7

Inequalities for Rational Function Spaces

Overview

Precise Markov- and Bernstein-type inequalities are given for various classes of rational functions in the first section of this chapter. Extensions of the inequalities of Lax, Schur, and Russak are also presented, as are inequalities for self-reciprocal polynomials. The second section of the chapter is concerned with metric inequalities for polynomials and rational functions.

7.1 Inequalities for Rational Function Spaces

Sharp extensions of most of the polynomial inequalities of Section 5.1 are established for rational function spaces on $K := \mathbb{R} \pmod{2\pi}$, on the interval $[-1, 1]$, on the unit circle of $\mathbb{C}$, and on the real line. The classical inequalities of Section 5.1 are then recovered as limiting cases. A sharp extension of Lax's inequality is also given. Essentially sharp Markov- and Bernstein-type inequalities for self-reciprocal and antiself-reciprocal polynomials are presented in the exercises.

Let $D := \{z \in \mathbb{C} : |z| < 1\}$ and $\partial D := \{z \in \mathbb{C} : |z| = 1\}$, as before. We study the rational function spaces:

$$\mathcal{T}_n(a_1, a_2, \ldots, a_{2n}; K) := \left\{ \frac{t(\theta)}{\prod_{k=1}^{2n} |\sin((\theta - a_k)/2)|} : t \in \mathcal{T}_n \right\}$$

and

$$\mathcal{T}_n^c(a_1, a_2, \ldots, a_{2n}; K) := \left\{ \frac{t(\theta)}{\prod_{k=1}^{2n} \sin((\theta - a_k)/2)} : t \in \mathcal{T}_n^c \right\}$$

on K with $a_1, a_2, \ldots, a_{2n} \in \mathbb{C} \setminus \mathbb{R}$;

$$\mathcal{P}_n(a_1, a_2, \ldots, a_n; [-1,1]) := \left\{ \frac{p(x)}{\prod_{k=1}^{n} |x - a_k|} : p \in \mathcal{P}_n \right\}$$

and

$$\mathcal{P}_n^c(a_1, a_2, \ldots, a_n; [-1,1]) := \left\{ \frac{p(x)}{\prod_{k=1}^{n} (x - a_k)} : p \in \mathcal{P}_n^c \right\}$$

on $[-1,1]$ with $a_1, a_2, \ldots, a_n \in \mathbb{C} \setminus [-1,1]$;

$$\mathcal{P}_n^c(a_1, a_2, \ldots, a_n; \partial D) := \left\{ \frac{p(z)}{\prod_{k=1}^{n} (z - a_k)} : p \in \mathcal{P}_n^c \right\}$$

on ∂D with $a_1, a_2, \ldots, a_n \in \mathbb{C} \setminus \partial D$; and

$$\mathcal{P}_n(a_1, a_2, \ldots, a_n; \mathbb{R}) := \left\{ \frac{p(x)}{\prod_{k=1}^{n} |x - a_k|} : p \in \mathcal{P}_n \right\}$$

and

$$\mathcal{P}_n^c(a_1, a_2, \ldots, a_n; \mathbb{R}) := \left\{ \frac{p(z)}{\prod_{k=1}^{n} (z - a_k)} : p \in \mathcal{P}_n^c \right\}$$

on $\mathbb{R}$ with $a_1, a_2, \ldots, a_n \in \mathbb{C} \setminus \mathbb{R}$.

The Chebyshev polynomials $\widetilde{T}_n$, $\widetilde{U}_n$, and

$$V := (\cos \alpha)\widetilde{T}_n + (\sin \alpha)\widetilde{U}_n\,, \qquad \alpha \in K$$

for the rational function space $\mathcal{T}_n(a_1, a_2, \ldots, a_{2n}; K)$ are defined in E.3 of Section 3.5, and they play a central role in this section.

Theorem 7.1.1 (Bernstein-Szegő-Type Inequality on K). *Given*

$$(a_k)_{k=1}^{2n} \subset \mathbb{C} \setminus \mathbb{R}, \qquad \operatorname{Im}(a_k) > 0,$$

let

$$\widetilde{B}_n(\theta) := \frac{1}{2} \sum_{k=1}^{2n} \frac{1 - |e^{ia_k}|^2}{|e^{ia_k} - e^{i\theta}|^2}.$$

Then

$$f'(\theta)^2 + \widetilde{B}_n^2(\theta) f^2(\theta) \le \widetilde{B}_n^2(\theta) \|f\|_K^2, \qquad \theta \in K$$

for every $f \in \mathcal{T}_n(a_1, a_2, \ldots, a_{2n}; K)$.

Equality holds if and only if either θ is a maximum point of $|f|$ (that is, $f(\theta) = \pm\|f\|_K$) or f is a linear combination of $\widetilde{T}_n$ and $\widetilde{U}_n$ (with real coefficients) as defined in E.3 of Section 3.5.

Corollary 7.1.2 (Bernstein-Type Inequality on K, Real Case). *Given*

$$(a_k)_{k=1}^{2n} \subset \mathbb{C} \setminus \mathbb{R}, \qquad \operatorname{Im}(a_k) > 0,$$

let the Bernstein factor $\widetilde{B}_n$ be defined as in Theorem 7.1.1. Then

$$|f'(\theta)| \le \widetilde{B}_n(\theta) \|f\|_K, \qquad \theta \in K$$

for every $f \in \mathcal{T}_n(a_1, a_2, \ldots, a_{2n}; K)$.

Equality holds if and only if f is a linear combination of $\widetilde{T}_n$ and $\widetilde{U}_n$ (with real coefficients) as defined in E.3 of Section 3.5, and $f(\theta) = 0$.

Theorem 7.1.1 and Corollary 7.1.2 can be easily obtained from the extension of Theorem 3.5.3 given by E.3 of Section 3.5, which gives explicit formulas for the Chebyshev polynomials for these classes $\mathcal{T}_n(a_1, a_2, \ldots, a_{2n}; K)$. The arguments are outlined in E.1.

The following two results can be obtained from Theorem 7.1.1 and Corollary 7.1.2 by the substitution $x = \cos\theta$; see E.2.

Corollary 7.1.3 (Bernstein-Szegő-Type Inequality on $[-1,1]$). *Associated with $(a_k)_{k=1}^{n} \subset \mathbb{C} \setminus [-1,1]$, let the Bernstein factor B_n be defined by*

$$B_n(x) := \sum_{k=1}^{n} \operatorname{Re}\left(\frac{\sqrt{a_k^2 - 1}}{a_k - x} \right),$$

where the choice of $\sqrt{a_k^2 - 1}$ is determined by $\left|a_k - \sqrt{a_k^2 - 1}\right| < 1$. Then

$$(1 - x^2) f'(x)^2 + B_n^2(x) f^2(x) \le B_n^2(x) \|f\|_{[-1,1]}^2, \qquad x \in [-1,1]$$

for every $f \in \mathcal{P}_n(a_1, a_2, \ldots, a_n; [-1,1])$.

Equality holds if and only if either x is a maximum point of $|f|$ (that is, $f(x) = \pm\|f\|_{[-1,1]}$) or $f = cT_n$ with $c \in \mathbb{R}$, where T_n is defined as in Section 3.5.

Corollary 7.1.4 (Bernstein-Type Inequality on $[-1,1]$, Real Case). *Given $(a_k)_{k=1}^n \subset \mathbb{C}\setminus[-1,1]$, let the Bernstein factor B_n be defined as in Corollary 7.1.3. Then*

$$|f'(x)| \le \frac{B_n(x)}{\sqrt{1-x^2}}\|f\|_{[-1,1]}\,, \qquad x \in (-1,1)$$

for every $f \in \mathcal{P}_n(a_1, a_2, \dots, a_n; [-1,1])$.

Equality holds if and only if $f = cT_n$ with $c \in \mathbb{R}$, where T_n is defined as in Section 3.5, and $f(x) = 0$. (Note that $B_n(x) > 0$ for every $x \in (-1,1)$.)

Our next result follows from Theorem 7.1.1; see the hints to E.3.

Corollary 7.1.5 (Bernstein-Szegő-Type Inequality on $\mathbb{R}$). *Given*

$$(a_k)_{k=1}^n \subset \mathbb{C}\setminus\mathbb{R}\,, \qquad \operatorname{Im}(a_k) > 0\,,$$

let the Bernstein factor B_n be defined by

$$B_n(x) := \sum_{k=1}^n \frac{\operatorname{Im}(a_k)}{|x-a_k|^2}\,.$$

Then

$$f'(x)^2 + B_n^2(x)f^2(x) \le B_n^2(x)\|f\|_{\mathbb{R}}^2\,, \qquad x \in \mathbb{R}$$

for every $f \in \mathcal{P}_n(a_1, a_2, \dots, a_n; \mathbb{R})$.

Equality holds if and only if either x is a maximum point of $|f|$ (that is, $|f(x)| = \pm\|f\|_{\mathbb{R}}$) or f is a linear combination of T_n and U_n (with real coefficients) defined in E.5 of Section 3.5.

Corollary 7.1.6 (Bernstein-Type Inequality on $\mathbb{R}$, Real Case). *Associated with $(a_k)_{k=1}^n \subset \mathbb{C}\setminus\mathbb{R}$, let the Bernstein factor B_n be defined as in Corollary 7.1.5. Then*

$$|f'(x)| \le B_n(x)\|f\|_{\mathbb{R}}\,, \qquad x \in \mathbb{R}$$

for every $f \in \mathcal{P}_n(a_1, a_2, \dots, a_n; \mathbb{R})$.

Equality holds if and only if f is a linear combination of T_n and U_n (with real coefficients) defined in E.5 of Section 3.5, and $f(x) = 0$.

To formulate our next theorem we introduce some notation. For a polynomial

$$q(z) := \prod_{k=1}^n (z - a_k)\,, \qquad a_k \in \mathbb{C},$$

we define

$$q^*(z) := \prod_{k=1}^{n} (1 - \overline{a}_k z) =: z^n \overline{q}_n(z^{-1}) .$$

Then

$$|q(z)| = |q^*(z)| , \qquad z \in \partial D . \tag{7.1.1}$$

The function

$$S_n(z) := \frac{q^*(z)}{q(z)} = \prod_{k=1}^{n} \frac{1 - \overline{a}_k z}{z - a_k}$$

is the Blaschke product associated with $(a_k)_{k=1}^n$.

Theorem 7.1.7 (Bernstein-Type Inequality on ∂D, Complex Case). *Given $(a_k)_{k=1}^n \subset \mathbb{C} \setminus \partial D$, let the Bernstein factor B_n be defined by*

$$B_n(z) := \max\{B_n^+(z), B_n^-(z)\}$$

with

$$B_n^+(z) := \sum_{\substack{k=1 \\ |a_k|>1}}^{n} \frac{|a_k|^2 - 1}{|a_k - z|^2} \qquad \text{and} \qquad B_n^-(z) := \sum_{\substack{k=1 \\ |a_k|<1}}^{n} \frac{1 - |a_k|^2}{|a_k - z|^2} .$$

Then

$$|f'(z)| \le B_n(z) \|f\|_{\partial D} , \qquad z \in \partial D$$

for every $f \in \mathcal{P}_n^c(a_1, a_2, \dots, a_n; \partial D)$.

If the first sum is not less than the second sum for a fixed $z \in \partial D$, then equality holds for $f = cS_n^+$ with $c \in \mathbb{C}$, where S_n^+ is the Blaschke product associated with those a_k for which $|a_k| > 1$. If the second sum is not less than the first sum for a fixed $z \in \partial D$, then equality holds for $f = cS_n^-$ with $c \in \mathbb{C}$, where S_n^- is the Blaschke product associated with those a_k for which $|a_k| < 1$.

