section 3.4 on orthonormal Müntz-Legendre polynomials, we prove an L_2 version of Newman's inequality for Müntz polynomials with complex exponents. Some Nikolskii-type inequalities for Müntz polynomials are studied. The exercises treat a number of other inequalities for Müntz polynomials and exponential sums. Throughout this section we use the notation introduced in Section 3.4. Unless stated otherwise, the span always denotes the linear span over $\mathbb{R}$.

Theorem 6.1.1 (Newman's Inequality). *Let $\Lambda := (\lambda_i)_{i=0}^{\infty}$ be a sequence of distinct nonnegative real numbers. Then*

$$\frac{2}{3}\sum_{j=0}^{n}\lambda_j \le \sup_{0\neq p\in M_n(\Lambda)}\frac{\|xp'(x)\|_{[0,1]}}{\|p\|_{[0,1]}} \le 9\sum_{j=0}^{n}\lambda_j$$

for every $n\in\mathbb{N}$, where $M_n(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}$.

Proof. It is equivalent to prove that

$$\frac{2}{3}\sum_{j=0}^{n}\lambda_j \le \sup_{0\neq P\in E_n(\Lambda)}\frac{\|P'\|_{[0,\infty)}}{\|P\|_{[0,\infty)}} \le 9\sum_{j=0}^{n}\lambda_j\,, \tag{6.1.1}$$

where $E_n(\Lambda) := \operatorname{span}\{e^{-\lambda_0 t}, e^{-\lambda_1 t}, \dots, e^{-\lambda_n t}\}$. Without loss of generality we may assume that $\lambda_0 := 0$. By a change of scale we may also assume that $\sum_{j=0}^{n}\lambda_j = 1$. We begin with the first inequality. We define the Blaschke product

$$B(z) := \prod_{j=1}^{n}\frac{z-\lambda_j}{z+\lambda_j}$$

and the function

$$T(t) := \frac{1}{2\pi i}\int_{\Gamma}\frac{e^{-zt}}{B(z)}\,dz\,, \qquad \Gamma := \{z\in\mathbb{C} : |z-1| = 1\}\,. \tag{6.1.2}$$

By the residue theorem

$$T(t) := \sum_{j=1}^{n}(B'(\lambda_j))^{-1}e^{-\lambda_j t}\,,$$

and hence $T\in E_n(\Lambda)$. We claim that

$$|B(z)| \ge \frac{1}{3}\,, \qquad z\in\Gamma\,. \tag{6.1.3}$$

Indeed, it is easy to see that $0\le\lambda_j\le 1$ implies

$$\left|\frac{z-\lambda_j}{z+\lambda_j}\right| \ge \frac{2-\lambda_j}{2+\lambda_j} = \frac{1-\frac{1}{2}\lambda_j}{1+\frac{1}{2}\lambda_j}\,, \qquad z\in\Gamma\,.$$

So, for $z\in\Gamma$,

$$|B(z)| \ge \prod_{j=1}^{n}\frac{1-\frac{1}{2}\lambda_j}{1+\frac{1}{2}\lambda_j} \ge \frac{1-\frac{1}{2}\sum_{j=1}^{n}\lambda_j}{1+\frac{1}{2}\sum_{j=1}^{n}\lambda_j} = \frac{1-\frac{1}{2}}{1+\frac{1}{2}} = \frac{1}{3}\,.$$

Here the inequality

$$\frac{1-x}{1+x}\,\frac{1-y}{1+y} = \frac{1-(x+y)}{1+(x+y)} + \frac{2xy(x+y)}{(1+x)(1+y)(1+(x+y))}$$
$$\geq \frac{1-(x+y)}{1+(x+y)}\,, \qquad x,y \geq 0$$

is used. From (6.1.2) and (6.1.3) we can deduce that

$$|T(t)| \leq \frac{1}{2\pi}\int_\Gamma \left|\frac{e^{-zt}}{B(z)}\right| |dz| \leq \frac{1}{2\pi} 3(2\pi) = 3\,, \quad t \geq 0\,. \tag{6.1.4}$$

Also

$$T'(t) = \frac{1}{2\pi i}\int_\Gamma \frac{-ze^{-zt}}{B(z)}\,dz$$

and

$$T'(0) = -\frac{1}{2\pi i}\int_\Gamma \frac{z}{B(z)}\,dz = -\frac{1}{2\pi i}\int_{|z|=1} \frac{z}{B(z)}\,dz\,. \tag{6.1.5}$$

Now, for $|z| > \max_{1\leq j\leq n}\lambda_j$, we have the Laurent series expansion

$$\begin{aligned}\frac{z}{B(z)} &= z\prod_{j=1}^n \frac{1+\lambda_j/z}{1-\lambda_j/z} = z\prod_{j=1}^n \left(1 + 2\sum_{k=1}^\infty \left(\frac{\lambda_j}{z}\right)^k\right)\\ &= z\left(1 + 2\left(\sum_{j=1}^n \lambda_j\right) z^{-1} + 2\left(\sum_{j=1}^n \lambda_j\right)^2 z^{-2} + \cdots\right)\\ &= z + 2 + 2z^{-1} + \cdots\,,\end{aligned} \tag{6.1.6}$$

which, together with (6.1.5), yields that $T'(0) = -2$. Hence, by (6.1.4),

$$\frac{|T'(0)|}{\|T\|_{[0,\infty)}} \geq \frac{2}{3} = \frac{2}{3}\sum_{j=1}^n \lambda_j\,,$$

so the lower bound of the theorem is proved.

To prove the upper bound in (6.1.1), first we show that if

$$U(t) := \frac{1}{2\pi i}\int_\Gamma \frac{e^{-zt}}{(1-z)B(z)}\,dz\,, \qquad \Gamma := \{z \in \mathbb{C} : |z-1| = 1\}\,,$$

then

$$\int_0^\infty |U''(t)|\,dt \leq 6\,. \tag{6.1.7}$$

Indeed, observe that if $z = 1 + e^{i\theta}$, then $|z|^2 = 2 + 2\cos\theta$, so (6.1.3) and Fubini's theorem yield that

$$\begin{aligned}\int_0^\infty |U''(t)|\,dt &= \int_0^\infty \frac{1}{2\pi}\left|\int_\Gamma \frac{z^2 e^{-zt}}{(1-z)B(z)}\,dz\right| dt \\ &\le \frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} \frac{|z|^2|e^{-zt}|}{|B(z)|}\,d\theta\,dt \\ &\le \frac{3}{2\pi}\int_0^\infty \int_0^{2\pi} (2+2\cos\theta)e^{-(1+\cos\theta)t}\,d\theta\,dt \\ &= \frac{3}{2\pi}\int_0^{2\pi} (2+2\cos\theta)\,\frac{1}{1+\cos\theta}\,d\theta = 6\,.\end{aligned}$$

Now we show that

$$\int_0^\infty e^{-\lambda_j t}U''(t)\,dt = \lambda_j - 3\,. \tag{6.1.8}$$

To see this we write the left-hand side as

$$\begin{aligned}\int_0^\infty e^{-\lambda_j t}U''(t)\,dt &= \int_0^\infty e^{-\lambda_j t}\frac{1}{2\pi i}\int_\Gamma \frac{z^2 e^{-zt}}{(1-z)B(z)}\,dz\,dt \\ &= \frac{1}{2\pi i}\int_0^\infty \int_\Gamma \frac{z^2 e^{-(z+\lambda_j)t}}{(1-z)B(z)}\,dz\,dt = \frac{1}{2\pi i}\int_\Gamma \frac{z^2}{(z+\lambda_j)(1-z)B(z)}\,dz \\ &= \frac{1}{2\pi i}\int_{|z|=2} \frac{z}{z+\lambda_j}\,\frac{z}{1-z}\,\frac{1}{B(z)}\,dz\,,\end{aligned}$$

where in the third equality Fubini's theorem is used again. Here, for $|z| > 1$, we have the Laurent series expansions

$$\begin{aligned}\frac{z}{z+\lambda_j} &= 1 - \lambda_j z^{-1} + \lambda_j^2 z^{-2} + \cdots\,, \\ \frac{z}{1-z} &= -1 - z^{-1} - z^{-2} - \cdots\,,\end{aligned}$$

and, as in (6.1.6),

$$\frac{1}{B(z)} = 1 + 2z^{-1} + 2z^{-2} + \cdots\,.$$

Now (6.1.8) follows from the residue theorem (see, for example, Ash [71]). Let $P \in E_n(\Lambda)$ be of the form

$$P(t) = \sum_{j=0}^{n} c_j e^{-\lambda_j t}\,, \qquad c_j \in \mathbb{R}\,.$$

Then

$$\begin{aligned}\int_0^\infty P(t+a)U''(t)\,dt &= \int_0^\infty \sum_{j=0}^n c_j e^{-\lambda_j a} e^{-\lambda_j t} U''(t)\,dt \\ &= \sum_{j=0}^n c_j e^{-\lambda_j a} \int_0^\infty e^{-\lambda_j t} U''(t)\,dt = \sum_{j=0}^n c_j(\lambda_j - 3)e^{-\lambda_j a} \\ &= -P'(a) - 3P(a)\end{aligned}$$

and so

$$|P'(a)| \le 3|P(a)| + \int_0^\infty |P(t+a)\,U''(t)|\,dt\,. \tag{6.1.9}$$

Combining this with (6.1.7) gives

$$\|P'\|_{[0,\infty)} \le 3\,\|P\|_{[0,\infty)} + 6\,\|P\|_{[0,\infty)} = 9\,\|P\|_{[0,\infty)}\,,$$

and the theorem is proved. □

The next theorem establishes an L_p extension of Newman's inequality.

Theorem 6.1.2 (Newman's Inequality in L_p). *Let $p \in [1,\infty)$. If $\Lambda := (\lambda_i)_{i=0}^\infty$ is a sequence of distinct real numbers greater than $-1/p$, then*

$$\|xP'(x)\|_{L_p[0,1]} \le \left(\frac{1}{p} + 12\sum_{j=0}^n \left(\lambda_j + \frac{1}{p}\right)\right) \|P\|_{L_p[0,1]}$$

for every $P \in M_n(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}$.

If $\Gamma := (\gamma_i)_{i=0}^\infty$ is a sequence of distinct positive real numbers, then

$$\|P'\|_{L_p[0,\infty)} \le 12\left(\sum_{j=0}^n \gamma_j\right) \|P\|_{L_p[0,\infty)}$$

for every $P \in E_n(\Gamma) := \operatorname{span}\{e^{-\gamma_0 t}, e^{-\gamma_1 t}, \dots, e^{-\gamma_n t}\}$.

Proof. First we show that the first statement of the theorem follows from the second. Indeed, if $(\lambda_i)_{i=0}^\infty$ is a sequence of distinct real numbers greater than $-1/p$ and $\gamma_i := \lambda_i + \frac{1}{p}$ for each i, then $(\gamma_i)_{i=1}^\infty$ is a sequence of distinct positive real numbers. Let $Q \in M_n(\Lambda)$. Applying the second inequality with

$$P(t) := Q(e^{-t})e^{-t/p} \in E_n(\Gamma)$$

and using the substitution $x = e^{-t}$, we obtain

$$\left(\int_0^1 \left| x\left(x^{1/p}Q(x)\right)'\right|^p x^{-1}\,dx\right)^{1/p} \le 12\left(\sum_{j=0}^n \left(\lambda_j+\frac{1}{p}\right)\right)\|Q\|_{L_p[0,1]}.$$

Now the product rule of differentiation and Minkowski's inequality yield

$$\|xQ'(x)\|_{L_p[0,1]} \le \left(\frac{1}{p}+12\sum_{j=0}^n \left(\lambda_j+\frac{1}{p}\right)\right)\|Q\|_{L_p[0,1]},$$

which is the first statement of the theorem.

We prove the second statement. Let $P \in E_n(\Gamma)$ and $p \in [1,\infty)$ be fixed. As in the proof of Theorem 6.1.1, by a change of scale, without loss of generality we may assume that $\sum_{j=0}^n \gamma_j = 1$. It follows from (6.1.9) and Hölder's inequality (see E.7 a] of Section 2.2) that

$$\begin{aligned}
|P'(a)|^p &\le 2^{p-1}\left(3^p|P(a)|^p + \left(\int_0^\infty |P(t+a)||U''(t)|\,dt\right)^p\right)\\
&\le 6^p|P(a)|^p\\
&\quad + 2^{p-1}\left(\left(\int_0^\infty |P(t+a)|^p|U''(t)|\,dt\right)^{1/p}\left(\int_0^\infty |U''(t)|\,dt\right)^{1/q}\right)^p
\end{aligned}$$

for every $a \in [0,\infty)$, where $q \in (1,\infty]$ is the conjugate exponent to p defined by $p^{-1}+q^{-1}=1$. Combining the above inequality with (6.1.7), we obtain

$$|P'(a)|^p \le 6^p|P(a)|^p + 2^{p-1}6^{p/q}\int_0^\infty |P(t+a)|^p|U''(t)|\,dt$$

for every $a \in [0,\infty)$. Integrating with respect to a, then using Fubini's theorem and (6.1.7), we conclude that

$$\begin{aligned}
\|P'\|^p_{L_p[0,\infty)} &\le 6^p\|P\|^p_{L_p[0,\infty)} + 2^{p-1}6^{p/q}\int_0^\infty\int_0^\infty |P(t+a)|^p|U''(t)|\,dt\,da\\
&\le 6^p\|P\|^p_{L_p[0,\infty)} + 2^{p-1}6^{p/q}\int_0^\infty\int_0^\infty |P(t+a)|^p|U''(t)|\,da\,dt\\
&\le 6^p\|P\|^p_{L_p[0,\infty)} + 2^{p-1}6^{p/q}\|P\|^p_{L_p[0,\infty)}\int_0^\infty |U''(t)|\,dt\\
&\le 6^p\|P\|^p_{L_p[0,\infty)} + 2^{p-1}6^{p/q+1}\|P\|^p_{L_p[0,\infty)}\\
&= (6^p+2^{p-1}6^p)\|P\|^p_{L_p[0,\infty)} \le 12^p\|P\|^p_{L_p[0,\infty)}
\end{aligned}$$

and the proof is finished. □

The following Nikolskii-type inequality follows from Theorem 6.1.1 quite simply:

Theorem 6.1.3 (Nikolskii-Type Inequality). *Suppose $0 < q < p \le \infty$. If $\Lambda := (\lambda_i)_{i=0}^{\infty}$ is a sequence of distinct real numbers greater than $-1/q$, then*

$$\|y^{1/q-1/p}P(y)\|_{L_p[0,1]} \le \left(18 \cdot 2^q \sum_{j=0}^{n} \left(\lambda_j + \frac{1}{q}\right)\right)^{1/q-1/p} \|P\|_{L_q[0,1]}$$

for every $P \in M_n(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}$.