Proof. For reasons of symmetry it is sufficient to prove the theorem only for $z = 1$. Without loss of generality we may assume that

$$\operatorname{Re}\left(\sum_{k=1}^{n} \frac{1}{1 - a_k} \right) \ne \frac{n}{2} ; \tag{7.1.2}$$

the remaining cases follow from this by a limiting argument. Let $Q := \partial D$ (equipped with the usual metric topology), $V := \mathcal{P}_n^c(a_1, a_2, \dots, a_n; \partial D)$, and $L(f) := f'(1)$ for $f \in V$. We show in this situation that $n + 1 \le r$ in Theorem A.3.3 (interpolation of linear functionals). Suppose to the contrary that $r \le n$. By Theorem A.3.3, there are distinct points $x_1, x_2, \dots, x_r$ on ∂D, and there are constants $c_1, c_2, \dots, c_r \in \mathbb{C}$ such that

$$\frac{p'(1)q(1)-q'(1)p(1)}{q(1)^2}=\sum_{k=1}^{r}c_k\frac{p(x_k)}{q(x_k)}\,,\qquad p\in\mathcal{P}_n^c\,, \tag{7.1.3}$$

where

$$q(z):=\prod_{k=1}^{n}(z-a_k)\,. \tag{7.1.4}$$

We claim that $x_k\neq 1$ for each $k=1,2,\dots,r$. Indeed, if there is index k such that $x_k=1$, then Theorem A.3.3 implies that

$$p(z):=(z+1)^{n-r}\prod_{k=1}^{r}(z-x_k)\in\mathcal{P}_n^c$$

has a zero at 1 with multiplicity at least two, which is a contradiction. Applying (7.1.3) with the above p, we obtain

$$p'(1)q(1)-q'(1)p(1)=0$$

and since $p(1)\neq 0$ and $q(1)\neq 0$, this is equivalent to

$$\frac{q'(1)}{q(1)}=\frac{p'(1)}{p(1)}\,,$$

that is, in terms of the zeros of p_n and q_n,

$$\sum_{k=1}^{n}\frac{1}{1-a_k}=\frac{n-r}{2}+\sum_{k=1}^{r}\frac{1}{1-x_k}\,. \tag{7.1.5}$$

Since $x_k\in\partial D$ and $x_k\neq 1$, we have

$$\operatorname{Re}\left(\frac{1}{1-x_k}\right)=\frac{1}{2}\,,\qquad k=1,2,\dots,r\,. \tag{7.1.6}$$

It follows from (7.1.5) and (7.1.6) that

$$\operatorname{Re}\left(\sum_{k=1}^{n}\frac{1}{1-a_k}\right)=\frac{n}{2}\,,$$

which contradicts assumption (7.1.2). So $n+1\leq r$, indeed.

A compactness argument shows that there is a function $\widetilde{f}\in V$ such that

$$\|\widetilde{f}\|_{\partial D}=1\qquad\text{and}\qquad L(\widetilde{f})=\|L\|:=\max_{0\neq f\in V}\frac{|L(f)|}{\|f\|_{\partial D}}\,.$$

Theorem A.3.3 implies $|\widetilde{f}(x_k)| = 1$ for every $k = 1, 2, \dots, r$. Hence, if

$$\widetilde{f} = \frac{\widetilde{p}}{q}\,, \qquad \widetilde{p} \in \mathcal{P}_n^c\,, \qquad q(z) = \prod_{k=1}^{n} (z - a_k)\,,$$

then

$$h(z) := |\widetilde{p}(z)|^2 - |q(z)|^2 \le 0\,, \qquad z \in \partial D \tag{7.1.7}$$

and

$$h(x_k) = 0\,, \qquad k = 1, 2, \dots, r\,. \tag{7.1.8}$$

Note that $t(\theta) := h(e^{i\theta}) \in \mathcal{T}_n$ vanishes at each θ_k, where the numbers $\theta_k \in [-\pi, \pi)$ are defined by $x_k = e^{i\theta_k}$, $k = 1, 2, \dots, r$. Because of (7.1.7), each of these zeros is of even multiplicity. Hence, $n + 1 \le r$ implies that $t \in \mathcal{T}_n$ has at least $2n + 2$ zeros and therefore $t = 0$. From this we can deduce that $h(z) = 0$ for every $z \in \partial D$, so

$$|\widetilde{p}(z)| = |q(z)|\,, \qquad z \in \partial D\,. \tag{7.1.9}$$

We now have

$$z^{-n}\widetilde{p}(z)\widetilde{p}^{\,*}(z) = |\widetilde{p}(z)|^2 = |q(z)|^2 = z^{-n} q(z) q^*(z)\,, \qquad z \in \partial D\,,$$

so by the unicity theorem for analytic functions (see E.1 e] of Section 1.2)

$$\widetilde{p}\,\widetilde{p}^{\,*} = q q^*\,.$$

From this, it follows that there exists a constant $0 \ne c \in \mathbb{C}$ such that

$$\widetilde{f}(z) = \frac{\widetilde{p}(z)}{q(z)} = c \prod_{k=1}^{m} \frac{z - \overline{\alpha}_k^{-1}}{z - \alpha_k}\,, \qquad z \in \mathbb{C}, \quad q(z) \ne 0$$

with some $m \le n$ and

$$\alpha_k := a_{j_k}\,, \qquad k = 1, 2, \dots, m\,, \qquad 1 \le j_1 < j_2 < \cdots < j_m \le n\,.$$

A straightforward calculation gives that

$$\begin{aligned} |\widetilde{f}'(1)| &= \left| \frac{\widetilde{f}'(1)}{\widetilde{f}(1)} \right| = \left| \sum_{k=1}^{m} \left(\frac{1}{1 - \overline{\alpha}_k^{-1}} - \frac{1}{1 - \alpha_k} \right) \right| \\ &= \left| \sum_{k=1}^{m} \frac{|\alpha_k|^2 - 1}{|\alpha_k - 1|^2} \right| \le \max\{B_n^+(z), B_n^-(z)\}\,, \end{aligned}$$

which finishes the proof. □

Corollary 7.1.8 (Bernstein-Type Inequality on K, Complex Case). *Given $(a_k)_{k=1}^{2n} \subset \mathbb{C} \setminus \mathbb{R}$, let the Bernstein factor B_n be defined by*

$$\widetilde{B}_n(\theta) := \max\{\widetilde{B}_n^+(\theta), \widetilde{B}_n^-(\theta)\}$$

with

$$\widetilde{B}_n^+(\theta) := \sum_{\substack{k=1 \\ \mathrm{Im}(a_k)<0}}^{2n} \frac{|e^{ia_k}|^2 - 1}{|e^{ia_k} - e^{i\theta}|^2} \quad \text{and} \quad \widetilde{B}_n^-(\theta) := \sum_{\substack{k=1 \\ \mathrm{Im}(a_k)>0}}^{2n} \frac{1 - |e^{ia_k}|^2}{|e^{ia_k} - e^{i\theta}|^2}\,.$$

Then

$$|f'(\theta)| \le B_n(\theta)\|f\|_K\,, \qquad \theta \in K$$

for every $f \in \mathcal{T}_n^c(a_1, a_2, \dots, a_{2n}; K)$.

If the first sum is not less than the second sum for a fixed $\theta \in K$, then equality holds for $f(\theta) = cS_{2n}^+(e^{i\theta})$ with $c \in \mathbb{C}$, where S_{2n}^+ is the Blaschke product associated with those e^{ia_k} for which $\mathrm{Im}(a_k) < 0$. If the second sum is not less than the first sum for a fixed $\theta \in K$, then equality holds for $f(\theta) = cS_{2n}^-(e^{i\theta})$ with $c \in \mathbb{C}$, where S_{2n}^- is the Blaschke product associated with those e^{ia_k} for which $\mathrm{Im}(a_k) > 0$. Note that

$$S_{2n}^{\pm}(e^{i\theta}) \in \mathcal{T}_n^c(a_1, a_2, \dots, a_{2n}; K)\,.$$

Proof. Observe that if

$$h(\theta) := \prod_{j=1}^{2n} \sin((\theta - a_j)/2) \in \mathcal{T}_n^c$$

and $t_n \in \mathcal{T}_n^c$, then there are $p \in \mathcal{P}_{2n}^c$ and $q \in \mathcal{P}_{2n}^c$ such that

$$\frac{t(\theta)}{h(\theta)} = \frac{p(e^{i\theta})e^{-in\theta}}{q(e^{i\theta})e^{-in\theta}} = \frac{p(e^{i\theta})}{q(e^{i\theta})}\,,$$

where q is of the form

$$q(z) = c\prod_{j=1}^{2n}(z - e^{ia_j})$$

with some $c \in \mathbb{C}$. So the corollary follows from Theorem 7.1.7. □

Corollary 7.1.9 (Bernstein-Type Inequality on $[-1,1]$, Complex Case). *Given $\{a_k\}_{k=1}^n \subset \mathbb{C} \setminus [-1,1]$, let the Bernstein factor $B_n(x)$ be defined by*

$$B_n(x) = \max\left\{\sum_{k=1}^n \frac{1 - |c_k|^2}{|c_k - z|^2}\,,\ \sum_{k=1}^n \frac{|c_k|^{-2} - 1}{|c_k^{-1} - z|^2}\right\},$$

where c_k and z are determined by

$$a_k := \tfrac{1}{2}(c_k + c_k^{-1}), \qquad |c_k| < 1;$$
$$x := \tfrac{1}{2}(z + z^{-1}), \qquad \operatorname{Im}(z) > 0.$$

Then

$$|f'(x)| \le \frac{B_n(x)}{\sqrt{1-x^2}} \|f\|_{[-1,1]}, \qquad x \in (-1,1)$$

for every $f \in \mathcal{P}_n^c(a_1, a_2, \dots, a_n; [-1,1])$. Note that

$$\sum_{k=1}^{n} \frac{1-|c_k|^2}{|c_k - z|^2} = \operatorname{Re}\left(\sum_{k=1}^{n} \frac{\sqrt{a_k^2 - 1}}{a_k - x} \right), \qquad x \in [-1,1],$$

where the choice of $\sqrt{a_k^2 - 1}$ is determined by $\left|a_k - \sqrt{a_k^2 - 1}\right| < 1$.

Proof. The corollary follows from Theorem 7.1.7 by the substitution $x = \frac{1}{2}(z + z^{-1})$. □

Bernstein's classical polynomial inequalities discussed in Section 5.1 are contained in Theorem 7.1.7 and Corollaries 7.1.8 and 7.1.9 as limiting cases. In Theorem 7.1.7 and Corollary 7.1.9 we take

$$(a_1^{(m)}, a_2^{(m)}, \dots, a_n^{(m)}) \subset \mathbb{C} \setminus D$$

so that

$$\lim_{m \to \infty} |a_k^{(m)}| = \infty, \qquad k = 1, 2, \dots, n.$$

In Corollary 7.1.8 we take

$$(a_1^{(m)}, a_2^{(m)}, \dots, a_{2n}^{(m)}) \subset \mathbb{C} \setminus \mathbb{R}$$

so that

$$a_{n+k}^{(m)} = \overline{a}_k^{(m)} \quad \text{and} \quad \lim_{m \to \infty} |\operatorname{Im}(a_k^{(m)})| = \infty, \qquad k = 1, 2, \dots, n.$$

To formulate our next result we introduce the Blaschke product

$$Q_n(z) := \prod_{k=1}^{n} \frac{z - \overline{a}_k}{z - a_k}$$

associated with $(a_1, a_2, \dots, a_n) \subset \mathbb{C} \setminus \mathbb{R}$. Obviously $|Q_n(z)| = 1$ for every $z \in \mathbb{R}$.

Corollary 7.1.10 (Bernstein-Type Inequality on $\mathbb{R}$, Complex Case). *Given $(a_k)_{k=1}^n \subset \mathbb{C} \setminus \mathbb{R}$, let the Bernstein factor $B_n(x)$ be defined by*

$$B_n(x) := \max\{B_n^+(x), B_n^-(x)\}$$

with

$$B_n^+(x) := \sum_{\substack{k=1 \\ \mathrm{Im}(a_k)>0}}^{n} \frac{2\,|\mathrm{Im}(a_k)|}{|x-a_k|^2} \quad \textit{and} \quad B_n^-(x) := \sum_{\substack{k=1 \\ \mathrm{Im}(a_k)<0}}^{n} \frac{2\,|\mathrm{Im}(a_k)|}{|x-a_k|^2}$$

for every $x \in \mathbb{R}$. Then

$$|f'(x)| \le B_n(x)\|f\|_{\mathbb{R}}, \qquad x \in \mathbb{R}$$

for every $f \in \mathcal{P}_n^c(a_1, a_2, \dots, a_n; \mathbb{R})$.

If the first sum is not less than the second sum for a fixed $x \in \mathbb{R}$, then equality holds for $f = cQ_n^+$ with $c \in \mathbb{C}$, where Q_n^+ is the Blaschke product associated with the poles a_k lying in the open upper half-plane

$$H^+ := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}.$$

If the second sum is not less than the first sum for a fixed $x \in \mathbb{R}$, then equality holds for $f = cQ_n^-$ with $c \in \mathbb{C}$, where Q_n^- is the Blaschke product associated with the poles a_k lying in the open lower half-plane

$$H^- := \{z \in \mathbb{C} : \mathrm{Im}(z) < 0\}.$$

Corollary 7.1.10 follows from Theorem 7.1.7; see E.4.

The next theorem improves the Bernstein-type inequality of Theorem 7.1.7 in the case when $\{a_k\}_{k=1}^n \subset \mathbb{C} \setminus \overline{D}$ and f has all its zeros in $\mathbb{C} \setminus D$. It extends Lax [44].

Theorem 7.1.11 (Lax-Type Inequality). *Given $(a_k)_{k=1}^n \subset \mathbb{C} \setminus \overline{D}$, let the Bernstein factor B_n be, as in Theorem 7.1.7, defined by*

$$B_n(z) := \sum_{k=1}^{n} \frac{|a_k|^2 - 1}{|a_k - z|^2}.$$

Then

$$|h'(z)| \le \tfrac{1}{2} B_n(z)\|h\|_{\partial D}, \qquad z \in \partial D$$

for every $h \in \mathcal{P}_n^c(a_1, a_2, \dots, a_n; \partial D)$ having all its zeros in $\mathbb{C} \setminus D$.

Equality holds for $h = c(S_n + 1)$ with $c \in \mathbb{C}$, where S_n is the Blaschke product associated with $(a_k)_{k=1}^n$.

Note that $B_n(z) = |S_n'(z)|$. *Note also that*

$$h := c(S_n + 1) \in \mathcal{P}_n^c(a_1, a_2, \dots, a_n; \partial D), \qquad c \neq 0$$

has all its zeros on ∂D.

Proof. First assume that each zero of h is on ∂D, the general case can be reduced to this (see E.5). Thus let $h := p/q$, where $p \in \mathcal{P}_n^c$ has all its zeros on ∂D and where

$$q(z) := \prod_{k=1}^{n} (z - a_k), \qquad |a_k| > 1.$$

Let

$$q^*(z) := \prod_{k=1}^{n} (1 - \overline{a}_k z).$$

We study

$$u(\theta) := \frac{p(e^{2i\theta})e^{-in\theta}}{|q(e^{2i\theta})|} = \frac{p(e^{2i\theta})}{\sqrt{q(e^{2i\theta})}\sqrt{q^*(e^{2i\theta})}}$$

for $\theta \in \mathbb{R}$, where the square roots are taken so that $\sqrt{q}$ is analytic in a neighborhood of the closed unit disk, and $\sqrt{q^*}$ is analytic in a neighborhood of the complement of the open unit disk. Since $p \in \mathcal{P}_n^c$ has all its zeros on ∂D, there exists a $0 \neq \beta \in \mathbb{C}$ such that

$$t(\theta) := \beta p(e^{2i\theta})e^{-in\theta}$$

is a real trigonometric polynomial of degree at most n (see E.5 a]). Also

$$\begin{aligned} |q(e^{2i\theta)}| = |q^*(e^{2i\theta})| &= \prod_{k=1}^{n} |1 - \overline{a}_k e^{2i\theta}| \\ &= \gamma \prod_{k=1}^{2n} |\sin((\theta - c_k)/2)|, \end{aligned}$$

where $\gamma > 0$,

$$e^{ic_k} = \overline{a}_k^{-1/2}, \qquad \operatorname{Im}(c_k) > 0, \quad k = 1, 2, \dots, n$$

and

$$e^{ic_k} = -\overline{a}_k^{-1/2}, \qquad \operatorname{Im}(c_k) > 0, \quad k = n+1, n+2, \dots, 2n.$$

Applying Theorem 7.1.1 to

$$\beta^{-1} u \in \mathcal{T}_n(c_1, c_2, \dots, c_{2n}; K),$$

we obtain

$$|u'(\theta) - i\widetilde{B}_n(\theta)u(\theta)| \le \widetilde{B}_n(\theta)\|u\|_K\,, \qquad \theta \in K\,, \tag{7.1.10}$$

where

$$\begin{aligned}\widetilde{B}_n(\theta) &= \frac{1}{2}\sum_{k=1}^{n}\left(\frac{1-|a_k|^{-1}}{|\overline{a}_k^{-1/2} - e^{i\theta}|^2} + \frac{1-|a_k|^{-1}}{|-\overline{a}_k^{-1/2} - e^{i\theta}|^2}\right) \\ &= \sum_{k=1}^{n}\frac{1-|a_k|^{-2}}{|\overline{a}_k^{-1} - e^{2i\theta}|^2} = \sum_{k=1}^{n}\frac{|a_k|^2-1}{|a_k - e^{2i\theta}|^2}\,.\end{aligned} \tag{7.1.11}$$

Observe that

$$u(\theta) = \frac{p(e^{2i\theta})}{q(e^{2i\theta})}\,\frac{\sqrt{q(e^{2i\theta})}}{\sqrt{q^*(e^{2i\theta})}} = \frac{p(e^{2i\theta})}{q(e^{2i\theta})}f_n(e^{i\theta})\,, \tag{7.1.12}$$

where

$$f_n(z) := \frac{\sqrt{q(z^2)}}{\sqrt{q^*(z^2)}}\,. \tag{7.1.13}$$

A simple calculation (see E.4. of Section 3.5) shows that

$$\widetilde{B}_n(\theta) = e^{i\theta}\frac{f_n'(e^{i\theta})}{f_n(e^{i\theta})}\,, \qquad \theta \in K\,. \tag{7.1.14}$$

Also, since $|f_n(e^{i\theta})| = 1$ for every $\theta \in K$, we have

$$\|u\|_K = \left\|\frac{p}{q}\right\|_{\partial D} = \|h\|_{\partial D}\,. \tag{7.1.15}$$

Now (7.1.10) to (7.1.15) yield

$$\left|\frac{d}{d\theta}\left(\frac{p(e^{2i\theta})}{q(e^{2i\theta})}f_n(e^{i\theta})\right) - ie^{i\theta}\frac{f_n'(e^{i\theta})}{f_n(e^{i\theta})}\,\frac{p(e^{2i\theta})}{q(e^{2i\theta})}f_n(e^{i\theta})\right| \le \widetilde{B}_n(\theta)\|h\|_{\partial D}\,.$$

So

$$\begin{aligned}|2ie^{2i\theta}h'(e^{2i\theta})f_n(e^{2i\theta}) &+ ie^{i\theta}f_n'(e^{i\theta})h(e^{2i\theta}) \\ &- ie^{i\theta}f_n'(e^{i\theta})h(e^{2i\theta})| \le \widetilde{B}_n(\theta)\|h\|_{\partial D}\,.\end{aligned}$$

Thus

$$2\,|h'(e^{2i\theta})| \le \widetilde{B}_n(\theta)\|h\|_{\partial D}\,,$$

which, together with (7.1.11), finishes the proof. □

Comments, Exercises, and Examples.

Most of the results in this section have been proved in Borwein and Erdélyi [to appear 4] and in Borwein, Erdélyi, and Zhang [94a]. A weaker version of Corollary 7.1.9 has been obtained by Russak (see Petrushev and Popov [87]). Theorem 7.1.11 contains, as a limiting case, an inequality of Lax [44] conjectured by Erdős. Lax's inequality establishes the sharp Bernstein-type inequality on the unit disk for polynomials $p \in \mathcal{P}_n^c$ having no zeros in the open unit disk. That is,

$$\|p'\|_D \le \frac{n}{2}\,\|p\|_D$$

for such polynomials. Various extensions of this inequality are given by Ankeny and Rivlin [55], Govil [73], Malik [69], and others. We discuss some of these in E.16 of Appendix 5.

E.1 Proof of Theorem 7.1.1 and Corollary 7.1.2. Given $(a_k)_{k=1}^{2n} \subset \mathbb{C} \setminus \mathbb{R}$, let

$$\mathcal{T}_{n,a} := \mathcal{T}_n(a_1, a_2, \dots, a_{2n}; K)\,.$$

a] Show that $\mathcal{T}_{n,a}$ is a Hermite interpolation space. That is, if the points $x_1, x_2, \dots, x_k \in K$ are distinct, and $m_1, m_2, \dots, m_k$ are positive integers with $\sum_{i=1}^{k} m_i \le 2n+1$, then for any choice of real numbers $y_{i,j}$, there is a function $f \in \mathcal{T}_{n,a}$ such that

$$f^{(j)}(x_i) = y_{i,j}\,, \qquad i = 1, 2, \dots, k\,, \quad j = 0, 1, \dots, m_i - 1\,.$$

Hint: See the hint to E.7 of Section 1.1. □

b] Show that for every fixed $\theta \in K$, the value

$$\max_{0 \ne f \in \mathcal{T}_{n,a}} \frac{f'(\theta)^2 + B_n^2(\theta) f^2(\theta)}{\|f\|_K^2}$$

is attained by an $\widetilde{f} \in \mathcal{T}_{n,a}$.

Hint: Use a compactness argument. □

c] Show that $\widetilde{f} = cV$, where $c \in \mathbb{R}$ and V is one of the Chebyshev polynomials for $\mathcal{T}_{n,a}$ defined in Theorem 3.5.3 and E.3 of Section 3.5.

Hint: Use a variational method with the help of part a]. □

d] Prove Theorem 7.1.1.

Hint: Use part c] and E.4 of Section 3.5. □

e] Prove Corollary 7.1.2.

E.2 Proof of Corollaries 7.1.3 and 7.1.4.

a] Prove the inequality of Corollary 7.1.3.

Hint: Use Theorem 7.1.1 with the substitution $x = \cos\theta$. Note that $f \in \mathcal{P}_n(a_1, a_2, \dots, a_n; [-1,1])$ implies

$$g(\theta) := f(\cos\theta) = \mathcal{T}_n(c_1, c_1, c_2, c_2, \dots, c_n, c_n; K)\,,$$

where the numbers $c_k \in \mathbb{C}$ are defined by

$$e^{ic_k} = a_k - \sqrt{a_k^2 - 1}\,, \qquad a_k = \tfrac{1}{2}(e^{ic_k} + e^{-ic_k}), \quad \operatorname{Im}(c_k) > 0\,.$$

Verify that if $\widetilde{B}_n$ is the Bernstein factor given in Theorem 7.1.1 associated with

$$(c_1, c_1, c_2, c_2, \dots, c_n, c_n)\,,$$

then

$$\widetilde{B}_n(\theta) = \sum_{k=1}^{n} \operatorname{Re}\left(\frac{\sqrt{a_k^2 - 1}}{a_k - \cos\theta}\right), \qquad \theta \in K\,,$$

where the choice of $\sqrt{a_k^2 - 1}$ is determined by $\left|a_k - \sqrt{a_k^2 - 1}\right| < 1$. □

b] Given $x \in [-1,1]$, prove that equality holds in the inequality of Corollary 7.1.3 if and only if either x is a maximum point of $|f|$ (that is, $f(x) = \pm\|f\|_{[-1,1]}$) or $f = cT_n$ with $c \in \mathbb{R}$, where T_n is defined in Section 3.5.