If $\Gamma := (\gamma_i)_{i=0}^{\infty}$ is a sequence of distinct positive real numbers, then

$$\|P\|_{L_p[0,\infty)} \le \left(18 \cdot 2^q \sum_{j=0}^{n} \gamma_j\right)^{1/q-1/p} \|P\|_{L_q[0,\infty)} \tag{6.1.10}$$

for every $P \in E_n(\Gamma) := \operatorname{span}\{e^{-\gamma_0 t}, e^{-\gamma_1 t}, \dots, e^{-\gamma_n t}\}$.

Proof. First we show that the first statement of the theorem follows from the second. If $(\lambda_i)_{i=0}^{\infty}$ is a sequence of distinct real numbers greater than $-1/q$ and $\gamma_i := \lambda_i + 1/q$ for each i, then $(\gamma_i)_{i=1}^{\infty}$ is a sequence of distinct positive real numbers. Let $Q \in M_n(\Lambda)$. Applying (6.1.10) with

$$P(t) := Q(e^{-t})e^{-t/q} \in E_n(\Gamma)$$

and using the substitution $x = e^{-t}$, we obtain

$$\begin{aligned}&\|y^{1/q-1/p}Q(y)\|_{L_p[0,1]}\\ &\qquad \le (18 \cdot 2^q)^{1/q-1/p} \left(\sum_{j=0}^{n} \left(\lambda_j + \frac{1}{q}\right)\right)^{1/q-1/p} \|Q\|_{L_q[0,1]},\end{aligned}$$

which is the first statement of the theorem.

It is sufficient to prove (6.1.10) when $p = \infty$, and then a simple argument gives the desired result for arbitrary $0 < q < p < \infty$. To see this, assume that there is a constant C so that

$$\|P\|_{[0,\infty)} \le C^{1/q}\|P\|_{L_q[0,\infty)}$$

for every $P \in E_n(\Gamma)$ and $0 < q < \infty$. Then

$$\begin{aligned}\|P\|_{L_p[0,\infty)}^p &= \int_0^{\infty} |P(t)|^{p-q+q}\,dt \le \|P\|_{[0,\infty)}^{p-q}\|P\|_{L_q[0,\infty)}^q\\ &\le C^{p/q-1}\|P\|_{L_q[0,\infty]}^{p-q}\|P\|_{L_q[0,\infty)}^q\end{aligned}$$

and therefore

$$\|P\|_{L_p[0,\infty)} \le C^{1/q-1/p}\|P\|_{L_q[0,\infty)}$$

for every $f \in E_n(\Gamma)$ and $0 < q < p \le \infty$.

When $p = \infty$, (6.1.10) can be proven as follows. Let $P \in E_n(\Gamma)$, and let $y \in [0,\infty)$ be chosen so that $|P(y)| = \|P\|_{[0,\infty)}$. From Theorem 6.1.1 and the Mean Value Theorem, we can deduce that $|P(t)| > \frac{1}{2}\|P\|_{[0,\infty)}$ for every

$$t \in I := \left[y, y + (18\gamma)^{-1}\right], \quad \text{where} \quad \gamma := \sum_{j=0}^{n} \gamma_j .$$

Thus

$$\|P\|^q_{L_q[0,\infty]} \geq \int_I |P(t)|^q \, dt \geq \left(18 \sum_{j=0}^{n} \gamma_j\right)^{-1} 2^{-q} \|P\|^q_{[0,\infty)} ,$$

and the result follows. □

Theorem 6.1.3 immediately implies the following result, which is a special case of Theorem 4.2.4:

Theorem 6.1.4 (Müntz-Type Theorem in L_p). *Let $p \in [1,\infty)$. Let $(\lambda_i)_{i=0}^{\infty}$ be a sequence of distinct real numbers greater than $-1/p$ satisfying*

$$\sum_{j=0}^{\infty} \left(\lambda_j + \frac{1}{p}\right) < \infty .$$

Then $\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ *is not dense in* $L_p[0,1]$.

The next theorem offers an L_2 analog of Theorem 6.1.1 even for complex exponents. It also improves the multiplicative constant 12 in the L_2 inequality of Theorem 6.1.2 and shows that the L_2 inequality of Theorem 6.1.2 is essentially sharp.

Theorem 6.1.5. *If $\Lambda := (\lambda_i)_{i=0}^{\infty}$ is a sequence of distinct complex numbers with $\operatorname{Re}(\lambda_i) > -1/2$ for each i, then*

$$\sup_{0 \neq p \in M_n(\Lambda)} \frac{\|xp'(x)\|_{L_2[0,1]}}{\|p\|_{L_2[0,1]}} \leq \left(\sum_{j=0}^{n} |\lambda_j|^2 + \sum_{j=0}^{n} (1 + 2\operatorname{Re}(\lambda_j)) \sum_{k=j+1}^{n} (1 + 2\operatorname{Re}(\lambda_k))\right)^{1/2}$$

for every $n \in \mathbb{N}$, where $M_n(\Lambda)$ denotes the linear span of $\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}$ over $\mathbb{C}$.

If $\Lambda := (\lambda_i)_{i=0}^{\infty}$ is a sequence of distinct nonnegative real numbers, then

$$\frac{1}{2\sqrt{30}} \sum_{j=0}^{n} \lambda_j \leq \sup_{0 \neq p \in M_n(\Lambda)} \frac{\|xp'(x)\|_{L_2[0,1]}}{\|p\|_{L_2[0,1]}} \leq \frac{1}{\sqrt{2}} \sum_{j=0}^{n} (1 + 2\lambda_j)$$

for every $n \in \mathbb{N}$, where $M_n(\Lambda)$ denotes the linear span of $\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}$ over $\mathbb{R}$.

Proof. Let $p \in M_n(\Lambda)$ with $\|p\|_{L_2[0,1]} = 1$. Then

$$p(x) = \sum_{k=0}^{n} a_k L_k^*(x) \qquad \text{with} \qquad \sum_{k=0}^{n} |a_k|^2 = 1$$

and

$$xp'(x) = \sum_{k=0}^{n} a_k x L_k^{*\prime}(x)\,,$$

where $L_k^* \in M_k(\Lambda)$ denotes the kth orthonormal Müntz-Legendre polynomials on $[0,1]$. Using the recurrence formula of Corollary 3.4.5 b] for the terms $xL_k^{*\prime}(x)$ in the above sum, we obtain

$$xp'(x) = \sum_{j=0}^{n} \left(a_j \lambda_j + \sqrt{1+\lambda_j+\overline{\lambda}_j} \sum_{k=j+1}^{n} a_k \sqrt{1+\lambda_k+\overline{\lambda}_k} \right) L_j^*(x)\,.$$

Hence

$$\|xp'(x)\|_{L_2[0,1]}^2 = \sum_{j=0}^{n} \left| a_j \lambda_j + \sqrt{1+\lambda_j+\overline{\lambda}_j} \sum_{k=j+1}^{n} a_k \sqrt{1+\lambda_k+\overline{\lambda}_k} \right|^2 .$$

If we apply the Cauchy-Schwarz inequality to each term in the first sum and recall that $\sum_{k=0}^{n} |a_k|^2 = 1$, we see that

$$\begin{aligned} \|xp'(x)\|_{L_2[0,1]}^2 &\le \sum_{j=0}^{n} \left(|\lambda_j|^2 + (1+\lambda_j+\overline{\lambda}_j) \right) \sum_{k=j+1}^{n} (1+\lambda_k+\overline{\lambda}_k) \\ &\le \frac{1}{2} \left(\sum_{j=0}^{n} (1+2|\lambda_j|) \right)^2 , \end{aligned}$$

which proves the first part and the upper bound in the second part of the theorem.

Now we prove the lower estimate in the second part of the theorem. With the sequence $\Lambda := (\lambda_i)_{i=0}^{\infty}$ of distinct nonnegative real numbers, we associate

$$q(x) := \sum_{k=0}^{n} \sqrt{\lambda_k} \left(\sum_{j=0}^{k} \lambda_j \right) L_k^*(x) \in M_n(\Lambda)\,.$$

Since the system $(L_k^*)_{k=0}^{\infty}$ is orthonormal on [0,1], we have

$$\|q\|_{L_2[0,1]}^2 = \sum_{k=0}^{n} \lambda_k \left(\sum_{j=0}^{k} \lambda_j \right)^2 \le \left(\sum_{j=0}^{n} \lambda_j \right)^3 . \tag{6.1.11}$$

Furthermore

$$xq'(x) = \sum_{k=0}^{n} \sqrt{\lambda_k} \left(\sum_{j=0}^{k} \lambda_j \right) x L_k^{*\prime}(x) = \sum_{m=0}^{n} b_m L_m^*(x) \,,$$

where, by the recurrence formula of Corollary 3.4.5 b],

$$\begin{aligned} b_m &:= \lambda_m \sqrt{\lambda_m} \sum_{j=0}^{m} \lambda_j + \sqrt{1+2\lambda_m} \sum_{k=m+1}^{n} \sqrt{\lambda_k(1+2\lambda_k)} \sum_{j=0}^{k} \lambda_j \\ &\geq \sqrt{\lambda_m} \sum_{k=m}^{n} \lambda_k \sum_{j=0}^{k} \lambda_j \,. \end{aligned}$$

Hence

$$\begin{aligned} \|xq'(x)\|_{L_2[0,1]}^2 &= \sum_{m=0}^{n} b_m^2 \geq \sum_{m=0}^{n} \lambda_m \left(\sum_{k=m}^{n} \lambda_k \sum_{j=0}^{k} \lambda_j \right)^2 \\ &= \sum_{\substack{0 \leq m \leq n \\ m \leq k,k' \leq n}} \sum_{\substack{0 \leq j \leq k \\ 0 \leq j' \leq k'}} \lambda_m \lambda_k \lambda_j \lambda_{k'} \lambda_{j'} \\ &\geq \sum_{0 \leq m \leq j \leq j' \leq k \leq k' \leq n} \lambda_m \lambda_k \lambda_j \lambda_{k'} \lambda_{j'} \geq \frac{1}{5!} \left(\sum_{j=0}^{n} \lambda_j \right)^5 . \end{aligned}$$

This, together with (6.1.11), yields the lower bound in the second part of the theorem. □

Comments, Exercises, and Examples.

Theorem 6.1.1 is due to Newman [76]. We presented a modified version of Newman's original proof of Theorem 6.1.1. He worked with T instead of U, and instead of (6.1.9) he established a more complicated identity involving the second derivative of P. Therefore, he needed an application of Kolmogorov's inequality (see E.1) to finish his proof. It can be proven that if the exponents λ_j are distinct nonnegative integers, then $\|xp'(x)\|_{[0,1]}$ in Theorem 6.1.1 can be replaced by $\|p'\|_{[0,1]}$ (see E.3). Theorems 6.1.2 to 6.1.4 were proved by Borwein and Erdélyi [to appear 6], while Theorem 6.1.5 is due to Borwein, Erdélyi, and J. Zhang [94b]. It is shown in E.8 that Theorem 6.1.2 is essentially sharp for every Λ with a gap condition, and for every $p \in [2, \infty)$.

The interval $[0,1]$ plays a special role in this section, analogs of the results on $[a,b]$, $a > 0$, cannot be obtained by a linear transformation. E.10 deals with the nontrivial extension of Newman's inequality to intervals $[a,b]$, $a > 0$.

A conjecture of Lorentz about the "right" Bernstein-type inequality for exponential sums with n terms is settled in E.4.

E.1 Kolmogorov's Inequality. Show that

$$\|f'\|_{[0,\infty)}^2 \le 4\|f\|_{[0,\infty)}\|f''\|_{[0,\infty)}$$

for every $f \in C^2[0,\infty)$.

Hint: By Taylor's theorem

$$f(x+h) = f(x) + f'(x)h + \tfrac{1}{2}f''(\xi)h^2\,, \qquad h > 0$$

with some $\xi \in (x, x+h)$. Hence

$$\|f'\|_{[0,\infty)} \le 2h^{-1}\|f\|_{[0,\infty)} + (h/2)\|f''\|_{[0,\infty)}\,.$$

Now minimize the right-hand side by taking

$$h := 2\left(\|f\|_{[0,\infty)}\|f''\|_{[0,\infty)}^{-1}\right)^{1/2}\,.$$

□

The constant 4 in E.1 is not the best possible. Kolmogorov [62] proved that

$$\|f^{(k)}\|_{\mathbb{R}} \le K(n,k)\|f\|_{\mathbb{R}}^{1-k/n}\|f^{(n)}\|_{\mathbb{R}}^{k/n}$$

for every $f \in C^n(\mathbb{R})$ and $0 < k < n$ and found the best possible constants $K(n,k)$; see also DeVore and Lorentz [93]. This generalizes a result of Landau, who proved the above inequality for $n = 2$, $k = 1$, and showed that $K(2,1) = \sqrt{2}$. Various multivariate extensions of Kolmogorov's inequality have also been established; see, for example, Ditzian [89].