Hint: Observe that

$$V = (\cos\alpha)\widetilde{T}_n + (\sin\alpha)\widetilde{U}_n$$

is even if and only if $V = \pm\widetilde{T}_n$ and use Theorem 7.1.1. □

c] Prove Corollary 7.1.4.

E.3 Proof of Corollaries 7.1.5 and 7.1.6.

a] Prove Corollary 7.1.5.

Hint: Use Theorem 7.1.1 and the substitution $x = i\dfrac{e^{i\theta} + 1}{e^{i\theta} - 1}$, which maps K onto $\mathbb{R} \cup \{\infty\}$. □

b] Prove Corollary 7.1.6.

E.4 Proof of Corollary 7.1.10.

Prove Corollary 7.1.10.

Hint: Use Theorem 7.1.7 and the substitution $x = i\dfrac{z+1}{z-1}$, which maps ∂D onto $\mathbb{R} \cup \{\infty\}$. □

E.5 Completion of the Proof of Theorem 7.1.11.

a] Show that if $p \in \mathcal{P}_n^c$ has all its zeros on the unit circle, then there is a $0 \neq \beta \in \mathbb{C}$ such that $g(\theta) := \beta e^{-in\theta} p(e^{2i\theta})$ is a real trigonometric polynomial of degree at most n.

For $f = p/q$ with

$$q(z) := \prod_{k=1}^{n} (z - a_k)\,, \qquad a_k \in \mathbb{C}$$

and

$$p(z) := \gamma \prod_{k=1}^{m} (z - b_k)\,, \qquad b_k \in \mathbb{C}, \quad \gamma \in \mathbb{C}, \quad m \le n\,,$$

we define $f^* := p^*/q$, where

$$p^*(z) := \overline{\gamma} \prod_{k=1}^{m} (1 - z\overline{b}_k)\,.$$

Let $D := \{z \in \mathbb{C} : |z| < 1\}$.

b] Show that $|f^*(z)| = |f(z)|$ for every $z \in \partial D$.

In each of the remaining parts of the exercise suppose that $|a_k| > 1$ for each k.

c] Show that if $\epsilon \in \partial D$ and $|b_k| \ge 1$ for each k, then $f + \epsilon f^*$ has all its zeros on the unit circle.

d] Show that if $|b_k| \ge 1$ for each k, then

$$|f'(z)| \le |f^{*\prime}(z)|\,, \qquad z \in \partial D\,.$$

Hint: First observe that it is sufficient to study the case $z = 1$. We have

$$\begin{aligned}
\left|\operatorname{Re}\left(\frac{f'(1)}{f(1)}\right)\right| &= \left|\operatorname{Re}\left(\sum_{k=1}^{n}\frac{1}{1-b_k}\right) - \operatorname{Re}\left(\sum_{k=1}^{n}\frac{1}{1-a_k}\right)\right| \\
&\le \left|\operatorname{Re}\left(\sum_{k=1}^{n}\frac{1}{1-b_k}\right) - \frac{n}{2}\right| + \left|\frac{n}{2} - \operatorname{Re}\left(\sum_{k=1}^{n}\frac{1}{1-a_k}\right)\right| \\
&= \frac{n}{2} - \operatorname{Re}\left(\sum_{k=1}^{n}\frac{1}{1-b_k}\right) + \frac{n}{2} - \operatorname{Re}\left(\sum_{k=1}^{n}\frac{1}{1-a_k}\right) \\
&= n - \operatorname{Re}\left(\sum_{k=1}^{n}\frac{1}{1-b_k}\right) - \operatorname{Re}\left(\sum_{k=1}^{n}\frac{1}{1-a_k}\right) \\
&= \operatorname{Re}\left(\sum_{k=1}^{n}\frac{1}{1-b_k^{-1}}\right) - \operatorname{Re}\left(\sum_{k=1}^{n}\frac{1}{1-a_k}\right) \\
&= \operatorname{Re}\left(\sum_{k=1}^{n}\frac{1}{1-\overline{b}_k^{-1}}\right) - \operatorname{Re}\left(\sum_{k=1}^{n}\frac{1}{1-a_k}\right) \\
&= \left|\operatorname{Re}\left(\frac{f^{*\prime}(1)}{f^*(1)}\right)\right|
\end{aligned}$$

and

$$\begin{aligned}\left|\operatorname{Im}\left(\frac{f'(1)}{f(1)}\right)\right| &= \left|\operatorname{Im}\left(\sum_{k=1}^{n}\frac{1}{1-b_k}\right) - \operatorname{Im}\left(\sum_{k=1}^{n}\frac{1}{1-a_k}\right)\right| \\ &= \left|\operatorname{Im}\left(n - \sum_{k=1}^{n}\frac{1}{1-b_k}\right) - \operatorname{Im}\left(\sum_{k=1}^{n}\frac{1}{1-a_k}\right)\right| \\ &= \left|-\operatorname{Im}\left(\sum_{k=1}^{n}\frac{1}{1-\overline{b}_k^{-1}}\right) + \operatorname{Im}\left(\sum_{k=1}^{n}\frac{1}{1-a_k}\right)\right| \\ &= \left|\operatorname{Im}\left(\frac{f^{*\prime}(1)}{f^*(1)}\right)\right| .\end{aligned}$$

The result now follows from the combination of part b] and the above two inequalities. □

e] Prove that if $|b_k| \geq 1$ for each k, then

$$2|f'(z)| \leq |f'(z)| + |f^{*\prime}(z)| \leq B_n(z)\|f\|_{\partial D}\,, \qquad z \in \partial D\,,$$

where $B_n(z)$ is the Bernstein factor defined in Theorem 7.1.11.

Hint: Use parts c] and d] and the already proved part of Theorem 7.1.11 (when f has all its zeros on the unit circle). □

f] Show that if $|b_k| \geq 1$ for each k, then

$$|f'(z)| \leq \frac{1}{2}B_n(z)\left(\max_{z\in\partial D}|f(z)| - \min_{z\in\partial D}|f(z)|\right)\,, \qquad z \in \partial D\,,$$

where $B_n(z)$ is the Bernstein factor defined in Theorem 7.1.7. This extends a result of Aziz and Dawood [88].

Hint: Assume that $\|f\|_{\partial D} = 1$. Let $m := \min_{z\in\partial D}|f(z)|$. Let α be a constant of modulus less than 1. Let $g(z) := f(z) - \alpha m$. Observe that the argument of α can be chosen so that

$$|g^{*\prime}(z)| = |f^{*\prime}(z)| - |\alpha| m B_n(z)\,.$$

By Rouché's theorem, g has no zeros in D. So parts d] and e] imply that

$$\begin{aligned}2\,|f^{*\prime}(z)| - 2\,|\alpha|\,mB_n(z) = 2\,|g^{*\prime}(z)| &\leq |g'(z)| + |g^{*\prime}(z)| \\ &= |f'(z)| + |f^{*\prime}(z)| - |\alpha| m B_n(z) \\ &\leq B_n(z) - |\alpha| m B_n(z)\,.\end{aligned}$$

Since $|\alpha|$ can be chosen arbitrarily close to 1, the result follows. □

E.6 Extensions of Russak's Inequalities.

a] Given $(a_k)_{k=1}^{2n} \subset \mathbb{C} \setminus \mathbb{R}$, show that

$$\|f'\|_{L_1(K)} \leq 2\pi n \, \|f\|_K$$

for every $f \in \mathcal{T}_n(a_1, a_2, \ldots, a_{2n}; K)$ and

$$\|f'\|_{L_1(K)} \leq 4\pi n \, \|f\|_K$$

for every $f \in \mathcal{T}_n^c(a_1, a_2, \ldots, a_{2n}; K)$.

b] Given $(a_k)_{k=1}^{n} \subset \mathbb{C} \setminus \mathbb{R}$, show that

$$\|f'\|_{L_1(\mathbb{R})} \leq \pi n \, \|f\|_{\mathbb{R}}$$

for every $f \in \mathcal{P}_n(a_1, a_2, \ldots, a_n; \mathbb{R},)$ and

$$\|f'\|_{L_1(\mathbb{R})} \leq 2\pi n \, \|f\|_{\mathbb{R}}$$

for every $f \in \mathcal{P}_n^c(a_1, a_2, \ldots, a_n; \mathbb{R})$.

Hint: Use Corollaries 7.1.2, 7.1.8, 7.1.6, and 7.1.10. Write the Bernstein factors in a form so that the integral (of each term in the maximum if the Bernstein factor is defined by a maximum) can be evaluated by the residue theorem (in part a]) and by finding the antiderivative (in part b]). □

c] Are any of the inequalities of parts a] and b] sharp? If so, in which cases?

E.7 Markov-Type Inequality. Given $(a_k)_{k=1}^{n} \subset \mathbb{R} \setminus [-1, 1]$, show that

$$\|f'\|_{[-1,1]} \leq \frac{n}{n-1} \left(\sum_{k=1}^{n} \frac{1 + |c_k|}{1 - |c_k|} \right)^2 \|f\|_{[-1,1]}$$

for every $f \in \mathcal{P}_n^c(a_1, a_2, \ldots, a_n; [-1, 1])$, where the numbers c_k are defined by

$$c_k := a_k - \sqrt{a_k^2 - 1}\,, \qquad a_k = \tfrac{1}{2}(c_k + c_k^{-1})\,, \qquad |c_k| < 1\,.$$

Proceed as follows:

a] Given $(a_k)_{k=1}^{n} \subset \mathbb{R} \setminus [-1, 1]$, let

$$a_k(y) := \begin{cases} \dfrac{2a_k}{1+y} + \dfrac{1-y}{1+y} & \text{if} \quad 0 \leq y \leq 1 \\ \dfrac{2a_k}{1-y} + \dfrac{1+y}{1-y} & \text{if} \quad -1 \leq y \leq 0 \end{cases}$$

and let $c_k(y)$ be defined by

$$a_k(y) =: \tfrac{1}{2}(c_k(y) + c_k(y)^{-1}), \qquad |c_k(y)| < 1.$$

Show that

$$|f'(y)| \le \frac{2}{1+|y|}\left(\sum_{k=1}^{n}\frac{1+c_k(y)}{1-c_k(y)}\right)^2 \|f\|_{[-1,1]}$$

for every $f \in \mathcal{P}_n^c(a_1, a_2, \dots, a_n; [-1,1])$.

Hint: Show by a variational method that

$$\max_f \frac{|f'(\pm 1)|}{\|f\|_{[-1,1]}} = |T_n'(\pm 1)|,$$

where the maximum is taken for all $0 \ne f \in \mathcal{P}_n(a_1, a_2, \dots, a_n; [-1,1])$, and T_n is the Chebyshev polynomial for $\mathcal{P}_n(a_1, a_2, \dots, a_n; [-1,1])$ defined in Section 3.5. Now the result follows from E.1 c] of Section 3.5 by a linear shift from $[-1,1]$ to $[-1,y]$ if $0 \le y \le 1$, or to $[y,1]$ if $-1 \le y \le 0$. □

b] Given $(a_k)_{k=1}^n \subset \mathbb{R} \setminus [-1,1]$, show that

$$|f'(y)| \le \frac{n}{1-|y|}\|f\|_{[-1,1]}, \qquad y \in (-1,1)$$

for every $f \in \mathcal{P}_n(a_1, a_2, \dots, a_n; [-1,1])$.