E.2 Nikolskii-Type Inequalities.

a] Suppose $(V, \|\cdot\|)$ is an $(n+1)$-dimensional real or complex Hilbert space, $(p_k)_{k=0}^n \subset V$ is an orthonormal system, and $\varphi \ne 0$ is a linear functional on V. Then

$$|\varphi(p)| \le \left(\sum_{k=0}^{n} |\varphi(p_k)|^2\right)^{1/2} \|p\|$$

for every $p \in V$. Equality holds if and only if

$$p = c\sum_{j=0}^{n} \overline{\varphi(p_k)}p_k\,, \qquad c \in \mathbb{R} \quad \text{or} \quad c \in \mathbb{C}.$$

Hint: Write p as a linear combination of the orthonormal elements p_k, use the linearity of φ, then apply the Cauchy-Schwarz inequality. □

b] Suppose $\Lambda := (\lambda_i)_{i=0}^{\infty}$ is a set of distinct complex numbers satisfying $\mathrm{Re}(\lambda_i) > -1/2$ for each i, and $y \in [0,\infty)$ is fixed. Show that

$$|p^{(m)}(y)| \le \left(\sum_{k=0}^{n} |L_k^{*(m)}(y)|^2 \right)^{1/2} \|p\|_{L_2[0,1]}$$

for every $p \in M_n(\Lambda)$, where $M_n(\Lambda)$ denotes the linear span of

$$\{x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_n}\}$$

over $\mathbb{C}$, and $L_k^* \in M_k(\Lambda)$ is the kth orthonormal Müntz-Legendre polynomial on $[0,1]$. Show that if there exists a $q \in M_n(\Lambda)$ with $q^{(m)}(y) \ne 0$, then equality holds if and only if

$$p = c \sum_{k=0}^{n} \overline{L_k^{*(m)}(y)} L_k^*, \qquad c \in \mathbb{C}.$$

c] Under the assumptions of part b] show that

$$\frac{|y^{1/2} p(y)|}{\|p\|_{L_2[0,1]}} \le \left(\sum_{j=0}^{n} (1 + 2\,\mathrm{Re}(\lambda_j)) \right)^{1/2}$$

and

$$\frac{|y^{3/2} p'(y)|}{\|p\|_{L_2[0,1]}} \le \left(\sum_{k=0}^{n} (1 + 2\,\mathrm{Re}(\lambda_k)) \Big| \lambda_k + \sum_{j=0}^{k-1} (1 + 2\,\mathrm{Re}(\lambda_j)) \Big|^2 \right)^{1/2}$$

for every $0 \ne p \in M_n(\Lambda)$ and $y \in [0,1]$.

Hint: When $y = 1$, use part b] and substitute the explicit values of $L_k^*(1)$ and $L_k^{*\prime}(1)$ (see Corollary 3.4.6 and formula (3.4.8)). If $0 < y < 1$, then the scaling $x \to yx$ reduces the inequality to the case $y = 1$. □

d] Show that if $n \ge 1$ and $p \ne 0$, then equality holds in the inequalities of part b] if and only if $y = 1$ and

$$p = c \sum_{k=0}^{n} L_k^*(1) L_k^* \qquad \text{or} \qquad p = c \sum_{k=0}^{n} L_k^{*\prime}(1) L_k^*,$$

respectively, with some $0 \ne c \in \mathbb{C}$.

e] Show that

$$\sup_{0\neq p\in\mathcal{P}_n} \frac{\|p\|_{[-1,1]}}{\|p\|_{L_2[-1,1]}} = \frac{n+1}{\sqrt{2}}\,.$$

Hint: Show that there is an extremal polynomial $\widetilde{p}$ for the above extremal problem for which $\|\widetilde{p}\|_{[-1,1]}$ is achieved at 1. Now use parts c] and d] to show that

$$|\widetilde{p}(1)|^2 = \frac{1}{2}\left(\sum_{k=0}^{n}(1+2k)\right)\int_{-1}^{1}\widetilde{p}^{\,2}(t)\,dt = \frac{1}{2}\,(n+1)^2\int_{-1}^{1}\widetilde{p}^{\,2}(t)\,dt$$

and the result follows. □

E.3 An Improvement of Newman's Inequality.

a] Suppose $\Lambda := (\lambda_k)_{k=0}^{\infty}$ is a sequence with $\lambda_0 = 0$ and $\lambda_{k+1} - \lambda_k \geq 1$ for each k. Show that

$$\|p'\|_{[0,1]} \leq 18\left(\sum_{j=1}^{n}\lambda_j\right)\|p\|_{[0,1]}$$

for every $p \in M_n(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_n}\}$.

Hint: Let $y \in [0,1]$. To estimate $|p'(y)|$ distinguish two cases. If $\frac{1}{2} \leq y \leq 1$, use Theorem 6.1.1 (Newman's inequality), and if $0 \leq y \leq \frac{1}{2}$, use E.3 f] of Section 3.3 and Theorem 5.1.8 (Markov's inequality) transformed to $[y,1]$ to show that

$$|p'(y)| \leq \frac{2n^2}{1-y}\|p\|_{[y,1]} \leq 8\left(\sum_{j=1}^{n}\lambda_j\right)\|p\|_{[0,1]}$$

for every $p \in M_n(\Lambda)$. □

The next exercise is based on an example given by Bos.

b] Show that for every $\delta \in (0,1)$ there exists a sequence $\Lambda := (\lambda_k)_{k=0}^{\infty}$ with $\lambda_0 = 0$, $\lambda_1 \geq 1$, and

$$\lambda_{k+1} - \lambda_k \geq \delta\,, \qquad i = 0, 1, 2, \ldots$$

such that

$$\lim_{n\to\infty}\sup_{0\neq p\in M_n(\Lambda)} \frac{|p'(0)|}{\left(\sum_{j=0}^{n}\lambda_j\right)\|p\|_{[0,1]}} = \infty\,.$$

Outline. Let Q_n be the Chebyshev polynomial T_n transformed linearly from $[-1,1]$ to $[0,1]$, that is,

$$Q_n(x) = \cos(n\arccos(2x-1))\,, \qquad x \in [0,1]\,.$$

Choose natural numbers u and v so that $\delta < u/v < 1$. Let $\Lambda := (\lambda_k)_{k=0}^{\infty}$ be defined by $\lambda_0 := 0$, $\lambda_1 := 1$, and

$$\lambda_k := 1 + \frac{ku}{v}, \qquad k = 1, 2, \ldots .$$

Let

$$p_n(x) := x^{1-u}\big(Q_n(x^{u/v}) - (-1)^n\big)^v \in M_{nv-v}(\Lambda).$$

Then

$$|p_n'(0)| = \big(2n^2\big)^v.$$

For the sake of brevity let

$$q_n(x) := Q_n(x^{u/v}) - (-1)^n.$$

Use Theorem 5.1.8 (Markov's inequality) and the Mean Value Theorem to show that

$$\|q_n\|_{[y,1]} \geq \tfrac{1}{2}\|q_n\|_{[0,1]}$$

with

$$y := \big(2n^2\big)^{-v/u}.$$

Thus, if n is odd, then

$$\begin{aligned}\|p_n\|_{[0,1]} &\geq \|p_n\|_{[y,1]} \geq y^{1-u}(\|q_n\|_{[y,1]})^v \\ &\geq y^{1-u}\left(\tfrac{1}{2}\|q_n\|_{[0,1]}\right)^v = \big(2n^2\big)^{(u-1)v/u}\end{aligned}$$

and

$$\begin{aligned}\frac{|p_n'(0)|}{\left(\sum_{j=0}^{nv-v}\lambda_j\right)\|p_n\|_{[0,1]}} &= \frac{(2n^2)^v}{\left(\sum_{j=0}^{nv-v}\left(1 + j\frac{u}{v}\right)\right)\big(2n^2\big)^{(u-1)v/u}} \\ &\geq \frac{\big(2n^2\big)^{v/u}}{(1+nu)nv} \geq \frac{\big(2n^2\big)^{v/u-1}}{uv} \underset{n\to\infty}{\longrightarrow} \infty .\end{aligned}$$

□

In his book *Nonlinear Approximation Theory*, Braess [86] writes the following: "The rational functions and exponential sums belong to those concrete families of functions which are the most frequently used in nonlinear approximation theory. The starting point of consideration of exponential sums is an approximation problem often encountered for the analysis of decay processes in natural sciences. A given empirical function on a real interval is to be approximated by sums of the form

$$\sum_{j=1}^{n} a_j e^{\lambda_j t},$$

where the parameters a_j and λ_j are to be determined, while n is fixed."

The next exercise treats inequalities for exponential sums of $n + 1$ terms.

E.4 Nikolskii- and Bernstein-Type Inequalities for Exponential Sums. Let

$$\Lambda := (\lambda_i)_{i=1}^{\infty}$$

be a sequence of distinct nonzero real numbers. Let

$$E_n(\Lambda) := \left\{ f : f(t) = a_0 + \sum_{j=1}^{n} a_j e^{\lambda_j t}, \quad a_j \in \mathbb{R} \right\}$$

and

$$E_n := \bigcup_{\Lambda} E_n(\Lambda) = \left\{ f : f(t) = a_0 + \sum_{j=1}^{n} a_j e^{\lambda_j t}, \quad a_j, \lambda_j \in \mathbb{R} \right\},$$

that is, E_n is the collection of all $(n+1)$-term exponential sums with constant first term. Schmidt [70] proved that there is a constant $c(n)$ depending only on n such that

$$\|f'\|_{[a+\delta,b-\delta]} \le c(n)\delta^{-1}\|f\|_{[a,b]}$$

for every $p \in E_n$ and $\delta \in \left(0, \frac{1}{2}(b-a)\right)$. Lorentz [89] improved Schmidt's result by showing that for every $\alpha > \frac{1}{2}$, there is a constant $c(\alpha)$ depending only on α such that $c(n)$ in the above inequality can be replaced by $c(\alpha)n^{\alpha \log n}$, and he speculated that there may be an absolute constant c such that Schmidt's inequality holds with $c(n)$ replaced by cn. Part d] of this exercise shows that Schmidt's inequality holds with $c(n) = 2n-1$. A weaker version of this showing that Schmidt's inequality holds with $c(n) = 8(n+1)^2$ is obtained in part b] and uses a Nikolskii-type inequality for exponential sums established in part a]. Part e] shows that the result of part d] is sharp up to a multiplicative absolute constant.

a] Let $p \in (0,2]$. Show that

$$\|f\|_{[a+\delta,b-\delta]} \le 2^{2/p^2} \left(\frac{n+1}{\delta}\right)^{1/p} \|f\|_{L_p[a,b]}$$

for every $f \in E_n$ and $\delta \in \left(0, \frac{1}{2}(b-a)\right)$.

Proof. Take the orthonormal sequence $(L_k)_{k=0}^{n}$ on $\left[-\frac{1}{2}, \frac{1}{2}\right]$, that is,

(1) $L_k \in \operatorname{span}\{1, e^{\lambda_1 t}, e^{\lambda_2 t}, \dots, e^{\lambda_k t}\}, \qquad k = 0, 1, \dots, n,$

and

(2) $\displaystyle\int_{-1/2}^{1/2} L_i L_j = \delta_{i,j}, \qquad 0 \le i \le j \le n,$

where $\delta_{i,j}$ is the Kronecker symbol. On writing $f \in E_n(\Lambda)$ as a linear combination of $L_0, L_1, \dots, L_n$, and using the Cauchy-Schwarz inequality

and the orthonormality of $(L_k)_{k=0}^n$ on $\left[-\frac{1}{2},\frac{1}{2}\right]$, we obtain in a standard fashion that

$$\max_{0\neq f\in E_n(\Lambda)} \frac{|f(t_0)|}{\|f\|_{L_2[-1/2,1/2]}} = \left(\sum_{k=0}^n L_k(t_0)^2\right)^{1/2}, \qquad t_0\in\mathbb{R}.$$

Since

$$\int_{-1/2}^{1/2} \textstyle\sum_{k=0}^n L_k^2(x)\,dx = n+1\,,$$

there exists a $t_0\in\left[-\frac{1}{2},\frac{1}{2}\right]$ such that

$$\max_{0\neq p\in E_n(\Lambda)} \frac{|f(t_0)|}{\|f\|_{L_2[-1/2,1/2]}} = \left(\sum_{k=0}^n L_k^2(t_0)\right)^{1/2} \le \sqrt{n+1}\,.$$

Observe that if $f\in E_n(\Lambda)$, then $g(t):=f(t-t_0)\in E_n(\Lambda)$, so

$$\max_{0\neq f\in E_n(\Lambda)} \frac{|f(0)|}{\|f\|_{L_2[-1,1]}} \le \sqrt{n+1}\,.$$

Let

$$C := \max_{0\neq f\in E_n(\Lambda)} \frac{|f(0)|}{\|f\|_{L_p[-2,2]}}\,.$$

Then

$$\max_{0\neq f\in E_n(\Lambda)} \frac{|f(y)|}{\|f\|_{L_p[-2,2]}} \le C\left(\frac{2}{2-|y|}\right)^{1/p} \le 2^{1/p}C\,, \qquad y\in[-1,1]\,.$$

Therefore, for every $f\in E_n(\Lambda)$,

$$\begin{aligned}
|f(0)| &\le \sqrt{n+1}\,\|f\|_{L_2[-1,1]}\\
&\le \sqrt{n+1}\left(\|f\|_{L_p[-1,1]}^p \|f\|_{[-1,1]}^{2-p}\right)^{1/2}\\
&\le \sqrt{n+1}\left(\|f\|_{L_p[-1,1]}^p \left(2^{1/p}C\right)^{2-p}\|f\|_{L_p[-2,2]}^{2-p}\right)^{1/2}\\
&\le \sqrt{n+1}\left(2^{1/p}C\right)^{1-p/2}\|f\|_{L_p[-2,2]}\\
&= 2^{1/p-1/2}\sqrt{n+1}\,C^{1-p/2}\|f\|_{L_p[-2,2]}\,.
\end{aligned}$$

Hence

$$C = \max_{0\neq f\in E_n(\Lambda)} \frac{|f(0)|}{\|f\|_{L_p[-2,2]}} \le 2^{1/p-1/2}\sqrt{n+1}\,C^{1-p/2}$$

and we conclude that $C\le 2^{2/p^2-1/p}(n+1)^{1/p}$. So

$$|f(0)| \le 2^{2/p^2-1/p}(n+1)^{1/p}\|f\|_{L_p[-2,2]}$$

for every $f \in E_n(\Lambda)$. Now let $f \in E_n(\Lambda)$ and $t_0 \in [a+\delta, b-\delta]$. If we apply the above inequality to

$$g(t) := f\left(\tfrac{1}{2}\delta t + t_0\right) \in E_n(\Lambda)\,,$$

we obtain

$$\|f\|_{[a+\delta,b-\delta]} \le 2^{2/p^2-1/p}(n+1)^{1/p}\left(\frac{2}{\delta}\right)^{1/p}\|f\|_{L_p[a,b]}$$

and the result follows. □

The following Bernstein-type inequality can be obtained as a simple corollary of part a]:

b] Show that

$$\|f'\|_{[a+\delta,b-\delta]} \le 8(n+1)^2\delta^{-1}\|f\|_{[a,b]}$$

for every $f \in E_n$ and $\delta \in (0, \frac{1}{2}(b-a))$.