Hint: When $y = 0$, this follows from Corollary 7.1.3. When $y \in (-1,1)$ is arbitrary, use a linear shift from $[-1,1]$ to $[2y-1,1]$ if $0 \le y \le 1$, or to $[-1, 2y+1]$ if $-1 < y \le 0$. □

c] Prove the Markov-type inequality of the exercise.

Hint: Combine parts a] and b]. Note that

$$|a_k(y)| \ge |a_k| \qquad \text{and} \qquad |c_k(y)| \le |c_k| < 1, \qquad k = 1, 2, \dots, n$$

holds for every $y \in [-1,1]$. □

E.8 Schur-Type Inequality. Given $\{a_k\}_{k=1}^n \subset \mathbb{R} \setminus [-1,1]$, show that

$$\|f\|_{[-1,1]} \le \max\{|U_n(1)|, |U_n(-1)|\} \cdot \left\|f(x)\sqrt{1-x^2}\right\|_{[-1,1]}$$

for every $f \in \mathcal{P}_n(a_1, a_2, \dots, a_n; [-1,1])$, where U_n is the Chebyshev polynomial (of the second kind) for $\mathcal{P}_n(a_1, a_2, \dots, a_n; [-1,1])$ defined in Section 3.5, and

$$|U_n(\pm 1)| = \left|\sum_{k=1}^{n}\frac{\sqrt{a_k^2-1}}{a_k \mp 1}\right|$$

with the choice of $\sqrt{a_k^2-1}$ determined by $\left|a_k - \sqrt{a_k^2-1}\right| < 1$. Show that equality holds if and only if $f = cU_n$, $c \in \mathbb{R}$.

Hint: First show that if $\widetilde{f} \in \mathcal{P}_n(a_1, a_2, \dots, a_n; [-1,1])$ is extremal for

$$\max_f \frac{|f(\pm 1)|}{\left\| f(x)\sqrt{1-x^2} \right\|_{[-1,1]}},$$

where the maximum is taken for all $0 \neq f \in \mathcal{P}_n(a_1, a_2, \dots, a_n; [-1,1])$, then $f = cU_n$ with some $c \in \mathbb{R}$. Observe that $U_n(\pm 1)$ can be evaluated by L'Hospital's rule since

$$U_n(x)^2 = \frac{1 - T_n(x)^2}{1 - x^2};$$

hence $|U_n(\pm 1)|^2 = |T_n'(\pm 1)| = |B_n(\pm 1)| \cdot |U_n(\pm 1)|$ with the notation of Section 3.5. Thus $|U_n(\pm 1)| = |B_n(\pm 1)|$.

If $y \in [-1,1]$ is arbitrary, then use a linear shift from $[-1,1]$ to $[-y,y]$ (some caution must be exercised about the change of poles). □

E.9 Extension of Lax's Inequality on the Half-Plane. Associated with $(a_k)_{k=1}^n \subset H^+ := \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$, let the Bernstein factor B_n be, as in Corollary 7.1.10, defined by

$$B_n(x) := \sum_{k=1}^n \frac{2\operatorname{Im}(a_k)}{|x - a_k|^2}.$$

Show that

$$|h'(x)| \leq \tfrac{1}{2} B_n(x) \|h\|_{\mathbb{R}}, \qquad x \in \mathbb{R}$$

for every $h \in \mathcal{P}_n^c(a_1, a_2, \dots, a_n; \mathbb{R})$ having all its zeros in H^+.

Equality holds for $h = c(\widetilde{S}_n + 1)$ with $c \in \mathbb{C}$, where $\widetilde{S}_n$ is the Blaschke product associated with $(a_k)_{k=1}^n$. Note that $B_n(z) = |\widetilde{S}_n'(z)|$. Note also that

$$h = c(\widetilde{S} + 1) \in \mathcal{P}_n^c(a_1, a_2, \dots, a_n; \mathbb{R}), \qquad c \neq 0$$

has all its zeros on $\mathbb{R}$.

Hint: Use Theorem 7.1.11 with the substitution $x = i\dfrac{z+1}{z-1}$. □

E.10 Remarks on Theorem 7.1.7 and Corollary 7.1.10.

a] Given $(a_k)_{k=1}^n \subset D$ and $z \in \partial D$, show that equality holds in the inequality of Theorem 7.1.7 if and only if $f = cS_n$ with $c \in \mathbb{C}$, where S_n is the Blaschke product associated with $(a_k)_{k=1}^n$.

Hint: Analyze the proof of Theorem 7.1.7. □

b] Given $(a_k)_{k=1}^n \subset H^+ = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ and $x \in \mathbb{R}$, show that equality holds in the inequality of Corollary 7.1.10 if and only if $f = cQ_n$ with $c \in \mathbb{C}$, where Q_n is the Blaschke product associated with $(a_k)_{k=1}^n$.

Note that the *only if* parts of E.10 a] and E.10 b] above are not claimed in the general case of Theorem 7.1.7 and Corollary 7.1.10 (why?).

E.11 Markov-Bernstein-Type Inequality for SR_n^c and ASR_n^c. Let SR_n^c denote the set of all self-reciprocal polynomials $p \in \mathcal{P}_n^c$ satisfying

$$p(z) \equiv z^n p(z^{-1}) .$$

Let SR_n denote the set of all real self-reciprocal polynomials of degree at most n, that is, $\mathrm{SR}_n := \mathrm{SR}_n^c \cap \mathcal{P}_n$. For a polynomial $p \in \mathcal{P}_n^c$ of the form

$$p(z) = \sum_{j=0}^{n} c_j z^j , \qquad c_j \in \mathbb{C} , \tag{7.1.16}$$

$p \in \mathrm{SR}_n^c$ if and only if

$$c_j = c_{n-j} , \qquad j = 0, 1, \dots , n .$$

Let ASR_n^c denote the set of all antiself-reciprocal polynomials $p \in \mathcal{P}_n^c$ satisfying

$$p(z) \equiv -z^n p(z^{-1}) .$$

Let ASR_n denote the set of all real antiself-reciprocal polynomials of degree at most n, that is, $\mathrm{ASR}_n := \mathrm{ASR}_n^c \cap \mathcal{P}_n$. Let $\mathrm{ASR}_n := \mathrm{ASR}_n^c \cap \mathcal{P}_n$. For a polynomial $p \in \mathcal{P}_n^c$ of the form (7.1.16) $p \in \mathrm{ASR}_n^c$ if and only if

$$c_j = -c_{n-j} , \qquad j = 0, 1, \dots , n .$$

a] There exists an absolute constant c such that

$$|p'(x)| \le cn \min \left\{ (1 + \log n), \ \log \left(\frac{e}{1 - x^2} \right) \right\} \|p\|_{[-1,1]}$$

holds for every $x \in [-1, 1]$ and for every $p \in \mathcal{P}_n^c$ satisfying

$$|p(x)| \le (1 + |x|^n) \|p\|_{[-1,1]} , \qquad x \in \mathbb{R}, \tag{7.1.17}$$

in particular, for every $p \in \mathrm{SR}_n^c$ and for every $p \in \mathrm{ASR}_n^c$.

The inequality

$$|p'(x)| \le cn(1 + \log n) \|p\|_{[-1,1]}$$

for all $p \in \mathrm{SR}_n$ and for all $p \in \mathrm{ASR}_n$ was first obtained by Kroó and Szabados [94a]. They also showed that up to the constant $c > 0$ the above inequality is sharp for both SR_n and ASR_n. Here we present a distinct proof. The sharpness is studied in E.12 f].

Outline. Suppose $p \in \mathcal{P}_n^c$ satisfies (7.1.17). Then

$$f(x) := \frac{p(x)}{1+x^{2n}} \tag{7.1.18}$$

satisfies

$$\|f\|_{\mathbb{R}} \le 2\,\|p\|_{[-1,1]}\,. \tag{7.1.19}$$

Let

$$a_k := e^{i(2k-1)\pi/(2n)}\,, \qquad k = 1, 2, \ldots, 2n \tag{7.1.20}$$

be the zeros of the equation $z^{2n}+1=0$. Now Corollary 7.1.10, together with (7.1.18) to (7.1.20), yields that

$$\begin{aligned} |f'(x)| &\le 2\left(\sum_{k=1}^{n} \frac{\operatorname{Im}(a_k)}{|x-a_k|^2}\right)\|f\|_{\mathbb{R}} \\ &\le 4\left(\sum_{k=1}^{n} \frac{\operatorname{Im}(a_k)}{|x-a_k|^2}\right)\|p\|_{[-1,1]}\,, \qquad x \in \mathbb{R}\,. \end{aligned} \tag{7.1.21}$$

Show that if $n \in \mathbb{N}$ and $x \in [-1,1]$, then

$$4\sum_{k=1}^{n} \frac{\operatorname{Im}(a_k)}{|x-a_k|^2} \;\sim\; n \min\left\{(1+\log n),\;\; \log\left(\frac{e}{1-x^2}\right)\right\}, \tag{7.1.22}$$

where here the $\sim$ symbol means that there are absolute constants $c_1 > 0$ and $c_2 > 0$ (independent of $n \in \mathbb{N}$ and $x \in [-1,1]$) such that the left-hand side is between c_1 times the right-hand side and c_2 times the right-hand side for every $n \in \mathbb{N}$ and $x \in [-1,1]$. Combining (7.1.18), (7.1.21), and (7.1.22), we conclude that there is an absolute constant c_2 such that

$$\begin{aligned} &\left|\frac{p'(x)}{1+x^{2n}} - \frac{2nx^{2n-1}}{1+x^{2n}}\frac{p(x)}{1+x^{2n}}\right| \\ &\qquad \le c_2 n \min\left\{(1+\log n),\;\; \log\left(\frac{e}{1-x^2}\right)\right\}\|p\|_{[-1,1]}\,, \qquad x \in \mathbb{R}\,. \end{aligned}$$

So if $x \in [-1,1]$, then

$$|p'(x)| \le \left(2c_2 n \min\left\{(1+\log n),\;\; \log\left(\frac{e}{1-x^2}\right)\right\} + 2n\right)\|p\|_{[-1,1]}$$

and the proof is finished. □

b] There exists an absolute constant $c > 0$ such that

$$\left\|\frac{p'(x)}{1+x^{2n}}\right\|_{\mathbb{R}} \le cn(1+\log n)\left\|\frac{p(x)}{1+x^{2n}}\right\|_{\mathbb{R}}$$

for every $p \in \mathcal{P}_{2n}^c$.