Proof. Note that $f \in E_n(\Lambda)$ implies $f' \in E_n(\Lambda)$. Applying part a] to f' with $p = 1$, we obtain

$$|f'(0)| \le 2(n+1)\|f'\|_{L_1[-2,2]} = 2(n+1)\mathrm{Var}_{[-2,2]}(f) \le 4(n+1)^2\|f\|_{[-2,2]}$$

for every $f \in E_n(\Lambda)$. If $f \in E_n(\Lambda)$ and $t_0 \in [a+\delta, b-\delta]$, then on applying the above inequality to

$$g(t) := f\left(\tfrac{1}{2}\delta t + t_0\right) \in E_n(\Lambda)\,,$$

we obtain the desired result. □

c] **Lorentz's Conjecture.** Show that

$$\sup_{0\ne f\in \widetilde{E}_{2n}} \frac{|f'(0)|}{\|f\|_{[-1,1]}} = 2n-1\,,$$

where

$$\widetilde{E}_{2n} := \left\{ f : f(t) = a_0 + \sum_{j=1}^{n}\left(a_j e^{\lambda_j t} + b_j e^{-\lambda_j t}\right), \quad a_j, b_j, \lambda_j \in \mathbb{R} \right\}.$$

Proof. First we prove that

$$|f'(0)| \le (2n-1)\,\|f\|_{[-1,1]}$$

for every $f \in \widetilde{E}_{2n}$. So let

$$f \in \operatorname{span}\{1, e^{\pm\lambda_1 t}, e^{\pm\lambda_2 t}, \dots, e^{\pm\lambda_n t}\}$$

with some nonzero real numbers $\lambda_1, \lambda_2, \dots, \lambda_n$, where, without loss of generality, we may assume that

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n .$$

Let

$$g(t) := \tfrac{1}{2}(f(t) - f(-t)) .$$

Observe that

$$g \in \operatorname{span}\{\sinh \lambda_1 t\,,\ \sinh \lambda_2 t\,,\ \dots\,,\ \sinh \lambda_n t\} .$$

It is also straightforward that

$$g'(0) = f'(0) \qquad \text{and} \qquad \|g\|_{[0,1]} \le \|f\|_{[-1,1]} .$$

For a given $\epsilon > 0$, let

$$H_{n,\epsilon} := \operatorname{span}\{\sinh \epsilon t\,,\ \sinh 2\epsilon t\,,\ \dots\,,\ \sinh n\epsilon t\}$$

and

$$K_{n,\epsilon} := \sup\left\{|h'(0)| : h \in H_{n,\epsilon}\,,\ \ \|h\|_{[0,1]} = 1\right\} .$$

The inequality of E.5 e] in Section 3.3 is the key to the proof. It shows that it is sufficient to prove that

$$\inf\{K_{n,\epsilon} : \epsilon > 0\} \le 2n - 1 .$$

Observe that every $h \in H_{n,\epsilon}$ is of the form

$$h(t) = e^{-n\epsilon t} P(e^{\epsilon t}) , \qquad P \in \mathcal{P}_{2n} .$$

Therefore, using E.23 c] of Section 5.1, we obtain for every $h \in H_{n,\epsilon}$ that

$$\begin{aligned} |h'(0)| &= |\epsilon P'(1) - n\epsilon P(1)| \\ &\le \frac{\epsilon(2n-1)}{1 - e^{-\epsilon}} \|P\|_{[e^{-\epsilon}, e^{\epsilon}]} + n\epsilon \|P\|_{[e^{-\epsilon}, e^{\epsilon}]} \\ &\le \left(\frac{\epsilon(2n-1)}{1 - e^{-\epsilon}} + n\epsilon\right) e^{n\epsilon} \|h\|_{[-1,1]} . \end{aligned}$$

It follows that

$$K_{n,\epsilon} \le \left(\frac{\epsilon(2n-1)}{1 - e^{-\epsilon}} + n\epsilon\right) e^{n\epsilon} .$$

So $\inf\{K_{n,\epsilon} : \epsilon > 0\} \le 2n - 1$, and the upper bound follows.

Now we prove that

$$\sup_{0\neq f\in \bar{E}_{2n}} \frac{|f'(0)|}{\|f\|_{[-1,1]}} \geq 2n-1\,.$$

Let $\epsilon > 0$ be fixed. We define

$$Q_{2n,\epsilon}(t) := e^{-n\epsilon t} T_{2n-1}\left(\frac{e^{\epsilon t}}{e^{\epsilon}-1} - \frac{1}{e^{\epsilon}-1}\right)\,,$$

where T_{2n-1} denotes the Chebyshev polynomial of degree $2n-1$ defined by

$$T_{2n-1}(x) = \cos((2n-1)\arccos x),\ \ x\in[-1,1]\,.$$

It is simple to check that $Q_{2n,\epsilon}\in \widetilde{E}_{2n}$,

$$\|Q_{2n,\epsilon}\|_{[-1,1]} \leq e^{n\epsilon t}$$

and

$$|Q'_{2n,\epsilon}(0)| \geq 2n-1-n\epsilon\,.$$

Now the result follows by letting ϵ decrease to 0. □

d] Show that

$$\|f'\|_{[a+\delta,b-\delta]} \leq (2n-1)\delta^{-1}\|f\|_{[a,b]}$$

for every $f\in E_n$ and $\delta\in\left(0,\frac{1}{2}(b-a)\right)$.

Proof. Observe that $E_n\subset\widetilde{E}_{2n}$. Hence the result follows from part c] by a linear substitution. □

e] Let $a<b$ and $y\in(a,b)$. Suppose that $n\in\mathbb{N}$ is odd. Let T_n be the Chebyshev polynomial of degree n defined by (2.1.1). Let

$$Q_n(t) := Q_{n,y}(t) := T_n\left(\frac{e}{e-1}\exp\left(\frac{t-b}{b-y}\right) - \frac{1}{e-1}\right)$$

and

$$R_n(t) := R_{n,y}(t) := T_n\left(\frac{e}{e-1}\exp\left(\frac{t-a}{a-y}\right) - \frac{1}{e-1}\right)\,.$$

Show that $Q_n, R_n\in E_n$ and

$$\frac{|Q'_n(y)|}{\|Q_n\|_{[a,b]}} = \frac{1}{e-1}\frac{n}{b-y} \qquad \text{and} \qquad \frac{|R'_n(y)|}{\|R_n\|_{[a,b]}} = \frac{1}{e-1}\frac{n}{y-a}$$

for every $y\in(a,b)$.

f] Let $a < b$ and $y \in (a,b)$. Conclude that

$$\frac{1}{e-1}\frac{n-1}{\min\{y-a,b-y\}} \le \sup_{0\not\equiv f\in E_n}\frac{|f'(y)|}{\|f\|_{[a,b]}} \le \frac{2n-1}{\min\{y-a,b-y\}}.$$

In the rest of the exercise let

$$E_n^* := \left\{ f : f(t) = \sum_{j=1}^{l} P_{k_j}(t)e^{\lambda_j t},\ \ \lambda_j \in \mathbb{R},\ \ P_{k_j} \in \mathcal{P}_{k_j},\ \ \sum_{j=1}^{l}(k_j+1) = n \right\}.$$

g] Extend the inequalities of parts a], c], and f] to E_n^*.

h] Let $[a,b]$ be a finite interval. Let $g \in C[a,b]$. Show that the value

$$\inf\{\|g-f\|_{[a,b]} : f \in E_n^*\}$$

is attained by an $\widetilde{f} \in E_n^*$.

Hint: Use Schmidt's inequality (or its improved form given by part c]). For the nontrivial details, see Braess [86]. □

i] Let $[a,b]$ be a finite interval. Let $p \in [1,\infty)$ and $g \in L_p[a,b]$. Show that the value

$$\inf\{\|g-f\|_{L_p[a,b]} : f \in E_n^*\}$$

is attained by an $\widetilde{f} \in E_n^*$.

Hint: Use part a] with $p = 1$ and Hölder's inequality. Once again, for the details, see Braess [86]. □

The following result is from Borwein and Erdélyi [95c]:

E.5 Upper Bound for the Derivative of Exponential Sums with Nonnegative Exponents. The equality

$$\sup_p \frac{|p'(a)|}{\|p\|_{[a,b]}} = \frac{2n^2}{b-a}$$

holds for every $a < b$, where the supremum is taken over all exponential sums $0 \not\equiv p \in E_n$ with nonnegative exponents. The equality

$$\sup_p \frac{|p'(a)|}{\|p\|_{[a,b]}} = \frac{2n^2}{a(\log b - \log a)}$$

also holds for every $0 < a < b$, where the supremum is taken over all Müntz polynomials $0 \not\equiv p$ of the form

$$p(x) = a_0 + \sum_{j=1}^{n} a_j x^{\lambda_j}, \qquad a_j \in \mathbb{R}, \quad \lambda_j > 0.$$

Outline. It is sufficient to prove only the second statement, the first can be obtained from it by the change of variable $x = e^t$. For $\epsilon > 0$, define $\Lambda_\epsilon := (j\epsilon)_{j=0}^{\infty}$. Let

$$T_{n,\epsilon} := T_n\{1, x^\epsilon, x^{2\epsilon}, \dots, x^{n\epsilon}; [a,b]\}$$

be the Chebyshev polynomial for $M_n(\Lambda_\epsilon)$ on $[a,b]$. Use E.3 b] and E.3 f] of Section 3.3 to show that

$$\frac{|p'(a)|}{\|p\|_{[a,b]}} \le \lim_{\epsilon\to 0+} \frac{|T'_{n,\epsilon}(a)|}{\|T_{n,\epsilon}\|_{[a,b]}} = \lim_{\epsilon\to 0+} |T'_{n,\epsilon}(a)|$$

for every p of the form

$$p(x) = a_0 + \sum_{j=1}^{n} a_j x^{\lambda_j}, \qquad a_j \in \mathbb{R}, \quad \lambda_j > 0.$$

From the definition and uniqueness of $T_{n,\epsilon}$ it follows that

$$T_{n,\epsilon}(x) = T_n\left(\frac{2}{b^\epsilon - a^\epsilon} x^\epsilon - \frac{b^\epsilon + a^\epsilon}{b^\epsilon - a^\epsilon}\right)$$

where $T_n(y) = \cos(n \arccos y)$, $y \in [-1,1]$. Therefore

$$\begin{aligned} |T'_{n,\epsilon}(a)| &= |T'_n(-1)| \frac{2}{b^\epsilon - a^\epsilon} \epsilon a^{\epsilon-1} \\ &= \frac{2n^2}{\epsilon^{-1}(b^\epsilon - 1) - \epsilon^{-1}(a^\epsilon - 1)} a^{\epsilon-1} \underset{\epsilon\to 0+}{\longrightarrow} \frac{2n^2}{a(\log b - \log a)} \end{aligned}$$

and the proof is finished. □

The next exercise follows Turán [84].

E.6 Turán's Inequalities for Exponential Sums.

a] Let

$$g(\nu) := \sum_{j=1}^{n} b_j z_j^\nu, \qquad b_j, z_j \in \mathbb{C}.$$

Suppose

$$|z_j| \ge 1, \qquad j = 1, 2, \dots, n.$$

Then

$$\max_{\nu=m+1,\dots,m+n} |g(\nu)| \ge \left(\frac{n}{2e(m+n)}\right)^2 |b_1 + b_2 + \cdots + b_n|$$

for every nonnegative integer m.

Proof. Let

$$f(z) := \prod_{j=1}^{n} \left(1 - \frac{z}{z_j}\right) =: \sum_{\nu=0}^{n} \alpha_\nu z^\nu . \tag{6.1.12}$$

Since $f(z)$ has all its zeros outside the open unit disk, $g := 1/f$ is of the form

$$g(z) = \sum_{\nu=0}^{\infty} \beta_\nu z^\nu , \qquad |z| < 1 . \tag{6.1.13}$$

Let

$$g_m(z) := \sum_{\nu=0}^{m} \beta_\nu z^\nu \tag{6.1.14}$$

and

$$h := 1 - f g_m \in \mathcal{P}^c_{n+m} . \tag{6.1.15}$$

Note that

$$\begin{aligned} h(z) &= 1 - f(z) \left(g(z) - \sum_{\nu=m}^{\infty} \beta_\nu z^\nu \right) \\ &= f(z) \sum_{\nu=m+1}^{\infty} \beta_\nu z^\nu \end{aligned} \tag{6.1.16}$$

so h is of the form

$$h(z) = \sum_{\nu=m+1}^{m+n} \gamma_\nu z^\nu . \tag{6.1.17}$$

Observe that $f(z_j) = 0$ and (6.1.13) imply $h(z_j) = 1$, that is,

$$\sum_{\nu=m+1}^{m+n} \gamma_\nu z_j^\nu = 1 , \qquad j = 1, 2, \dots , n .$$

Multiplication with b_j and summation over j yield the fundamental identity

$$\sum_{\nu=m+1}^{m+n} \gamma_\nu g(\nu) = \sum_{j=1}^{n} b_j . \tag{6.1.18}$$

This immediately gives

$$\max_{\nu=m+1,\dots,m+n} |g(\nu)| \left(\sum_{\nu=m+1}^{n} |\gamma_\nu| \right) \geq \left| \sum_{j=1}^{n} b_j \right| . \tag{6.1.19}$$

It follows from (6.1.15), (6.1.17), (6.1.12), and (6.1.14) that

$$\sum_{\nu=m+1}^{m+n} |\gamma_\nu| \leq \left(\sum_{\nu=0}^{n} |\alpha_\nu| \right) \left(\sum_{\nu=0}^{m} |\beta_\nu| \right) . \tag{6.1.20}$$

Since each z_j is of modulus at least 1, (6.1.12) yields that

$$\sum_{\nu=0}^{n} |\alpha_\nu| \leq 2^n . \tag{6.1.21}$$

Also, (6.1.13) implies that

$$\beta_\nu = \sum_{i_1+\cdots+i_n=\nu} \frac{1}{z_1^{i_1} z_2^{i_2} \cdots z_n^{i_n}} .$$

Again using that each z_j is of modulus at least 1, we obtain

$$|\beta_\nu| \leq \sum_{i_1+\cdots+i_n=\nu} 1 = \binom{\nu+n-1}{n-1} .$$

Hence

$$\sum_{\nu=0}^{m} |\beta_\nu| \leq \binom{m+n}{n} \leq \left(e\, \frac{m+n}{n} \right)^n . \tag{6.1.22}$$

By (6.1.20) to (6.1.22) we conclude that

$$\sum_{\nu=m+1}^{m+n} |\gamma_\nu| \leq \left(2e\, \frac{m+n}{n} \right)^n ,$$

which, together with (6.1.19), finishes the proof. □

b] Let

$$f(t) := \sum_{j=1}^{n} b_j e^{\lambda_j t} , \qquad b_j , \lambda_j \in \mathbb{C}.$$

Suppose

$$\mathrm{Re}(\lambda_j) \geq 0 , \qquad j = 1, 2, \dots , n .$$

Show that

$$|f(0)| \leq \left(\frac{2e(a+d)}{d} \right)^n \|f\|_{[a,a+d]}$$

for every $a > 0$ and $d > 0$.

Hint: First observe that the result of part a] can be formulated as

$$\max_{\substack{m\le\nu\le m+n\\ \nu\in\mathbb{N}}} |g(\nu)| \ge \left(\frac{n}{2e(m+n)}\right)^n |b_1+b_2+\cdots+b_n|\,,$$

where m is an arbitrary positive (not necessarily integer) number. Now apply the above inequality with

$$m := \frac{an}{d}\,, \quad z_j := \exp\left(\frac{d\lambda_j}{n}\right)\,, \quad j=1,2,\dots,n\,.$$

□

c] Let

$$p(z) := \sum_{j=1}^{n} b_j z^{\lambda_j}\,, \qquad b_j\in\mathbb{C}, \quad \lambda_j\in\mathbb{R}\,, \quad z=e^{i\theta}.$$

Show that

$$\max_{|z|=1}|p(z)| \le \left(\frac{4e\pi}{\delta}\right)^n \max_{\substack{|z|=1\\ \alpha\le\arg(z)\le\alpha+\delta}} |p(z)|$$

for every $0\le\alpha<\alpha+\delta\le 2\pi$.

Hint: Use part b]. □

The inequalities of the above exercise and their variants play a central role in the book of Turán [83], where many applications are also presented. The main point in these inequalities is that the exponent on the right-hand side is only the number of terms n, and so it is independent of the numbers λ_j. An inequality, say in part c], of type

$$\max_{|z|=1}|p(z)| \le c(\delta)^{\lambda_n} \max_{\substack{|z|=1\\ \alpha\le\arg(z)\le\alpha+\delta}} |p(z)|\,,$$

where $0\le\lambda_1<\lambda_2<\cdots<\lambda_n$ are integers and $c(\delta)$ depends only on δ, could be obtained by a simple direct argument, but it is much less useful than the inequality of E.6 c].

E.7 Nikolskii-Type Inequality for Müntz Polynomials. Suppose that $\Lambda := (\lambda_i)_{i=0}^{\infty}$ is a sequence with $\lambda_0 := 0$ and $\lambda_{i+1}-\lambda_i \ge 1$ for each i. Show that

$$\|P\|_{L_p[0,1]} \le \left(18\cdot 2^q \sum_{j=0}^{n}\lambda_j\right)^{1/q-1/p} \|P\|_{L_q[0,1]}$$

for every $P\in M_n(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1},\dots,x^{\lambda_n}\}$ and $0<q<p\le\infty$.

Proof. It is sufficient to study the case when $p = \infty$ (see the comment in the proof of Theorem 6.1.3). Let $P \in M_n(\Lambda)$ and let $x_0 \in [0,1]$ be chosen so that $|P(x_0)| = \|P\|_{[0,1]}$. Combining E.3 a] and the Mean Value Theorem, we obtain

$$|P(x)| \geq \frac{1}{2}\|P\|_{[0,1]}, \qquad x \in I,$$

where

$$I := \left[x_0 - (36\lambda)^{-1}, x_0 + (36\lambda)^{-1}\right] \quad \text{with} \quad \lambda := \sum_{j=0}^{n} \lambda_j .$$

So

$$\left(\int_0^1 |P(x)|^q\,dx\right)^{1/q} \geq \left(\int_I |P(x)|^q\,dx\right)^{1/q} \geq \left((18\lambda)^{-1}\left(\frac{1}{2}\|P\|_{[0,1]}\right)^q\right)^{1/q}$$

and the result follows. □

E.8 Sharpness of Theorem 6.1.2. Suppose $\Lambda := (\lambda_i)_{i=0}^{\infty}$ is a sequence with $\lambda_0 := 0$ and $\lambda_{i+1} - \lambda_i \geq 1$ for each i. Show that there exists an absolute constant $c > 0$ (independent of Λ and p) such that

$$\sup_{P \in M_n(\Lambda)} \frac{\|xP'(x)\|_{L_p[0,1]}}{\|P\|_{L_p[0,1]}} \geq c \sum_{k=0}^{n} \lambda_k$$

for every $p \in [2, \infty)$, where $M_n(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_n}\}$.

Proof. Let $L_k^* \in M_k(\Lambda)$ be the kth orthonormal Müntz-Legendre polynomial on $[0,1]$. Let $p \in [2, \infty)$ and

$$P := \sum_{k=0}^{n} L_k^{*\prime}(1) L_k^* .$$

For the sake of brevity let

$$\lambda := \sum_{j=0}^{n} \lambda_j .$$

Using Theorem 3.4.3 (orthogonality), (3.4.8), and Corollary 3.4.6, we obtain

$$\begin{aligned} P'(1) = \sum_{k=0}^{n} (L_k^{*\prime}(1))^2 &\geq \sum_{k=0}^{n} (2\lambda_k + 1)\left(\sum_{j=0}^{k} \lambda_j\right)^2 \\ &\geq 2\lambda\left(\sum_{j=0}^{k} \lambda_j\right)^2 \geq \tfrac{1}{8}\lambda^3 \end{aligned} \tag{6.1.23}$$

and

$$\|P\|_{L_2[0,1]} = \left(\sum_{k=0}^{n}(L_k^{*\prime}(1))^2\right)^{1/2} \tag{6.1.24}$$
$$\leq \left(\sum_{k=0}^{n}(2\lambda_k+1)\left(\sum_{j=0}^{k}(2\lambda_j+1)\right)^2\right)^{1/2}$$
$$\leq \sqrt{32}\,\lambda^{3/2}\,.$$

A combination of E.7 and (6.1.24) yields

$$\|P\|_{L_p[0,1]} \leq (72\,\lambda)^{1/2-1/p}\|P\|_{L_2[0,1]} \tag{6.1.25}$$
$$\leq 72^{1/2-1/p}\sqrt{32}\,\lambda^{2-1/p} \leq 48\,\lambda^{2-1/p}\,.$$

From E.3 a], E.7, and (6.1.24) we can deduce that

$$\|P'\|_{[0,1]} \leq 18\,\lambda\,\|P\|_{[0,1]} \tag{6.1.26}$$
$$\leq 18\,\lambda(72\,\lambda)^{1/2}\|P\|_{L_2[0,1]}$$
$$\leq 18\,\lambda(72\,\lambda)^{1/2}\sqrt{32}\,\lambda^{3/2}$$
$$\leq 18\cdot 48\,\lambda^3\,.$$

Applying E.3 a] with P', we get

$$\|P''\|_{[0,1]} \leq 18\,\lambda\,\|P'\|_{[0,1]} \leq 18^2\cdot 48\,\lambda^4\,. \tag{6.1.27}$$

Now (6.1.23), (6.1.27), and the Mean Value Theorem give

$$|P'(x)| \geq \tfrac{1}{16}\lambda^3\,, \qquad x\in I\,,$$

where

$$I := \left[1-\left(18^2\cdot 48\cdot 16\,\lambda\right)^{-1},1\right].$$

So

$$\|xP'(x)\|_{L_p[0,1]} \geq \left(\int_I \left|xP'(x)\right|^p dx\right)^{1/p} \tag{6.1.28}$$
$$\geq \left(\left(18^2\cdot 48\cdot 16\,\lambda\right)^{-1}\left(\tfrac{1}{32}\lambda^3\right)^p\right)^{1/p}$$
$$\geq \left(18^2\cdot 48\cdot 16\right)^{-1/p} 32^{-1/2}\lambda^{3-1/p}$$
$$\geq \left(128\cdot 9\sqrt{3}\right)^{-1}\lambda^{3-1/p}\,.$$

Combining (6.1.25) and (6.1.28), we obtain the required result with a constant $c = (128\cdot 9\sqrt{3})^{-1}$. □

E.9 On the Interval Where the Sup Norm of a Müntz Polynomial Lives. Let $\Lambda := (\lambda_j)_{j=0}^{\infty}$ be an increasing sequence of nonnegative real numbers. Let $0 \not\equiv p \in M_n(\Lambda) := \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots x^{\lambda_n}\}$, and let $q(x) := x^{k\alpha}p(x)$, where $\alpha > 0$ and k is a nonnegative integer. Let $\xi \in [0,1]$ be chosen so that $|q(\xi)| = \|q\|_{[0,1]}$.

a] Suppose $\alpha = 1$ and each λ_j is an integer. Then

$$\left(\frac{k}{k+n}\right)^2 < \xi\,.$$

Proof. See Saff and Varga [81]. □

The above result is sharp in a certain limiting sense, which is described in detail in Saff and Varga [78].

b] Suppose $\lambda_j = \alpha j$ for each j. Use part a] to show that

$$\left(\frac{k}{k+n}\right)^{2/\alpha} < \xi\,.$$

c] Suppose $\lambda_j = \alpha j$ for each j. Use E.11 of Section 5.1 to show, without using part a], that there exists an absolute constant $c > 0$ such that

$$\left(\frac{ck}{k+n}\right)^{2/\alpha} < \xi\,.$$

d] Suppose $\lambda_j \geq \alpha j$ for each j. Use part b] and E.3 g] of Section 3.3 to show that the conclusion of part b] remains valid.

e] Suppose $\lambda_j \geq \alpha j$ for each j. Use part c] and E.3 g] of Section 3.3 to show that the conclusion of part c] remains valid.

The following extension of Newman's inequality is in Borwein and Erdélyi [to appear 3].

E.10 Newman's Inequality on $[a,b] \subset (0,\infty)$.

a] Let $\Lambda := (\lambda_j)_{j=0}^{\infty}$ be an increasing sequence of nonnegative real numbers. Assume that there exists an $\alpha > 0$ such that $\lambda_j \geq \alpha j$ for each j. Suppose that $[a,b] \subset (0,\infty)$. Show that there exists a constant $c(a,b,\alpha)$ depending only on a, b, and α such that

$$\|p'\|_{[a,b]} \leq c(a,b,\alpha)\left(\sum_{j=0}^{n}\lambda_j\right)\|p\|_{[a,b]}$$

for every $p \in M_n(\Lambda) := \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots x^{\lambda_n}\}$.

Proof. We base the proof on E.9 d], however; it may also be based on E.9 e]. Let $p \in M_n(\Lambda)$. We want to estimate $p'(y)$ for every $y \in [a,b]$. First let $y \in \left[\frac{1}{2}(a+b), b\right]$. We define $q(x) := x^{mn\alpha}p(x)$, where m is the smallest positive integer satisfying

$$a \le \frac{a+b}{2}\left(\frac{m}{m+1}\right)^{2/\alpha}.$$

Scaling Newman's inequality from $[0,1]$ to $[0,y]$, then using E.9 d], we obtain

$$\begin{aligned}
|q'(y)| &\le \frac{9}{y}\sum_{j=0}^{n}(\lambda_j + mn\alpha)\|q\|_{[0,y]} \\
&= \frac{9}{y}\sum_{j=0}^{n}(\lambda_j + mn\alpha)\|q\|_{\left[y\left(\frac{m}{m+1}\right)^{2/\alpha},y\right]} \\
&\le c_1(a,b,\alpha)\left(\sum_{j=0}^{n}\lambda_j\right)\|q\|_{[a,y]}
\end{aligned}$$

with a constant $c_1(a,b,\alpha)$ depending only on a, b, and α. Hence

$$\begin{aligned}
|p'(y)| &\le \left|q'(y)y^{-mn\alpha}\right| + \frac{mn\alpha}{y}|p(y)| \\
&\le y^{-mn\alpha}c_1(a,b,\alpha)\left(\sum_{j=0}^{n}\lambda_j\right)\|q\|_{[a,y]} + \frac{mn}{y}\|p\|_{[a,y]} \\
&\le c_2(a,b,\alpha)\left(\sum_{j=0}^{n}\lambda_j\right)\|p\|_{[a,y]} \\
&\le c_2(a,b,\alpha)\left(\sum_{j=0}^{n}\lambda_j\right)\|p\|_{[a,b]}
\end{aligned}$$

with a constant $c_2(a,b,\alpha)$ depending only on a, b, and α.

Now let $y \in \left[a, \frac{1}{2}(a+b)\right]$. Then, by E.3 b] and f] of Section 3.3, we can deduce that

$$\begin{aligned}
|p'(y)| &\le \left|T_n'\{x^0, x^\alpha, x^{2\alpha}, \ldots, x^{n\alpha}; [y,b]\}(y)\right| \|p\|_{[y,b]} \\
&= \frac{2\alpha y^{\alpha-1}}{b^\alpha - y^\alpha}n^2\|p\|_{[y,b]} \le c_3(a,b,\alpha)n^2\|p\|_{[y,b]} \\
&\le c_4(a,b,\alpha)\left(\sum_{j=0}^{n}\lambda_j\right)\|p\|_{[y,b]}
\end{aligned}$$

with constants $c_3(a,b,\alpha)$ and $c_4(a,b,\alpha)$ depending only on a, b, and α. This finishes the proof. □

b] Show that if the gap condition $\lambda_j \geq j\alpha$ in part a] is dropped, then the conclusion of part a] fails to hold with $c(a,b,\alpha)$ replaced by a constant $c(a,b)$ depending only on a and b.

Hint: Study

$$T_n' \left\{ x^0, x^\alpha, x^{2\alpha}, \dots, x^{n\alpha}; \left[\tfrac{1}{2}, 1\right] \right\}$$

and let $\alpha \to 0+$. □

6.2 Nondense Müntz Spaces

Throughout this section we assume that $\Lambda := (\lambda_i)_{i=0}^\infty$ is a sequence of distinct nonnegative real numbers, $M_n(\Lambda)$ denotes the linear span of $\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}$ over $\mathbb{R}$, and

$$M(\Lambda) := \bigcup_{n=0}^\infty M_n(\Lambda) = \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}.$$

If $A \subset [0,1]$ is compact, then a combination of Tietze's and Müntz's theorems yields that $M(\Lambda)$ is dense in $C(A)$ whenever $\sum_{i=1}^\infty 1/\lambda_i = \infty$. (Recall that Tietze's theorem guarantees that if $A \subset [0,1]$ is compact, then for every $f \in C(A)$ there exists an $\widetilde{f} \in C[0,1]$ such that $\widetilde{f}(x) = f(x)$ for all $x \in A$.) If the Lebesgue measure $m(A)$ of A is positive, then the converse is also true. More precisely, we have the following.