Proof. Associated with $p \in \mathcal{P}_{2n}^c$, let

$$f(x) := \frac{p(x)}{1+x^{2n}}\,.$$

Let

$$a_k := e^{i(2k-1)\pi/(2n)}\,, \qquad k = 1, 2, \ldots, 2n$$

be the zeros of the equation $z^{2n}+1=0$. Using Corollary 7.1.10, we can deduce that there are absolute constants c_1 and c_2 such that

$$\begin{aligned}\|f'\|_{\mathbb{R}} &\le \left(\sum_{k=1}^{n} 2\,|\mathrm{Im}(a_k)|^{-1}\right)\|f\|_{\mathbb{R}} \le c_1\left(\sum_{k=1}^{n}\left(\frac{k}{n}\right)^{-1}\right)\|f\|_{\mathbb{R}}\\ &\le c_2 n(1+\log n)\|f\|_{\mathbb{R}}\,.\end{aligned}$$

Therefore

$$\left|\frac{p'(x)}{1+x^{2n}} - \frac{2nx^{2n-1}}{1+x^{2n}}\frac{p(x)}{1+x^{2n}}\right| \le c_2 n(1+\log n)\left\|\frac{p(x)}{1+x^{2n}}\right\|_{\mathbb{R}}$$

for every $x \in \mathbb{R}$, which implies

$$\left\|\frac{p'(x)}{1+x^{2n}}\right\|_{\mathbb{R}} \le (c_2 n(1+\log n)+2n)\left\|\frac{p(x)}{1+x^{2n}}\right\|_{\mathbb{R}}\,,$$

and the result follows. □

c] For every $m \in \mathbb{N}$, there exists a constant $c(m)$ depending only on m so that

$$\|p^{(m)}\|_{[-1,1]} \le c(m)(n(1+\log n))^m \|p\|_{[-1,1]}$$

for every $p \in \mathcal{P}_n^c$ satisfying

$$|p(x)| \le (1+|x|^n)\|p\|_{[-1,1]}\,. \tag{7.1.23}$$

Proof. Using part b] and induction on m, we see that there exists a constant $c_1(m)$ depending only on m such that

$$\left\|\frac{p^{(m)}(x)}{1+x^{2n}}\right\|_{\mathbb{R}} \le c_1(m)(n(1+\log n))^m \left\|\frac{p(x)}{1+x^{2n}}\right\|_{\mathbb{R}}$$

for every $p \in \mathcal{P}_{2n}^c$. Note that if $p \in \mathcal{P}_n^c$ satisfies (7.1.23), then

$$\left\|\frac{p(x)}{1+x^{2n}}\right\|_{\mathbb{R}} \le 2\left\|\frac{p(x)}{1+|x|^n}\right\|_{\mathbb{R}} \le 2\|p\|_{[-1,1]}\,,$$

and the result follows. □

E.12 Quasi-Chebyshev Polynomials for SR_{2n} and ASR_{2n}. Let

$$a_k := e^{i(2k-1)\pi/(2n)}\,, \qquad k = 1, 2, \dots n$$

be the zeros of the equation $z^{2n} + 1 = 0$ in the upper half-plane. Let

$$M_{2n}(z) := \prod_{k=1}^{n} (z - a_k)^2\,, \qquad z \in \mathbb{C},$$
$$P_{2n}(x) := \mathrm{Re}(M_{2n}(x))\,, \qquad x \in \mathbb{R}\,,$$

and

$$Q_{2n}(x) := \mathrm{Im}(M_{2n}(x))\,, \qquad x \in \mathbb{R}\,.$$

a] Show that if n is even, then $P_{2n} \in \mathrm{SR}_{2n}$ and $Q_{2n} \in \mathrm{ASR}_{2n}$, while if n is odd, then $Q_{2n} \in \mathrm{SR}_{2n}$ and $P_{2n} \in \mathrm{ASR}_{2n}$.

b] Show that

$$|M_{2n}(x)| = 1 + x^{2n}\,, \qquad x \in \mathbb{R}$$

and

$$P_{2n}(x)^2 + Q_{2n}(x)^2 = (1 + x^{2n})^2\,, \qquad x \in \mathbb{R}\,,$$

in particular

$$\|P_{2n}\|_{[-1,1]} \le 2 \qquad \text{and} \qquad \|Q_{2n}\|_{[-1,1]} \le 2\,.$$

Proof. Note that

$$\frac{M_{2n}(x)}{1 + x^{2n}} = \prod_{k=1}^{n} \frac{x - a_k}{x - \overline{a}_k}\,, \tag{7.1.24}$$

which implies the first equality. The rest is straightforward from the definitions. □

c] Show that there are extended real numbers

$$\infty = z_0 > z_1 > \cdots > z_{2n} = -\infty$$

such that

$$\frac{M_{2n}(z_j)}{1 + z_j^{2n}} = (-1)^j\,, \qquad j = 0, 1, \dots, 2n$$

(the value of the left-hand side at $\pm\infty$ is defined by taking the limit when $x \to \pm\infty$).

Hint: Use (7.1.24) and the argument principle (see, for example, Ash [71]). □

d] Let $n \in \mathbb{N}$ be even. Show that there exist

$$1 = x_0 > y_1 > x_1 > y_2 > \cdots > x_{n-1} > y_n > x_n = -1$$

such that

$$\frac{P_{2n}(x_j)}{1 + x_j^{2n}} = (-1)^{j+n/2}, \qquad Q_{2n}(x_j) = 0, \quad j = 0, 1, \ldots, n$$

and

$$\frac{Q_{2n}(y_j)}{1 + y_j^{2n}} = (-1)^{j+1+n/2}, \qquad P_{2n}(y_j) = 0, \quad j = 1, 2, \ldots n.$$

Formulate the analog statement when $n \in \mathbb{N}$ is odd.

e] Show that there exists an absolute constant $c > 0$ such that

$$|P'_{2n}(x)| + |Q'_{2n}(x)| \geq cn \min\left\{(1 + \log n), \ \log\left(\frac{e}{1 - x^2}\right)\right\}$$

for every $n \in \mathbb{N}$ and $x \in [-1, 1]$.

Proof. If $x \in [-1, 1]$, then

$$\begin{aligned} |P'_{2n}(x)| + |Q'_{2n}(x)| &\geq |M'_{2n}(x)| = |M'_{2n}(x)| \left|\frac{1 + x^{2n}}{M_{2n}(x)}\right| \\ &\geq \left|\frac{M'_{2n}(x)}{M_{2n}(x)}\right| = \left|\sum_{k=1}^{n} \frac{2}{x - a_k}\right| \geq 2 \sum_{k=1}^{n} \frac{\operatorname{Im}(a_k)}{|x - a_k|^2} \\ &\geq cn \min\left\{(1 + \log n), \ \log\left(\frac{e}{1 - x^2}\right)\right\} \end{aligned}$$

with an absolute constant $c > 0$, where the last inequality follows from (7.1.22). □

The next part shows the sharpness of the inequality of E.11 a].

f] Let $c > 0$ be the same absolute constant as in part e]. For the sake of brevity let

$$\delta_n(x) := \left(cn \min\left\{(1 + \log n), \ \log\left(\frac{e}{1 - x^2}\right)\right\}\right)^{-1}.$$

Show that for every interval

$$I_{n,x} := [x, x + 8\delta_n(x)] \subset [0, 1],$$

there exist $y_1 \in I_{n,x}$ and $y_2 \in I_{n,x}$ such that

$$|P'_{2n}(y_1)| \geq \tfrac{1}{2}\delta_n(x)^{-1} \qquad \text{and} \qquad |Q'_{2n}(y_2)| \geq \tfrac{1}{2}\delta_n(x)^{-1}\,.$$

Proof. Suppose that

$$|P'_{2n}(y)| < \tfrac{1}{2}\delta_n(x)^{-1}$$

for every $y \in I_{n,x}$. Then, from part e] we can deduce that

$$|Q'_{2n}(y)| > \tfrac{1}{2}\delta_n(x)^{-1}$$

for every $y \in I_{n,x}$. Therefore

$$\|Q_{2n}\|_{[-1,1]} \geq \tfrac{1}{2}\int_{I_{n,x}} |Q'_{2n}(y)|\,dy > \tfrac{1}{2}8\delta_n(x)\tfrac{1}{2}\delta_n(x)^{-1} = 2\,,$$

which contradicts the last inequality of part b]. This finishes the proof of the first inequality. The second inequality can be proven in the same way. □

g] Show that

$$p'(1) = \tfrac{1}{2}n\,p(1)$$

for every $p \in \mathrm{SR}_n^c$.

By using the quasi Chebyshev polynomials for SR_{2n} and ASR_{2n}, it can be shown that the inequality of E.11 c] is essentially sharp for the classes SR_n and ASR_{2n} for every m. This has been pointed out to us by Szabados. The argument requires some more technical details than the proof in the $m = 1$ case discussed in the above exercise.

7.2 Inequalities for Logarithmic Derivatives

We derive a series of metric inequalities of the form

$$m\left(\left\{x \in \mathbb{R} : \frac{r'(x)}{r(x)} \geq \alpha\right\}\right) \leq \frac{cn}{\alpha}\,, \qquad \alpha > 0\,,$$

where r is a rational function of type (n,n) and c is a constant independent of n. Here m is the Lebesgue measure, although, since the sets in question are usually just finite unions of intervals, this is mostly a notational convenience. One of the interesting features of these inequalities is their easy extension from the polynomial case to the rational case.

The basic inequality is due to Loomis [46]. Note the invariance of the measure of the set in this case.

Theorem 7.2.1. *If $p \in \mathcal{P}_n$ has n real roots, then*

$$m\left(\left\{x \in \mathbb{R} : \frac{p'(x)}{p(x)} \geq \alpha\right\}\right) = \frac{n}{\alpha}, \qquad \alpha > 0.$$

Proof. By considering $\alpha^{-1}p$ instead of p, it is sufficient to prove the theorem for $\alpha = 1$. We first consider the case where p has distinct roots, which are denoted by $\alpha_1 < \alpha_2 < \cdots < \alpha_n$. Then

$$\frac{p'(x)}{p(x)} = \sum_{k=1}^{n} \frac{1}{x - \alpha_i}.$$

Let $\beta_1 < \beta_2 < \cdots < \beta_n$ be the roots of $p - p'$, which must all be real. Note that these are the points where $p'(x)/p(x) = 1$. It is now easy to see from the graph that

$$\left\{x \in \mathbb{R} : \frac{p'(x)}{p(x)} \geq 1\right\} = [\alpha_1, \beta_1] \cup [\alpha_2, \beta_2] \cup \cdots \cup [\alpha_n, \beta_n]$$

and

$$m\left(\left\{x \in \mathbb{R} : \frac{p'(x)}{p(x)} \geq 1\right\}\right) = \sum_{i=1}^{n} \beta_i - \sum_{i=1}^{n} \alpha_i .$$

However, if $p(x) := x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then

$$\sum_{i=1}^{n} \alpha_i = -a_{n-1} ,$$

while

$$\sum_{i=1}^{n} \beta_i = -(a_{n-1} - n)$$

is -1 times the second coefficient of $p - p'$.

This gives the result for distinct roots. The case when some of the roots of p are repeated can be handled by an easy limiting argument. □

Corollary 7.2.2. *If $\alpha_i \in \mathbb{R}$, $c_i > 0$, $i = 1, 2, \ldots, n$, and $\sum_{i=1}^{n} c_i = 1$, then*

$$m\left(\left\{x \in \mathbb{R} : \sum_{i=1}^{n} \frac{c_i}{x - \alpha_i} \geq \alpha\right\}\right) = \frac{1}{\alpha}, \qquad \alpha > 0.$$

Proof. For c_i rational this follows immediately from Theorem 7.2.1 on clearing the denominators of c_i by multiplying by an integer. The real case is an obvious limiting argument. □

In order to extend Theorem 7.2.1 to arbitrary polynomials we need the following generalization of E.3 of Section 2.4 due to Videnskii [51]. The proof is indicated in the exercises.

Lemma 7.2.3. *If $p \in \mathcal{P}_n$ is positive on $[a,b]$, then there exists $q,s \in \mathcal{P}_n$ nonnegative on $[a,b]$ with all real roots (in $[a,b]$) so that*

$$p(x) = q(x) + s(x)\,.$$

We now prove the unrestricted case of Theorem 7.2.1.

Theorem 7.2.4. *Let $p \in \mathcal{P}_n$. Then*

$$m\left(\left\{x \in \mathbb{R} : \frac{p'(x)}{p(x)} \geq \alpha\right\}\right) \leq \frac{2n}{\alpha}\,, \qquad \alpha > 0\,.$$

Proof. Let $\alpha > 0$ and let $p_n \in \mathcal{P}_n$. Choose a and b such that

$$\left\{x \in \mathbb{R} : \frac{p'(x)}{p(x)} \geq \alpha\right\} \subset [a,b]\,.$$

By Lemma 7.2.3 we can find polynomials $q \in \mathcal{P}_{2n}$ and $s \in \mathcal{P}_{2n}$ such that

$$p^2(x) = q(x) + s(x)$$

where, for $x \in [a,b]$,

$$0 \leq q(x) \leq p^2(x) \qquad \text{and} \qquad 0 \leq s(x) \leq p^2(x)$$

and both q and s have only real roots. Now

$$\left\{x \in \mathbb{R} : \frac{p'(x)}{p(x)} \geq \alpha\right\} = \left\{x \in \mathbb{R} : \frac{(p^2)'(x)}{p^2(x)} \geq 2\alpha\right\}\,.$$

Also,

$$(p^2)'(x) \geq 2\alpha\, p^2(x)$$

holds exactly when

$$q'(x) + s'(x) \geq 2\alpha(q(x) + s(x))\,.$$

By Theorem 7.2.1

$$m\left(\left\{x \in \mathbb{R} : \frac{q'(x)}{q(x)} \geq 2\alpha\right\}\right) = m\left(\left\{x \in \mathbb{R} : \frac{s'(x)}{s(x)} \geq 2\alpha\right\}\right) = \frac{n}{\alpha}\,.$$

Since q and s are nonnegative on $[a,b]$, it follows that

$$m(\{x \in [a,b] : q'(x) + s'(x) \geq 2\alpha(q(x) + s(x))\}) \leq \frac{2n}{\alpha}\,,$$

and the proof is finished. □

It can be shown that this inequality is asymptotically sharp to the extent that the constant 2 cannot be replaced by any smaller constant for large n; see Kristiansen [82].

Theorem 7.2.4 extends easily to rational functions.

Theorem 7.2.5. *If $r = p/q$, where $p, q \in \mathcal{P}_n$, then*

$$m\left(\left\{x \in \mathbb{R} : \frac{r'(x)}{r(x)} \geq \alpha\right\}\right) \leq \frac{8n}{\alpha}, \qquad \alpha > 0.$$

Proof. We have

$$\frac{r'}{r} = \frac{p'}{p} - \frac{q'}{q}$$

and for $\alpha > 0$,

$$\left\{x \in \mathbb{R} : \frac{r'(x)}{r(x)} \geq \alpha\right\} \subset \left\{x \in \mathbb{R} : \frac{p'(x)}{p(x)} \geq \frac{\alpha}{2}\right\} \cup \left\{x \in \mathbb{R} : \frac{q'(x)}{q(x)} \leq -\frac{\alpha}{2}\right\}.$$

By Theorem 7.2.4

$$m\left(\left\{x \in \mathbb{R} : \frac{p'(x)}{p(x)} \geq \frac{\alpha}{2}\right\}\right) \leq \frac{4n}{\alpha},$$

and with $s(x) := q(-x)$,

$$m\left(\left\{x \in \mathbb{R} : \frac{q'(x)}{q(x)} \leq -\frac{\alpha}{2}\right\}\right) = m\left(\left\{x \in \mathbb{R} : \frac{s'(x)}{s(x)} \geq \frac{\alpha}{2}\right\}\right) \leq \frac{4n}{\alpha}.$$

It follows that

$$m\left(\left\{x \in \mathbb{R} : \frac{r'(x)}{r(x)} \geq \alpha\right\}\right) \leq \frac{8n}{\alpha}$$

for every $\alpha > 0$. □

This inequality probably does not have the exact constant. It can be shown (see Borwein, Rakhmanov, and Saff [to appear]) that the constant 8 cannot be replaced by any constant less than or equal to 2π.

Comments, Exercises, and Examples.

Many variants on the inequalities of this section are presented in E.2, E.3, E.4, and E.5. Some of these are in Borwein [82].

E.5 explores some metric properties of the lemniscate

$$E(p) := \{z \in \mathbb{C} : |p(z)| \leq 1\}$$

of a monic polynomial $p \in \mathcal{P}_n^c$ of the form

$$p(z) = \prod_{i=1}^{n} (z - z_i), \qquad z_i \in \mathbb{C}.$$

The reader is referred to Erdős, Herzog, and Piranian [58] for many results and open problems concerning the lemniscate of monic polynomials. One in particular, which is still open, conjectures that for monic polynomials $p \in \mathcal{P}_n^c$, the length of the boundary of $E(p)$ is maximal for $p(z) := z^n - 1$ and so is $O(n)$. Pommerenke [61] has shown that this length is $O(n^2)$. Borwein [95] improves this to $O(n)$; see E.7. E.9 solves another problem of Erdős, namely, the diameter of $E(p)$ for a monic polynomial $p \in \mathcal{P}_n^c$ is always at least 2.

Erdős [76] contains several other related open problems.

E.1 Polynomials as Sums of Polynomials with Real Roots.

a] Suppose $p \in \mathcal{P}_{2n} \setminus \mathcal{P}_{2n-1}$, and suppose that $p > 0$ on $[a, b]$. Then

$$p(x) = (x - a)(b - x)u^2(x) + v^2(x)$$

for some $u \in \mathcal{P}_{n-1}$ and $v \in \mathcal{P}_n$, which have all their zeros in $[a, b]$.

b] Suppose $p \in \mathcal{P}_{2n+1} \setminus \mathcal{P}_{2n}$ and suppose that $p > 0$ on $[a, b]$. Then

$$p(x) = (b - x)u^2(x) + (x - a)v^2(x)$$

for some $u, v \in \mathcal{P}_n$, which have all their zeros in $[a, b]$.

Hint: Let p be a polynomial of degree $2n$ that is strictly positive on $[a, b]$. Let T_n be the Chebyshev polynomial for the Chebyshev system

$$\left\{ \frac{1}{\sqrt{p(x)}}, \frac{x}{\sqrt{p(x)}}, \ldots, \frac{x^n}{\sqrt{p(x)}} \right\}.$$

Then $T_n(x) = v(x)/\sqrt{p(x)}$ with some $v \in \mathcal{P}_n$. Show that, for part a], v and u defined by

$$(x - a)(b - x)u^2(x) = p(x) - v^2(x)$$

are the required polynomials. Use a similar construction for part b]. □

E.2 Various Specializations. For the next exercises we use the notation $\mathcal{P}_n^+$ to denote the polynomial of degree at most n with nonnegative coefficients, and $\mathcal{P}_n^\uparrow$ to denote those elements of $\mathcal{P}_n$ that are nondecreasing on $[0, \infty)$.

a] If $r = p/q$, where $p, q \in \mathcal{P}_n$ and both p and q have only real roots, then

$$m\left(\left\{ x \in \mathbb{R} : \frac{r'(x)}{r(x)} \geq \alpha \right\} \right) \leq \frac{4n}{\alpha}, \qquad \alpha > 0.$$

b] Let $r(x) := x^n/(4n-x)^n$. Then

$$m\left(\left\{x \in \mathbb{R} : \frac{r'(x)}{r(x)} \geq 1\right\}\right) = 4n\,.$$

c] If $r = p/q$, where $p \in \mathcal{P}_n$ and $q \in \mathcal{P}_n^{\uparrow}$, then

$$m\left(\left\{x \geq 0 : \frac{r'(x)}{r(x)} \geq \alpha\right\}\right) \leq \frac{2n}{\alpha}\,, \qquad \alpha > 0\,.$$

d] If $r = p/q$, where $p \in \mathcal{P}_n^{+}$ and $q \in \mathcal{P}_n^{\uparrow}$, then

$$m\left(\left\{x \geq 0 : \frac{r'(x)}{r(x)} \geq \alpha\right\}\right) \leq \frac{n}{\alpha}\,, \qquad \alpha > 0\,.$$

e] Let $r(x) := x^n$. Then

$$m\left(\left\{x \geq 0 : \frac{r'(x)}{r(x)} \geq \alpha\right\}\right) = \frac{n}{\alpha}\,, \qquad \alpha > 0\,.$$

E.3 Another Metric Inequality. If $p \in \mathcal{P}_n$ has n real roots lying in the interval (a,b), then

$$m\left(\left\{x \in \mathbb{R} : \left|\frac{p'(x)}{p(x)}\right| \leq \frac{\alpha}{|(x-a)(b-x)|}\right\}\right) = \frac{2n}{\alpha}\,, \qquad \alpha > 0\,.$$

Outline. Prove that

$$m\left(\left\{x \in \mathbb{R} : 0 \leq \frac{(x-a)(b-x)p'(x)}{p(x)} \leq \alpha\right\}\right) = \frac{\alpha}{n}$$

and

$$m\left(\left\{x \in \mathbb{R} : 0 \geq \frac{(x-a)(b-x)p'(x)}{p(x)} \geq -\alpha\right\}\right) = \frac{\alpha}{n}\,.$$

First consider the case when p has distinct zeros. Let $y_0 < y_1 < \cdots < y_n$ denote the $n+1$ roots of $(x-a)(b-x)p'(x)$, and let $x_0 < x_1 < \cdots < x_{n-1}$ denote the n roots of $p(x)$. Then

$$y_0 < x_0 \leq y_1 < \cdots x_{n-1} < y_n < x_n := \infty\,.$$

Since

$$\lim_{x \to x_i -} (x-a)(b-x)\frac{p'(x)}{p(x)} = -\infty\,,$$

we can deduce that for each interval (y_i, x_i) there exists a point $\delta_i \in (y_i, x_i)$ such that

$$(\delta_i - a)(b-\delta_i)p'(\delta_i) = -\alpha p(\delta_i)\,.$$

Since the above equation can have at most $n+1$ solutions, we have

$$m\left(\left\{x \in \mathbb{R} : 0 \geq \frac{(x-a)(b-x)p'(x)}{p(x)} \geq -\alpha\right\}\right) = \sum_{i=0}^{n}(\delta_i - y_i)\,.$$

Now proceed as in the proof of Theorem 7.2.1. □

E.4 Extensions to $\mathcal{P}_n^c$.

a] If $p \in \mathcal{P}_n^c$, then

$$m\left(\left\{x \in \mathbb{R} : \left|\frac{p'(x)}{p(x)}\right| \geq \alpha\right\}\right) \leq \frac{8\sqrt{2}\,n}{\alpha}, \qquad \alpha > 0.$$

Hint: Write p as the sum of its real and imaginary parts. □

b] If $r = p/q$ with $p, q \in \mathcal{P}_n^c$, then

$$m\left(\left\{x \in \mathbb{R} : \left|\frac{r'(x)}{r(x)}\right| \geq \alpha\right\}\right) \leq \frac{32\sqrt{2}\,n}{\alpha}, \qquad \alpha > 0.$$

E.5 On the Lemniscate $E(p)$. Let

$$p(z) := \prod_{i=1}^{n} (z - z_i), \qquad z_i \in \mathbb{C}$$

and let

$$E(p) := \{z \in \mathbb{C} : |p(z)| \leq 1\}.$$

a] Show that

$$m(E(p) \cap \mathbb{R})) \leq 4 \cdot 2^{-1/n}$$

with equality only for the Chebyshev polynomial of degree n normalized to have lead coefficient 1; see Pólya [28].

Hint: Analogously to the proof of the Remez inequality of Section 5.1, show that the Chebyshev polynomial transformed to an interval of length 4 is extremal for this problem. □

b] Let $m_2(\cdot)$ denote the planar Lebesgue measure. Show that

$$m_2(E(p)) \leq 4\pi.$$

(In fact, $m_2(E(p)) \leq \pi$, which is exact for z^n; this is due to Pólya [28].)

E.6 Cartan's Lemma. *Let*

$$p(z) := \prod_{j=1}^{n} (z - z_j), \qquad z_j \in \mathbb{C}$$

and $\beta > 0$ be fixed. Then there exist at most n open disks, the sum of whose radii is at most 2β, so that if $z \in \mathbb{C}$ is outside the union of these open disks then

$$|p(z)| > \left(\frac{\beta}{e}\right)^n.$$

Prove Cartan's Lemma as follows:

a] Let $\alpha_1, \alpha_2, \ldots, \alpha_\nu$ be fixed complex numbers. Let $\beta > 0$. Show that there exists a positive integer μ less than or equal to ν for which there exists an open disk with radius $\mu\beta/n$ containing α_j for exactly μ distinct values of $j = 1, 2, \ldots, \nu$.