Theorem 6.2.1 (Müntz Theorem on Compact Sets of Positive Measure). *Suppose $\lambda_0 := 0$ and $\sum_{i=1}^\infty 1/\lambda_i < \infty$. Let $A \subset [0,1]$ be a compact set with positive Lebesgue measure. Then $M(\Lambda)$ is not dense in $C(A)$. Moreover, if the gap condition*

$$\inf\{\lambda_i - \lambda_{i-1} : i \in \mathbb{N}\} > 0$$

holds and

$$r_A := \sup\{x \in [0,\infty) : m(A \cap (x,\infty)) > 0\},$$

then every function $f \in C(A)$ from the uniform closure of $M(\Lambda)$ on A is of the form

$$f(x) = \sum_{j=0}^\infty a_j x^{\lambda_j}, \qquad x \in A \cap [0, r_A).$$

If the above gap condition does not hold, then every function $f \in C(A)$ from the uniform closure of $M(\Lambda)$ on A can still be extended analytically throughout the region

$$\{z \in \mathbb{C} \setminus (-\infty, 0] : |z| < r_A\}.$$

The proof of the above theorem rests on the following bounded Remez-type inequality:

Theorem 6.2.2 (Remez-Type Inequality for Nondense Müntz Spaces). *Let $\lambda_0 := 0$ and $\sum_{i=0}^{\infty} 1/\lambda_i < \infty$. Then there exists a constant c depending only on Λ and s (and not on ϱ, A, or the number of terms in p) such that*

$$\|p\|_{[0,\varrho]} \le c\,\|p\|_A$$

for every $p \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ and for every compact set $A \subset [\varrho, 1]$ of Lebesgue measure at least $s > 0$.

To prove Theorem 6.2.2 we need a few lemmas. We use the notation introduced in Section 3.3. However, for notational convenience, we let

$$T_n\{\lambda_0, \lambda_1, \dots, \lambda_n; A\} := T_n\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}; A\}$$

for compact sets $A \subset [0, \infty)$.

By the unique interpolation property of Chebyshev spaces (see Proposition 3.1.2), associated with

$$0 < x_0 < x_1 < \cdots < x_n\,,$$

we can define

$$\ell_k\{x_0, x_1, \dots, x_n\} \in M_n(\Lambda)\,, \qquad k = 0, 1, \dots, n$$

such that

$$\ell_k\{x_0, x_1, \dots, x_n\}(x_j) = \delta_{j,k} := \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \ne k\,. \end{cases}$$

Lemma 6.2.3. *Let*

$$0 < x_0 < x_1 < \cdots < x_n \qquad \text{and} \qquad 0 < \widetilde{x}_0 < \widetilde{x}_1 < \cdots < \widetilde{x}_n\,.$$

Suppose $0 \le k \le n$ and

$$\begin{array}{ll} x_j \le \widetilde{x}_j & \text{if } j = 0, 1, \dots, k-1\,; \\ x_j = \widetilde{x}_j & \text{if } j = k\,; \\ x_j \ge \widetilde{x}_j & \text{if } j = k+1, k+2, \dots, n\,. \end{array}$$

For notational convenience, let

$$\ell_k := \ell_k\{x_0, x_1, \dots, x_n\} \qquad \text{and} \qquad \widetilde{\ell}_k := \ell_k\{\widetilde{x}_0, \widetilde{x}_1, \dots, \widetilde{x}_n\}\,.$$

Then

$$|\ell_k(0)| \le |\widetilde{\ell}_k(0)|\,.$$

Proof. It is sufficient to prove the lemma in the case where there is an index m such that $1 \le m \le n$, $m \ne k$, and

$$\begin{aligned} x_j &= \widetilde{x}_j \quad \text{if } j = 0, 1, \dots, n, \ \ j \ne m\,; \\ x_m &< \widetilde{x}_m \quad \text{if } m < k\,; \\ x_m &> \widetilde{x}_m \quad \text{if } m > k\,. \end{aligned}$$

The general case of the lemma then follows from repeated applications of the above special cases. Note that in the above special cases

$$\ell_k - \widetilde{\ell}_k \in M_n(\Lambda)$$

has a zero at each of the points

$$x_j\,, \quad j = 0, 1, \dots, n\,, \ \ j \ne m\,;$$

hence it changes sign at each of these points, and it has no other zero in $[0, \infty)$ (see E.10 of Section 3.1). It is also obvious that

$$\operatorname{sign}(\ell_k(x)) = \operatorname{sign}(\widetilde{\ell}_k(x))\,, \qquad x \in (0, x_0)\,,$$

which, together with the previous observation and the inequality $x_0 \le \widetilde{x}_0$, yields that

$$|\ell_k(0)| \le |\widetilde{\ell}_k(0)|\,,$$

which finishes the proof. □

By a simple scaling we can extend Lemma 6.2.3 as follows. We use the notation introduced in Lemma 6.2.3.

Lemma 6.2.4. *Let*

$$0 < x_0 < x_1 < \cdots < x_n \qquad \textit{and} \qquad 0 < \widetilde{x}_0 < \widetilde{x}_1 < \cdots < \widetilde{x}_n\,.$$

Suppose $0 \le k \le n$, $\gamma \ge 0$, *and*

$$\begin{aligned} x_j &\le \widetilde{x}_j - \gamma \quad \textit{if} \ \ j = 0, 1, \dots, k-1\,; \\ x_j &= \widetilde{x}_j - \gamma \quad \textit{if} \ \ j = k\,; \\ x_j &\ge \widetilde{x}_j - \gamma \quad \textit{if} \ \ j = k+1, k+2, \dots, n\,. \end{aligned}$$

Then

$$|\ell_k(0)| \le |\widetilde{\ell}_k(0)|\,.$$

Proof. If $\gamma = 0$, then Lemma 6.2.3 yields the lemma. So we may assume that $\gamma > 0$. Let

$$\beta := \frac{x_k}{\widetilde{x}_k} = \frac{\widetilde{x}_k - \gamma}{\widetilde{x}_k}\,,$$
$$x_j^* := \beta \widetilde{x}_j\,, \qquad j = 0, 1, \dots, n\,,$$

and

$$\ell_k^* := \ell_k\{x_0^*, x_1^*, \dots, x_n^*\}\,, \qquad k = 0, 1, \dots, n\,.$$

Obviously

$$\widetilde{\ell}_k(\beta x) = \ell_k^*(x)\,, \qquad x \in [0, \infty)$$

and

$$\begin{aligned} x_j &\le x_j^* \quad \text{if } j = 0, 1, \dots, k-1\,,\\ x_j &= x_j^* \quad \text{if } j = k\,,\\ x_j &\ge x_j^* \quad \text{if } j = k+1, k+2, \dots, n\,. \end{aligned}$$

Hence Lemma 6.2.3 implies that

$$|\ell_k(0)| \le |\ell_k^*(0)| = |\widetilde{\ell}_k(0)|$$

which finishes the proof. □

Lemma 6.2.5. *Let A be a closed subset of $[0,1]$ with Lebesgue measure at least $s \in (0,1)$. Then*

$$|p(0)| \le |T_n\{\lambda_0, \lambda_1, \dots, \lambda_n; [1-s, 1]\}(0)| \cdot \|p\|_A$$

for every $p \in M(A)$.

Proof. If $0 \in A$, then the statement is trivial. So assume that $0 \notin A$. Let

$$\widetilde{x}_0 < \widetilde{x}_1 < \cdots < \widetilde{x}_n$$

denote the extreme points of

$$T_n := T_n\{\lambda_0, \lambda_1, \dots, \lambda_n; [1-s, 1]\}$$

in $[1-s, 1]$, that is,

$$T_n(\widetilde{x}_j) = (-1)^{n-j}\,, \qquad j = 0, 1, \dots, n\,.$$

Let $x_j \in A$, $j = 0, 1, \dots, n$, be defined so that

$$m([x_j, 1] \cap A) = m([\widetilde{x}_j, \widetilde{x}_n]) = \widetilde{x}_n - \widetilde{x}_j\,.$$

Since A is a closed subset of $[0,1]$ with $m(A) \ge s$, such points $x_j \in A$ exist. Let $p \in M_n(A)$. Then, by Lemma 6.2.4, we can deduce that

$$\begin{aligned}
|p(0)| &= \left| \sum_{k=0}^{n} p(x_k)\ell_k(0) \right| \\
&\le \left(\sum_{k=0}^{n} |\ell_k(0)| \right) \|p\|_A \\
&\le \left(\sum_{k=0}^{n} |\widetilde{\ell}_k(0)| \right) \|p\|_A \\
&= \left(\sum_{k=0}^{n} (-1)^{n-k} \widetilde{\ell}_k(0) \right) \|p\|_A \\
&= \left(\sum_{k=0}^{n} T_n(\widetilde{x}_k) \widetilde{\ell}_k(0) \right) \|p\|_A \\
&= |T_n(0)| \cdot \|p\|_A
\end{aligned}$$

which proves the lemma. □

Lemma 6.2.6. *Suppose* $\lambda_0 := 0$. *Let* A *be a closed subset of* $[0,1]$ *with Lebesgue measure at least* $s \in (0,1)$. *Then*

$$|p(y)| \le |T_n\{\lambda_0, \lambda_1, \dots, \lambda_n; [1-s,1]\}(0)| \cdot \|p\|_A$$

for every $p \in M_n(\Lambda)$ *and* $y \in [0, \inf A]$.

Proof. For notational convenience, let

$$T_{n,A} := T_n\{\lambda_0, \lambda_1, \dots, \lambda_n; A\}.$$

Note that $\lambda_0 = 0$ implies that $|T_{n,A}|$ is decreasing on $[0, \inf A]$, otherwise

$$T'_{n,A} \in \operatorname{span}\{x^{\lambda_1 - 1}, x^{\lambda_2 - 1}, \dots, x^{\lambda_n - 1}\}$$

would have at least $n+1$ zeros in $(0,1]$, which is impossible. Hence, it follows from E.3 and Lemma 6.2.5 that

$$\begin{aligned}
\frac{|p(y)|}{\|p\|_A} \le \frac{|T_{n,A}(y)|}{\|T_{n,A}\|_A} &= |T_{n,A}(y)| \le |T_{n,A}(0)| \\
&\le |T_n\{\lambda_0, \lambda_1, \dots, \lambda_n; [1-s,1]\}(0)|
\end{aligned}$$

for every $0 \ne p \in M_n(\Lambda)$. This finishes the proof. □

Proof of Theorem 6.2.2. Lemma 6.2.6 and E.5 a] of Section 4.2 yield the theorem. □

Proof of Theorem 6.2.1. The theorem follows from E.5 of this section and E.3 e] and E.8 b] of Section 4.2. □

Our next theorem is an interesting characterization of lacunary sequences.

Theorem 6.2.7 (Characterization of Lacunary Müntz Spaces). *Suppose* $\Lambda := (\lambda_i)_{i=0}^{\infty}$ *with* $0 \le \lambda_0 < \lambda_1 < \cdots$. *There exists a constant* c *depending only on* Λ *such that*

$$|a_{j,n}| \le c \, \|p\|_{[0,1]} , \qquad j = 0, 1, \ldots , n , \quad n \in \mathbb{N}$$

for every $p \in M(\Lambda)$ *of the form*

$$p(x) = \sum_{j=0}^{n} a_{j,n} x^{\lambda_j}$$

if and only if Λ *is lacunary, that is, if and only if the elements* λ_i *of* Λ *satisfy*

$$\inf\{\lambda_{i+1}/\lambda_i : i \in \mathbb{N}\} > 1 .$$

To prove Theorem 6.2.7 we need the following result of Hardy and Littlewood [26] whose proof we do not reproduce:

Theorem 6.2.8. *Suppose* $0 = \gamma_0 < \gamma_1 < \cdots$ *is a lacunary sequence, that is,*

$$\inf\{\gamma_{i+1}/\gamma_i : i \in \mathbb{N}\} > 1 .$$

Suppose the function f *is of the form*

$$f(x) = \sum_{i=0}^{\infty} a_i x^{\gamma_i} , \qquad a_i \in \mathbb{R} , \quad x \in [0,1)$$

and $A := \lim_{x \to 1-} f(x)$ *exists and is finite. Then* $\sum_{i=0}^{\infty} a_i = A$.

Proof of Theorem 6.2.7. Suppose Λ is lacunary and suppose there exists a sequence $(P_k)_{k=1}^{\infty} \subset M(\Lambda)$ such that if P_k is of the form

$$P_k(x) = \sum_{j=0}^{n_k} a_{j,n_k} x^{\lambda_j} , \qquad a_{j,n_k} \in \mathbb{R} ,$$

then

$$\|P_k\|_{[0,1]} = 1 \qquad \text{and} \qquad \max_{0 \le j \le n_k} |a_{j,n_k}| \ge k^2, \qquad k = 1, 2, \ldots . \tag{6.2.1}$$

We may assume, without loss of generality, that $\lambda_0 = 0$. Choose a sequence $(\alpha_k)_{k=1}^{\infty}$ of positive integers such that

$$\alpha_1 = 1 , \qquad \lambda_1 \alpha_{k+1} > 2 \alpha_k \lambda_{n_k} , \qquad k = 1, 2, \ldots .$$

Now let the function f be defined by

$$f(x) = \sum_{k=1}^{\infty} k^{-2} P_k(x^{\alpha_k}) \le \sum_{k=1}^{\infty} k^{-2} < \infty\,, \qquad 0 \le x \le 1\,. \tag{6.2.2}$$

Note that $f \in C[0,1]$ by the Weierstrass M-test. For notational convenience, let

$$m_0 := 0, \qquad m_k := \sum_{i=1}^{k} n_i\,, \qquad k = 1, 2, \ldots\,.$$

Further, let

$$\gamma_0 := 0 \qquad \text{and} \qquad a_0 := \sum_{k=1}^{\infty} k^{-2} a_{0,n_k} = \sum_{k=1}^{\infty} k^{-2} P_k(0)$$

and

$$\gamma_{m_{k-1}+j} := \alpha_k \lambda_j\,, \quad j = 1, 2, \ldots, n_k\,, \quad k = 1, 2, \ldots\,.$$

Observe that $a_0 \in \mathbb{R}$ is well-defined since $|P_k(0)| \le 1$ for each $k \in \mathbb{N}$. Also

$$\inf\{\gamma_{i+1}/\gamma_i : i \in \mathbb{N}\} \ge \min\{2, \inf\{\lambda_{i+1}/\lambda_i : i \in \mathbb{N}\}\} > 1\,.$$

Let $\Gamma := (\gamma_i)_{i=0}^{\infty}$. Then $f \in C[0,1]$ defined by (6.2.2) is in the uniform closure of $M(\Gamma)$ on $[0,1]$; hence, by the Clarkson-Erdős theorem (see E.3 e] of Section 4.2), f is of the form

$$f(x) = \sum_{i=0}^{\infty} a_i x^{\gamma_i}\,, \quad x \in [0,1)\,.$$

Since $f \in C[0,1]$, Theorem 6.2.8 implies that $A := \sum_{i=0}^{\infty} a_i$ exists and is finite. Recalling (6.2.2) and the choice of α_k, and using E.3 e] of Section 4.2, we can deduce that each

$$k^{-2} a_{j,n_k}\,, \qquad j = 1, 2, \ldots, n_k\,, \quad k = 1, 2, \ldots$$

is equal to one of the coefficients $a_1, a_2 \ldots$. Since $|a_{0,n_k}| = |P_k(0)| \le 1$ for each $k \in \mathbb{N}$, from (6.2.1) and (6.2.2) we see that $|a_i| \ge 1$ holds for infinitely many $i \in \mathbb{N}$, which contradicts the fact that $\sum_{i=0}^{\infty} a_i$ converges. This finishes the *if* part of the theorem.

Now assume that Λ is not lacunary. Then for every $\epsilon > 0$ there is an $n \in \mathbb{N}$ such that $\lambda_{n-1}/\lambda_n > 1 - \epsilon$. Observe that $P_n(x) := x^{\lambda_n} - x^{\lambda_{n-1}}$ achieves its maximum modulus on $[0,1]$ at

$$x = \left(\frac{\lambda_{n-1}}{\lambda_n}\right)^{1/(\lambda_n - \lambda_{n-1})}$$

and hence

$$\|P_n\|_{[0,1]} \le \left(\frac{\lambda_{n-1}}{\lambda_n}\right)^{\lambda_{n-1}/(\lambda_n-\lambda_{n-1})} \left(1-\frac{\lambda_{n-1}}{\lambda_n}\right) \le 1-\frac{\lambda_{n-1}}{\lambda_n} < \epsilon\,,$$

which shows that the lead coefficient $a_{n,n}$ of

$$T_n\{\lambda_0,\lambda_1,\dots,\lambda_n;[0,1]\}$$

is at least $1/\epsilon$, otherwise $a_{n,n}^{-1}T_n - P_n \in M_{n-1}(\Lambda)$ would have at least n zeros on $(0,1)$, which is a contradiction. The proof of the *only if* part of the theorem is now finished. □

From the above proof it also follows that under the assumptions of Theorem 6.2.7, the Chebyshev polynomials

$$T_n\{\lambda_0,\lambda_1,\dots,\lambda_n;[0,1]\}$$

have uniformly bounded coefficients if and only if Λ is lacunary.

As an application of Theorem 6.2.7 we derive the following Bernstein-type inequality.

Theorem 6.2.9 (Bernstein-Type Inequality). *Suppose $\lambda_0 := 0$, $\lambda_1 \ge 1$, and suppose $\Lambda := (\lambda_i)_{i=0}^{\infty}$ is lacunary, that is,*

$$\inf\{\lambda_{i+1}/\lambda_i : i \in \mathbb{N}\} > 1\,.$$

Then there exists a constant c depending only on Λ (and not on y or the number of terms in p) such that

$$|p'(y)| \le \frac{c}{1-y}\,\|p\|_{[0,1]}$$

for every $p \in M(\Lambda) = \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ and for every $y \in [0,1)$.

Proof. Let $p \in M(\Lambda)$ be of the form

$$p(x) = a_{0,n} + \sum_{j=1}^{n} a_{j,n}x^{\lambda_j}\,, \qquad \|p\|_{[0,1]} = 1\,.$$

Theorem 6.2.7 and the assumptions on Λ yield

$$\begin{aligned} |p'(y)| &\le \left|\sum_{j=1}^{n} a_{j,n}\lambda_j y^{\lambda_j-1}\right| \le \sum_{j=1}^{n} |a_{j,n}|\,\lambda_j y^{\lambda_j-1} \\ &\le c_1 \sum_{j=1}^{n} \lambda_j y^{\lambda_j-1} \le c_2 \sum_{j=0}^{\infty} y^j = \frac{c_2}{1-y}\,, \end{aligned}$$

where c_1 and c_2 depend only on Λ, and the theorem is proved. □

Comments, Exercises, and Examples.

The results of this section have been obtained by Borwein and Erdélyi [91, 93, 95b, to appear 1]. In E.1 we present some of the several important consequences of our central result, Theorem 6.2.2. In E.6 we offer another proof of the first part of Theorem 6.2.1 when Λ is lacunary, while E.9 shows that the Bernstein-type inequality of Theorem 6.2.9 "almost" characterizes the lacunary Müntz spaces. Note that if $A \subset [0,1]$ contains an interval, then the first part of Theorem 6.2.1 follows immediately from E.3 of Section 4.2. A typical case that does not follow from that exercise is when $A \subset [0,1]$ is a "fat" Cantor-type set of positive measure.

E.1 Some Consequences of Theorem 6.2.2. Let $A \subset [0,\infty)$ be a set of positive Lebesgue measure, and let r_A be the essential supremum of A as defined in Theorem 6.2.1. Suppose $q \in (0,\infty)$ and suppose w is a nonnegative-valued, integrable weight function on A with $\int_A w > 0$. Let $L_q(w) := L_q(\mu)$, where $d\mu = w\,dt$, and where $L_q(\mu)$ is defined in E.7 of Section 2.2. Let $\Lambda := (\lambda_i)_{i=0}^{\infty}$ be a sequence of distinct nonnegative real numbers with $\lambda_i \neq 0$ for each $i = 1, 2, \ldots$.

a] Suppose $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$. Then $M(\Lambda)$ is not dense in $L_q(w)$. Moreover, if the gap condition

$$\inf\{\lambda_i - \lambda_{i-1} : i \in \mathbb{N}\} > 0$$

holds, then every function $f \in L_q(w)$ belonging to the $L_q(w)$ closure of $M(\Lambda)$ can be represented as

$$f(x) = \sum_{i=0}^{\infty} a_i x^{\lambda_i}\,, \qquad a_i \in \mathbb{R}\,, \quad x \in A \cap [0, r_w)\,,$$

where

$$r_w := \sup\left\{ y \in [0,\infty) : \int_{A\cap(y,\infty)} w(x)\,dx > 0 \right\}\,.$$

If the above gap condition does not hold, then every function $f \in L_q(w)$ belonging to the $L_q(w)$ closure of $M(\Lambda)$ can still be represented as an analytic function on

$$\{z \in \mathbb{C} \setminus (-\infty, 0] : |z| < r_w\}$$

restricted to A.

Proof. Suppose $f \in L_q(w)$ and suppose there is a sequence $(p_i)_{i=1}^{\infty} \subset M(\Lambda)$ such that

$$\lim_{i\to\infty} \|f - p_i\|_{L_q(w)} = 0\,.$$

Minkowski's inequality (see E.7 b] and E.7 i] of Section 2.2.) yields that $(p_i)_{i=1}^{\infty}$ is a Cauchy sequence in $L_q(w)$. The assumptions on w imply that for every $\delta \in (0, r_w)$ there exists an $\alpha > 0$ such that

$$B := \{x \in A \cap (\delta, \infty) : w(x) > \alpha\}$$

is of positive Lebesgue measure. Let $s := m(B) > 0$. Observe that if $\|p\|_{L_q(w)} < \varepsilon$, then

$$m\left(\left\{x \in B : |p(x)| \geq \left(\frac{2\varepsilon}{\alpha s}\right)^{1/q}\right\}\right) \leq \frac{s}{2},$$

so

$$m\left(\left\{x \in B : |p(x)| < \left(\frac{2\varepsilon}{\alpha s}\right)^{1/q}\right\}\right) > \frac{s}{2}.$$

Hence, by Theorem 6.2.2, $(p_i)_{i=1}^{\infty}$ is uniformly Cauchy on $[0, \delta]$. The proof can now be finished as that of Theorem 6.2.1. □

b] Müntz-Type Theorem in $L_q(w)$. *$M(\Lambda)$ is dense in $L_q(w)$ if and only if $\sum_{i=1}^{\infty} 1/\lambda_i = \infty$.*

Proof. Suppose $\sum_{i=1}^{\infty} 1/\lambda_i = \infty$. Let $f \in L_q(w)$. It is standard measure theory to show that for every $\varepsilon > 0$ there exists a $g \in C[0,1]$ such that $g(0) = 0$ and

$$\|f - g\|_{L_q(w)} < \frac{\varepsilon}{2}.$$

Now Theorem 4.2.1 (full Müntz theorem in $C[0,1]$) implies that there exists a $p \in M(\Lambda)$ such that

$$\|g - p\|_{L_q(w)} \leq \|g - p\|_A \left(\int_A w\right)^{1/q} \leq \|g - p\|_{[0,1]} \left(\int_A w\right)^{1/q} < \frac{\varepsilon}{2}.$$

Therefore $M(\Lambda)$ is dense in $L_q(w)$.

Suppose now that $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$. Then part a] yields that $M(\Lambda)$ is not dense in $L_q(w)$. □

c] Convergence in $M(\Lambda)$. Suppose $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$, $(p_i)_{i=1}^{\infty} \subset M(\Lambda)$, and

$$p_i(x) \to f(x), \qquad x \in A.$$

Then $(p_i)_{i=1}^{\infty}$ converges uniformly on every closed subinterval of $[0, r_A)$.

Proof. Let $\delta \in (0, r_A)$ be fixed. Egoroff's theorem (see, for example, Royden [88]) and the definition of r_A imply the existence of a set $B \subset A \cap (\delta, \infty)$ of positive Lebesgue measure so that $(p_i)_{i=1}^{\infty}$ converges uniformly on B and hence is uniformly Cauchy on B. Now Theorem 6.2.2 yields that $(p_i)_{i=1}^{\infty}$ is uniformly Cauchy on $[0, \delta]$, and the result follows. □

d] Suppose $\sum_{i=1}^{\infty} 1/\lambda_i = \infty$. Show that there is a sequence $(p_i)_{i=1}^{\infty} \subset M(\Lambda)$ that converges pointwise on $[0,\infty)$ but does not converge uniformly on $A \cap [0,a]$ for some $a \in (0, r_A)$.

Hint: Use Theorem 4.2.1 (Müntz's theorem). □

e] Suppose $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$ and

$$\inf\{\lambda_i - \lambda_{i-1} : i \in \mathbb{N}\} > 0\,.$$

Let $P(\Lambda)$ denote the collection of all real-valued functions f defined on $[0,1)$ by a power series

$$f(x) = \sum_{i=0}^{\infty} a_i x^{\lambda_i}\,, \qquad a_i \in \mathbb{R}\,, \quad x \in [0,1)\,.$$

Suppose that $A \subset [0,1]$ with $r_A = 1$. Show that if $(f_i)_{i=1}^{\infty} \subset P(\Lambda)$ and

$$f_i(x) \to f(x)\,, \qquad x \in A\,,$$

then

$$f_i(x) \to \widetilde{f}(x)\,, \qquad x \in [0,1)\,,$$

where $\widetilde{f} \in P(\Lambda)$.

Hint: Use part c] and E.3 e] of Section 4.2. □

E.2 On the Smallest Zero of Chebyshev Polynomials in Nondense Müntz Spaces. Suppose $\lambda_0 := 0$ and $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$. Show that there exists a constant $c > 0$ depending only on $\Lambda := (\lambda_i)_{i=0}^{\infty}$ (and not on n) such that the smallest positive zero of

$$T_n\{0, \lambda_1, \lambda_2, \ldots, \lambda_n; [0,1]\}\,, \qquad n = 1, 2, \ldots$$

is greater than c.

Hint: If $\lambda_1 \geq 1$, then use the Mean Value Theorem, E.1 a] of Section 3.3, and E.5 b] of Section 4.2. If $0 < \lambda_1 < 1$, then the scaling $x \to x^{1/\lambda_1}$ reduces the problem to the case $\lambda_1 = 1$. □

E.3 Extremal Functions for the Remez-Type Inequality of Theorem 6.2.2. Suppose $0 \leq \lambda_0 < \lambda_1 < \cdots < \lambda_n$, $0 < \varrho$, $A \subset [\varrho, \infty)$ is a compact set containing at least $n+1$ points, and $y \in (0, \varrho)$ is fixed. Let

$$M_n(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_n}\}\,.$$

a] Show that there is a $0 \neq p^* \in M_n(\Lambda)$ such that

$$\frac{|p^*(y)|}{\|p^*\|_A} = \sup_{0 \neq p \in M_n(\Lambda)} \frac{|p(y)|}{\|p\|_A}\,.$$

Hint: Use a compactness argument. □

b] Show that $p^* = cT_n\{\lambda_0, \lambda_1, \ldots, \lambda_n; A\}$ for some $c \in \mathbb{R}$.

Hint: Use a perturbation argument. □

E.4 A Lexicographic Property of Chebyshev Polynomials in Different Müntz Spaces. Let $0 := \lambda_0 < \lambda_1 < \cdots < \lambda_n$, $0 := \gamma_0 < \gamma_1 < \cdots < \gamma_n$, and

$$\lambda_j \le \gamma_j\,, \qquad j = 0, 1, \dots, n\,.$$

Let $\varrho > 0$ and let $A \subset [\varrho, \infty)$ be compact containing at least $n+1$ points, and let

$$T_{n,\Lambda} := T_n\{\lambda_0, \lambda_1, \dots, \lambda_n; A\} \quad \text{and} \quad T_{n,\Gamma} := T_n\{\gamma_0, \gamma_1, \dots, \gamma_n; A\}\,.$$

a] Show that $|T_{n,\Gamma}(y)| \le |T_{n,\Lambda}(y)|$ for every $y \in [0, \varrho)$.