Hint: Suppose to the contrary that there is no such positive integer μ. Show that this would imply the existence of an open disk with radius $\nu\beta/n$ containing α_j for at least $\nu + 1$ distinct values of $j = 1, 2, \ldots, \nu$, which is a contradiction. □

b] Show that there exist open disks $D_1, D_2, \ldots, D_k$ and positive integers $m_1, m_2, \ldots, m_k$ with the following properties:

(1) $\sum_{j=1}^{k} m_j = n\,, \quad m_1 \geq m_2 \geq \cdots \geq m_k\,;$

(2) D_j has radius $\dfrac{m_j\beta}{n}\,;$

(3) $D_j \cap E_j$ contains exactly m_j zeros of p, where

$$E_j := \mathbb{C} \setminus (D_1 \cup D_2 \cup \ldots \cup D_{j-1})\,;$$

(4) for every integer $m > m_j$, no open disk of radius $\frac{m\beta}{n}$ contains exactly m zeros of p in E_j.

Hint: Use part a]. □

c] Let $D_1, D_2, \ldots, D_k$ be the open disks specified in part b]. For $j = 1, 2, \ldots, k$, let D_j^* be the disk with the same center as D_j and with twice its radius. Show that for every

$$z \in E^* := \mathbb{C} \setminus (D_1^* \cup D_2^* \cup \cdots \cup D_k^*)$$

there is a permutation $\widetilde{z}_1, \widetilde{z}_2, \ldots, \widetilde{z}_n$ of the zeros $z_1, z_2, \ldots, z_n$ such that

$$|z - \widetilde{z}_j| \geq \frac{j\beta}{n}\,, \qquad j = 1, 2, \ldots, n\,.$$

Hint: Let $z \in E^*$ be fixed. Show, by induction on i, that for every $i = 0, 1, \ldots, n-1$, there are at least $i+1$ zeros of p outside the open disk with center z and radius $\frac{(n-i)\beta}{n}$. Distinguish the cases

(1) $n - i > m_1\,;$

(2) $m_j \geq n - i > m_{j+1}, \qquad j = 1, 2, \ldots, k-1\,;$

(3) $m_k \geq n - i > 0\,.$ □

d] Finish the proof of Cartan's lemma.

E.7 The Length of the Boundary of $E(p)$. Let

$$p(z) := \prod_{j=1}^{n} (z - z_j), \qquad z_j \in \mathbb{C}$$

and let

$$E := E(p) := \{z \in \mathbb{C} : |p(z)| \le 1\}.$$

Show that the boundary ∂E of E is of length at most $4e\pi n$. (In fact, with E.10 a], the estimate can be improved to $(5.2)\,\pi n$.)

Outline. Proceed as follows:

a] Let L be an arbitrary line in the complex plane. Show that the set $\partial E \cap L$ contains at most $2n$ distinct points.

Hint: By performing a translation and a rotation, if necessary, we may assume that $L = \mathbb{R}$. Now observe that there is a polynomial $P(x, y)$ of degree at most $2n$, in two real variables x and y, with complex coefficients, such that

$$\begin{aligned}\partial E &= \{z \in \mathbb{C} : |p(z)|^2 = 1\} \\ &= \{z \in \mathbb{C} : p(z)\overline{p(z)} = 1\} \\ &= \{z = x + iy : x, y \in \mathbb{R},\ \ P(x, y) = 1\}.\end{aligned}$$

Hence

$$E \cap \mathbb{R} = \{x \in \mathbb{R} : P(x, 0) = 1\}$$

and since $P(x, 0) \in \mathcal{P}^c_{2n}$, the result follows. □

b] For $a \in \mathbb{C}$ and $r > 0$, let Q be the square

$$Q := \{z \in \mathbb{C} : |\mathrm{Re}(z - a)| < r,\ \ |\mathrm{Im}(z - a)| < r\}.$$

Show that $Q \cap \partial E$ is of length at most $8rn$.

Hint: Divide $Q \cap \partial E$ into subcurves $C_1, C_2, \ldots, C_m$ so that every vertical and horizontal line contains at most one point of each C_j. Let $l_x^{(j)}$ and $l_y^{(j)}$ denote the length of the interval obtained by projecting C_j to the x axes and y axes, respectively. Let

$$l_x := \sum_{j=1}^{m} l_x^{(j)} \qquad \text{and} \qquad l_y := \sum_{j=1}^{m} l_y^{(j)}.$$

Use part a] to show that $l_x \le 4nr$ and $l_y \le 4rn$. Hence, if l denotes the length of $Q \cap \partial E_n$, then

$$l \le l_x + l_y \le 8rn.$$

□

c] Show that the boundary ∂E of E is of length at most $16en$.

Hint: Combine Cartan's lemma (see E.7) and part b]. □

The rest of the exercise is about improving $16en$ to $4\pi en$.

d] Let B be the open disk with center $a \in \mathbb{C}$ and radius $r > 0$. Show that $B \cap \partial E$ is of length at most $2\pi rn$.

Outline. Let Q be the square

$$Q := \{z \in \mathbb{C} : |\mathrm{Re}(z)| < r\,,\ \ |\mathrm{Im}(z)| < r\}\,.$$

For $\alpha \in [0, 2\pi)$ let Q_α be the square

$$Q_\alpha := \{a + e^{i\alpha}(z - a) : z \in Q\}\,.$$

Let $l(\alpha)$ denote the length of $Q_\alpha \cap \partial E$. Let $l_x(\alpha)$ and $l_y(\alpha)$ denote the total length of the intervals obtained by projecting $Q_\alpha \cap \partial E$ to the lines

$$\{z = re^{i\alpha} : r \in \mathbb{R}\} \qquad \text{and} \qquad \{z = re^{i(\alpha+\pi/2)} : r \in \mathbb{R}\}\,,$$

respectively (counting multiplicities). The precise definition of $l_x(\alpha)$ and $l_y(\alpha)$ can be formulated in the same way as in the hint to part b], which is left to the reader. Use part a] to show that

$$l_x(\alpha) + l_y(\alpha) \le 8rn\,, \qquad \alpha \in [0, 2\pi)\,.$$

Hence

$$\begin{aligned}\frac{4}{\pi}\, l(0) &= \frac{1}{2\pi}\int_0^{2\pi} l(0)(|\sin\alpha| + |\cos\alpha|)\, d\alpha \\ &= \frac{1}{2\pi}\int_0^{2\pi} (l_x(\alpha) + l_y(\alpha))\, d\alpha \le 8rn\,,\end{aligned}$$

and $l(0) \le 2\pi rn$ follows. □

e] Prove the initial statement of the exercise.

Hint: Combine Cartan's lemma (see E.6) and part d]. □

E.8 On the Length of Another Lemniscate. Suppose $p \in \mathcal{P}_n^c$. Show that the length of the lemniscate

$$F = F(p) := \left\{z \in \mathbb{C} : \left|\frac{p'(z)}{p(z)}\right| = n\right\}$$

is at most $16n(1 + \log n)$ (actually at most $4\pi n(1 + \log n)$).

Hint: The arguments are very similar to those given in the outline to E.7. First show that if L is an arbitrary line in the complex plane, then $F \cap L$ contains at most $2n$ distinct points. Next prove that if D_r is an open disk of radius r in the complex plane, then $D_r \cap F$ is of length at most $8rn$ (actually at most $2\pi rn$). Now use E.6 c] with $\beta = 1 + \log n$. □

The proof of the following exercise requires some familiarity with harmonic functions.

E.9 On the Diameter of $E(p)$. The *diameter* $\operatorname{diam}(A)$ of a nonempty set $A \subset \mathbb{C}$ is defined by

$$\operatorname{diam}(A) := \sup\{|z_1 - z_2| : z_1, z_2 \in A\}.$$

Let $p \in \mathcal{P}_n^c$ be an arbitrary monic polynomial of degree n, that is,

$$p(z) := \prod_{j=1}^{n} (z - z_j), \qquad z_j \in \mathbb{C}.$$

Then the diameter of the set

$$E(p) := \{z \in \mathbb{C} : |p(z)| \le 1\}$$

is at least 2.

Proceed as follows:

a] Let

$$\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\} \qquad \text{and} \qquad \Delta := \{z \in \widehat{\mathbb{C}} : |z| > 1\}.$$

Let $p \in \mathcal{P}_n^c$ be monic. Let $E := \operatorname{co}(E(p))$, that is, the convex hull of $E(p)$. Use the Riemann mapping theorem (see Ahlfors [53]) to show that there exists a function g of the form

$$g(z) = bz + \sum_{j=0}^{\infty} b_j z^{-j}, \qquad z \in \Delta, \quad b, b_j \in \mathbb{C}$$

such that g is analytic and one-to-one on Δ, and

$$g(\Delta) = \widehat{\mathbb{C}} \setminus E.$$

Hint: Note that $\widehat{\mathbb{C}} \setminus E$ is simply connected. □

b] Let b be the same as in part a]. Show that $|b| \ge 1$.

Outline. Because of the definition of E, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$-\varepsilon \le \log|z^{-n} p(g(z))|, \qquad |z| = 1 + \delta.$$

Since $G(z) := \log|z^{-n} p(g(z))|$ is harmonic on Δ, we have

$$G(\infty) = \frac{1}{2\pi(1+\delta)} \int_0^{2\pi} G((1+\delta)e^{i\theta})\, d\theta \ge \frac{-\varepsilon}{2\pi(1+\delta)}$$

for every $\varepsilon > 0$, so $G(\infty) \ge 0$. On the other hand, since $p \in \mathcal{P}_n^c$ is a monic polynomial of degree n,

$$G(\infty) = \log |b|^n = n \log |b| \,,$$

from which $|b| \geq 1$ follows. □

c] Show that $\operatorname{diam}(E) \geq 2$.

Outline. Assume to the contrary that $\operatorname{diam}(E) < 2$. Then there exists a $\delta > 0$ such that

$$|z^{-1}(g(z) - g(-z))| < 2 \,, \qquad |z| = 1 + \delta \,.$$

Since

$$F(z) := z^{-1}(g(z) - g(-z)) = 2b + 2b_1 z^{-2} + 2b_3 z^{-4} + \cdots$$

is analytic on Δ, the maximum principle (E.1 d] of Section 1.2) yields

$$2|b| = |F(\infty)| \leq \max_{|z|=1+\delta} |F(z)| = \max_{|z|=1+\delta} |z^{-1}(g(z) - g(-z))| < 2 \,,$$

that is, $|b| < 1$, which contradicts part b]. □

d] Note that $\operatorname{diam}(A) = \operatorname{diam}(\operatorname{co}(A))$ for every nonempty $A \subset \mathbb{C}$, in particular, $\operatorname{diam}(E(p)) = \operatorname{diam}(E)$.

The more general result that $\operatorname{diam}(A) \geq 2 \operatorname{cap}(A)$ for every nonempty $A \subset \mathbb{C}$ is observed in Pommerenke [75].

E.10 More on $E(p)$. Suppose p is a monic polynomial with complex coefficients. As before, let

$$E := E(p) := \{z \in \mathbb{C} : |p(z)| \leq 1\} \,.$$

a] It follows from E.6 (Cartan's lemma) that the set $E(p)$ can be covered by disks the sum of whose radii is at most $2e$. It is conjectured in Erdős, Herzog, and Piranian [58] that the correct value in this problem is 2. (The current best constant is less than 2.6.)

b] If $E(p)$ is connected, then it is contained in a disk with radius 2 centered at $\frac{1}{n}\sum_{k=1}^{n} z_k$, where $z_1, z_2, \ldots, z_n$ are the zeros of p.

Proof. This is conjectured in Erdős, Herzog, and Piranian [58] and proved in Pommerenke [59b]. □

c] If $E(p)$ is connected, then its circumference is at least 2π.

Proof. This is also conjectured in Erdős, Herzog, and Piranian [58], and proved in Pommerenke [59b]. □