Hint: Suppose, without loss of generality, that there is an index m, $1 \le m \le n$, such that $\lambda_m < \gamma_m$ and $\lambda_j = \gamma_j$ if $j \ne m$. We choose an $R_{n,\Lambda} \in M_n(\Lambda)$ that interpolates $T_{n,\Gamma}$ at the n zeros of $T_{n,\Gamma}$ and is normalized so that $R_{n,\Lambda}(0) = T_{n,\Gamma}(0)$. Use Theorem 3.2.5 to show that $|R_{n,\Lambda}(x)| \le |T_{n,\Gamma}(x)|$ for every $x \in [0, \infty)$, in particular for every $x \in A$. Now use E.3 to show that $|T_{n,\Gamma}(0)| = |R_{n,\Lambda}(0)| \le |T_{n,\Lambda}(0)|$, which gives the desired result for $y = 0$. Using this, we can deduce that $|T_{n,\Gamma}(y)| \le |T_{n,\Lambda}(y)|$ for every $y \in [0, \varrho)$, otherwise

$$T_{n,\Lambda} - T_{n,\Gamma} \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}, x^{\gamma_m}\}$$

would have at least $(n+2)$ zeros in $(0, \infty)$, which is a contradiction. □

b] Show that

$$\max_{p \in M_n(\Gamma)} \frac{|p(y)|}{\|p\|_A} \le \max_{p \in M_n(\Lambda)} \frac{|p(y)|}{\|p\|_A}$$

for every $y \in [0, \varrho)$, where

$$M_n(\Gamma) := \operatorname{span}\{x^{\gamma_0}, x^{\gamma_1}, \dots, x^{\gamma_n}\}$$

and

$$M_n(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}\,.$$

Hint: Combine part a] and E.4. □

E.5 Theorem 6.2.1 Follows from Theorem 6.2.2. Under the assumptions of Theorem 6.2.1 show that if $(p_j)_{j=1}^{\infty} \subset M(\Lambda)$ is uniformly Cauchy in $C(A)$, then it is uniformly Cauchy in $C[0, y]$ for every $y \in (0, r_A)$, where r_A is defined as in Theorem 6.2.1.

Hint: Use Theorem 6.2.2. □

E.6 Some Corollaries of Theorem 6.2.9 in the Lacunary Case. Suppose $\lambda_0 := 0$, $\lambda_1 \geq 1$, and $\Lambda := (\lambda_i)_{i=0}^{\infty}$ is lacunary.

a] Show that the Chebyshev polynomials

$$T_n := T_n\{\lambda_0, \lambda_1, \dots, \lambda_n : [0,1]\}, \qquad n = 1, 2, \dots$$

have the following property: There is a constant $c_1 \in (0,1)$ depending only on Λ (and not on n) such that if $y \in [0,1)$ and $|T_n(y)| = 1$, then $|T_n(x)| \geq \frac{1}{2}$ for every $x \in [y, y + c_1(1-y)]$.

Hint: Use the Mean Value Theorem and the Bernstein-type inequality of Theorem 6.2.9. □

b] Show that there is a constant $c_2 \in (0,1)$ depending only on Λ (and not on n) so that if $a < b$ are two consecutive zeros of T_n, then $1 - b < c_2(1-a)$.

c] Let $\epsilon \in (0,1)$. Show that there is an $n_0 \in \mathbb{N}$ depending only on ϵ (and not on n) so that every T_n has at most n_0 zeros in $[0, 1-\epsilon]$.

d] Give a new proof of the first part of Theorem 6.2.1 based on parts a] and c].

Outline. By Lebesgue's density theorem (see Royden [88]), it may be supposed, without loss of generality, that the left-hand side Lebesgue density of A at 1 is 1. Choose $\epsilon \in (0,1)$ so that $A \cap [0, 1-\epsilon)$ contains infinitely many points and

$$\frac{m(A \cap [y,1])}{1-y} > c_1 \tag{6.2.5}$$

for every $y \in [1-\epsilon, 1]$, where $c_1 \in (0,1)$ is the same as in part a]. For this ϵ, choose n_0 according to part c]. Now define $g \in C(A)$ so that g alternates $n_0 + 3$ times in $A \cap [0, 1-\epsilon)$ between 2 and -2 and is identically zero on $[1-\epsilon, 1]$. Assume that there exists a $p \in M_n(\Lambda)$ such that $\|p - g\|_A \leq \frac{1}{4}$. Use part a] and (6.2.5) to show that $p - T_n \in M_n(\Lambda)$ has more than n distinct zeros in $[0,1]$, which is a contradiction. □

The following simple application of Theorem 6.2.9 was pointed out by Wojcieszyk:

e] Suppose $A \subset [0,1]$ is a measurable set and the left-hand side Lebesgue density of A at 1 is 1. Show that there is a constant $c > 0$ depending only on Λ and A so that

$$\|p\|_{[0,1]} \leq c \, \|p\|_A$$

for every $p \in M(\Lambda) = \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$.

Hint: Use the Mean Value Theorem, the Bernstein-type inequality of Theorem 6.2.9, and the Chebyshev-type inequality of E.3 f] of Section 4.2. □

f] Use part e] to give another proof of Theorem 6.2.1.

The following exercise constructs quasi-Chebyshev polynomials P_n for $M_n(\Lambda)$ if the lacunarity constant of Λ is large:

E.7 Quasi-Chebyshev Polynomials in Very Lacunary Müntz Spaces. Let $\lambda_0 = 0$, $\lambda_1 = 2$, and $\lambda_{i+1}/\lambda_i \geq 16$ for $i = 1, 2, \ldots$. Let

$$P_n(x) := 1 + 2\sum_{j=1}^{n}(-1)^j x^{\lambda_j}, \qquad n = 1, 2, \ldots .$$

Let $y_i := (4\lambda_i)^{-1}$. Prove the following statements:

a] $\|P_n\|_{[0,1]} = 1$ and $P_n(1) = (-1)^n$.

b] P_n has exactly n zeros, $x_{1,n} < x_{2,n} < \cdots < x_{n,n}$, in $(0,1)$.

c] $|P_n'(\xi)| \leq 2\lambda_n$ for every $\xi \in [x_{n,n}, 1]$.

d] We have

$$P_n(x) \leq -\frac{1}{5} \quad \text{if} \quad 1 - y_{2k} \leq x \leq 1 - \frac{y_{2k}}{2} \quad \text{and} \quad 1 \leq 2k \leq n$$

and

$$P_n(x) \geq \frac{1}{5} \quad \text{if} \quad 1 - y_{2k+1} \leq x \leq 1 - \frac{y_{2k+1}}{2} \quad \text{and} \quad 1 \leq 2k+1 \leq n\,.$$

Hint: Part a] is obvious. Prove the rest together, by induction on n. □

The next exercise follows Borwein and Erdélyi [95b].

E.8 Products of Müntz Spaces. Associated with $\Lambda := (\lambda_j)_{j=0}^{\infty}$, let

$$M^k(\Lambda) := \left\{ p = \prod_{j=1}^{k} p_j : p_j \in M(\Lambda) \right\}, \qquad k = 1, 2, \ldots .$$

Is $M^2(\Lambda)$ dense in $C[0,1]$ for $\Lambda := (j^2)_{j=0}^{\infty}$?

Note that $M^k(\Lambda)$, $k \geq 2$ is not the linear span of monomials, and Müntz's theorem does not give the answer. This exercise establishes Remez-, Bernstein-, and Nikolskii-type inequalities for $M^k(\Lambda)$. From any of these it follows immediately that if $\sum_{j=1}^{\infty} 1/\lambda_j < \infty$ and $A \subset [0,1]$ is a set of positive Lebesgue measure, then $M^k(\Lambda)$ is not dense in $C(A)$.

Throughout parts a] to d] of the exercise we assume $0 = \lambda_0 < \lambda_1 < \cdots$, $\sum_{j=1}^{\infty} 1/\lambda_j < \infty$, and $s \in (0,1)$.

a] Remez-Type Inequality for $M^k(\Lambda)$. There exists a constant c depending only on Λ, s, and k (and not on ϱ or A) such that

$$\|p\|_{[0,\varrho]} \leq c\,\|p\|_A$$

for every $p \in M^k(\Lambda)$ and for every compact set $A \subset [\varrho, 1]$ of Lebesgue measure at least $s > 0$.

Proof. Theorem 6.2.2 implies that there exists a constant $\alpha > 0$ depending only on Λ, s, and k such that

$$m(\{x \in [y,1] : |p(x)| > \alpha^{-1}|p(y)|\}) \geq 1 - y - \frac{s}{2k}$$

for every $p \in M(\Lambda)$ and $y \in [0, 1-s]$. Now let $p \in M^k(\Lambda)$, that is,

$$p = \prod_{j=1}^{k} p_j\,, \qquad p_j \in M(\Lambda)\,.$$

Then, for every $y \in [0, 1-s]$,

$$\begin{aligned} m(\{x \in [y,1] : |p(x)| &> \alpha^{-k}|p(y)|\}) \\ &\geq m\left(\bigcap_{j=1}^{k} \{x \in [y,1] : |p_j(x)| > \alpha^{-1}|p_j(y)|\}\right) \\ &\geq 1 - y - k\frac{s}{2k} = 1 - y - \frac{s}{2}\,. \end{aligned}$$

Hence $y \in [0, \inf A]$ and $m(A) \geq s$ imply that

$$m(\{x \in A : |p(x)| > \alpha^{-k}|p(y)|\}) \geq \frac{s}{2} > 0$$

and the inequality follows with $c = \alpha^k$. □

b] Solution to Newman's Problem. Let $A \subset [0,1]$ be a set of positive Lebesgue measure. Then $M^k(\Lambda)$ is not dense in $C(A)$.

Proof. This follows from part a]. □

c] Bernstein-Type Inequality for $M^k(\Lambda)$. Suppose $\lambda_1 \geq 1$. There exists a constant c depending only on Λ, s, and k (and not on ϱ and A) such that

$$\|p'\|_{[0,\varrho]} \leq c\,\|p\|_A$$

for every $p \in M^k(\Lambda)$ and for every compact set $A \subset [\varrho, 1]$ of Lebesgue measure at least $s > 0$.

Hint: Use the product rule of differentiation, and estimate each term separately. Proceed as in the proof of part a]. Use Theorem 6.2.2 and E.5 a] and b] of Section 4.2. □

d] Nikolskii-Type Inequality for $M^k(\Lambda)$. There exists a constant c depending only on Λ, s, k, q, and w (and not on ϱ and A) such that

$$\|p\|_{[0,\varrho]}^q \leq c \int_A |p(x)|^q w(x)\,dx$$

for every $p \in M^k(\Lambda)$, for every compact set $A \subset [\varrho, 1]$ of Lebesgue measure at least $s > 0$, for every function w measurable and positive a.e. on $[0,1]$, and for every $q \in (0, \infty)$.

Hint: Use part a]. □

e] Associated with

$$\Lambda_j = (\lambda_{i,j})_{i=0}^{\infty}, \qquad j = 1, 2, \dots, k,$$

let

$$M(\Lambda_1, \Lambda_2, \dots, \Lambda_k) := \left\{ p = \prod_{j=1}^{k} p_j : p_j \in M(\Lambda_j) \right\}.$$

Formulate and prove the analogs of parts a] to d] for $M(\Lambda_1, \Lambda_2, \dots, \Lambda_k)$.

E.9 A Weak Converse of Theorem 6.2.9. Suppose $\Lambda := (\lambda_i)_{i=0}^{\infty}$ is a (strictly) increasing sequence of nonnegative real numbers with $\lambda_0 := 0$ and $\lambda_1 \geq 1$. Suppose also that there exists a constant c depending only on Λ (and not on y or the number of terms in p) such that

$$|p'(y)| \leq \frac{c}{1-y} \|p\|_{[0,1]}$$

for every $p \in M(\Lambda) = \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ and for every $y \in [0,1)$. Show that there is a constant $\lambda > 1$ depending only on Λ such that $\lambda_n \geq \lambda^n$.

Outline. Let

$$T_n := T_n\{\lambda_0, \lambda_1, \dots, \lambda_n; [0,1]\}$$

and denote its zeros in $(0,1)$ by $x_{1,n} > x_{2,n} > \cdots > x_{n,n}$. Use the Mean Value Theorem and the assumed Bernstein-type inequality to show that there is a constant $\gamma \in (0,1)$ depending only on Λ such that

$$1 - x_{j,n} \leq \gamma(1 - x_{j+1,n}), \qquad j = 1, 2, \dots, n-1, \quad n \in \mathbb{N};$$

hence $1 - x_{1,n} \leq \gamma^n$. On the other hand, use the Mean Value Theorem and Theorem 6.1.1 (Newman's inequality) to show that

$$1 - x_{1,n} \geq \left(1 + 9\sum_{j=1}^{n} \lambda_j\right)^{-1} \geq (9(n+1)\lambda_n)^{-1}.$$

Finally, combine the lower and upper bounds for $1 - x_{1,n}$ to conclude that

$$\lambda_n \geq \frac{\gamma^{-n}}{9(n+1)}.$$

□

E.10 Polynomials in x^{λ_n}. Given $n \in \mathbb{N}$ and $\lambda_n \in \mathbb{R}$, let

$$\mathcal{P}_n(\lambda_n) := \{p_n(x^{\lambda_n}) : p_n \in \mathcal{P}_n\}$$

(as in E.6 of Section 4.1). Suppose $\lambda_n \geq 1$ for all $n \in \mathbb{N}$. Let $\delta \in \mathbb{R}$ be defined by

$$\limsup_n \frac{\log n}{\lambda_n} = \frac{1}{2} \log \frac{1}{\delta}\,.$$

Suppose $\delta > 0$.

a] Bounded Remez-Type Inequality. Suppose $0 < \widetilde{\delta} < \delta$. Show that there exists a constant c depending only on $\widetilde{\delta}$ (and not on n, y, or A) such that

$$|p(y)| \leq c \, \|p\|_A$$

for every $p \in \cup_{n=1}^{\infty} \mathcal{P}_n(\lambda_n)$, for every $A \subset [0,1]$ of Lebesgue measure at least $1 - \widetilde{\delta}$, and for every $y \in [0, \inf A]$.

Hint: Use Lemma 6.2.6 and E.6 a] of Section 4.1. □

b] Müntz-Type Theorem. *If $0 \leq \widetilde{\delta} < \delta$ and $A \subset [0,1]$ is a set of Lebesgue measure at least $1 - \widetilde{\delta}$, then $\cup_{n=1}^{\infty} \mathcal{P}_n(\lambda_n)$ is not dense in $C(A)$.*

Hint: Use part a]. □