A4

Inequalities for Generalized Polynomials in L_p

Overview

Many inequalities for generalized polynomials are given in this appendix. Of particular interest are the extensions of virtually all the basic inequalities to L_p spaces. The principal tool is a generalized version of Remez's inequality.

Inequalities for Generalized Polynomials in L_p

Generalized (nonnegative) polynomials are defined by (A.4.1) and (A.4.3). The basic inequalities of Chapter 5 are extended to these functions by replacing the degree with the generalized degree. The crucial observation is that Remez's inequality extends naturally to this setting. This Remez inequality then plays a central role in the extensions of the other inequalities. These generalizations allow for a simple general treatment of L_p inequalities, which is one main feature of this appendix.

The function

$$f(z) = |\omega| \prod_{j=1}^{m} |z - z_j|^{r_j} \tag{A.4.1}$$

with $0 < r_j \in \mathbb{R}$, $z_j \in \mathbb{C}$, and $0 \neq \omega \in \mathbb{C}$ is called a generalized nonnegative (algebraic) polynomial of (generalized) degree

$$N := \sum_{j=1}^{m} r_j \,. \tag{A.4.2}$$

The set of all generalized nonnegative algebraic polynomials of degree at most N is denoted by GAP_N.

The function

$$f(z) = |\omega| \prod_{j=1}^{m} |\sin((z - z_j)/2)|^{r_j} \tag{A.4.3}$$

with $0 < r_j \in \mathbb{R}$, $z_j \in \mathbb{C}$, and $0 \neq \omega \in \mathbb{C}$ is called a generalized nonnegative trigonometric polynomial of degree

$$N := \frac{1}{2} \sum_{j=1}^{m} r_j \,. \tag{A.4.4}$$

The set of all generalized nonnegative trigonometric polynomials of degree at most N is denoted by GTP_N. Throughout this section we will study generalized nonnegative polynomials restricted to the real line. If the exponents r_j in (A.4.1) or (A.4.3) are even integers, then f is a nonnegative algebraic or trigonometric polynomial, respectively. Note that the classes GAP_N and GTP_N are not linear spaces. Note also that if $f \in \mathrm{GAP}_N$ or $f \in \mathrm{GTP}_N$ is of the form (A.4.1) or (A.4.3), respectively, with all $r_j \geq 1$, then the one-sided derivatives of f exist at every $x \in \mathbb{R}$ with the same modulus, hence $|f'(x)|$ is well-defined for every $x \in \mathbb{R}$. We use the notation $|f'(x)|$ for $f \in \mathrm{GAP}_N$ or $f \in \mathrm{GTP}_N$ and $x \in \mathbb{R}$ throughout this section with this understanding. If $f \in \mathrm{GAP}_N$ is of the form (A.4.1) or $f \in \mathrm{GTP}_N$ is of the form (A.4.3), where the zeros $z_j \in \mathbb{C}$, $j = 1, 2, \ldots, m$, are distinct, then r_j is called the multiplicity of z_j in f. Our intention in this section is to extend most of the classical inequalities of Section 5.1 to generalized nonnegative polynomials. In addition, we prove Nikolskii-type inequalities for GAP_N and GTP_N.

Theorem A.4.1 (Remez-Type Inequality for GAP_N). *The inequality*

$$\|f\|_{[-1,1]} \leq \left(\frac{\sqrt{2} + \sqrt{s}}{\sqrt{2} - \sqrt{s}} \right)^N$$

holds for every $f \in \mathrm{GAP}_N$ and $s \in (0, 2)$ satisfying

$$m(\{x \in [-1, 1] : f(x) \leq 1\}) \geq 2 - s \,.$$

E.5 shows that this inequality is sharp. Note that if $0 < s \leq 1$, then

$$\left(\frac{\sqrt{2} + \sqrt{s}}{\sqrt{2} - \sqrt{s}} \right)^N \leq \exp(5N\sqrt{s}).$$

Throughout this section, as before, $K := \mathbb{R} \pmod{2\pi}$.

Theorem A.4.2 (Remez-Type Inequality for GTP_N). *The inequality*

$$\|f\|_K \le \exp\left(N\left(s+\tfrac{7}{4}s^2\right)\right) \le \exp(4Ns)$$

holds for every $f \in \mathrm{GTP}_N$ and $s \in (0,\pi/2]$ satisfying

$$m(\{x \in [-\pi,\pi) : f(x) \le 1\}) \ge 2\pi - s\,. \tag{A.4.5}$$

The inequality

$$\|f\|_K \le \left(\frac{\sqrt{2}+\sqrt{\sigma}}{\sqrt{2}-\sqrt{\sigma}}\right)^N, \qquad \sigma = 1-\cos(s/2)$$

holds for every even $f \in \mathrm{GTP}_N$ and $s \in (0,2\pi)$ satisfying (A.4.5).

We do not discuss what happens when $s \in (\pi/2, 2\pi)$ in the general case because the case when $s \in (0,\pi/2]$ is satisfactory for our needs.

Proof of Theorem A.4.1. First assume that $f \in \mathrm{GAP}_N$ is of the form (A.4.1) with rational exponents $r_j = q_j/q$, where $q_j, q \in \mathbb{N}$. Let $k \in \mathbb{N}$ be an integer. Then (restricted to $\mathbb{R}$) $p := f^{2kq} \in \mathcal{P}_{2kqN}$ and

$$m\left(\{x \in [-1,1] : |p(x)| \le 1\}\right) \ge 2 - s\,.$$

Hence Theorem 5.1.1 yields

$$\|f\|_{[-1,1]} = \|p\|_{[-1,1]}^{1/(2kq)} \le \left(T_{2kqN}\left(\frac{2+s}{2-s}\right)\right)^{1/(2kq)}.$$

Since by E.4,

$$\lim_{k\to\infty}\left(T_{2kqN}\left(\frac{2+s}{2-s}\right)\right)^{1/(2kq)} = \left(\frac{\sqrt{2}+\sqrt{s}}{\sqrt{2}-\sqrt{s}}\right)^N, \tag{A.4.6}$$

the theorem is proved. The case when the exponents $r_j > 0$ are arbitrary real numbers can be easily reduced to the already proved rational case by a straightforward density argument. □

Theorem A.4.2 follows from Theorem 5.1.2 in exactly the same way that Theorem A.4.1 follows from Theorem 5.1.1; see E.6.

Theorem A.4.3 (Nikolskii-Type Inequality for GTP_N). *Let χ be a nonnegative nondecreasing function defined on $[0,\infty)$ such that $\chi(x)/x$ is nonincreasing on $[0,\infty)$. Then there is an absolute constant $c_1 > 0$ such that*

$$\|\chi(f)\|_{L_p(K)} \le (c_1(1+qN))^{1/q-1/p}\|\chi(f)\|_{L_q(K)}$$

for every $f \in \mathrm{GTP}_N$ and $0 < q < p \le \infty$. If $\chi(x) = x$, then $c_1 \le e(4\pi)^{-1}$.

Theorem A.4.4 (Nikolskii-Type Inequality for GAP_N). *Let χ be a nonnegative nondecreasing function defined on $[0,\infty)$ such that $\chi(x)/x$ is nonincreasing on $[0,\infty)$. Then there is an absolute constant $c_2 > 0$ such that*

$$\|\chi(f)\|_{L_p[-1,1]} \le (c_2(2+qN))^{2/q-2/p}\|\chi(f)\|_{L_q[-1,1]}$$

for every $f \in \mathrm{GAP}_N$ and $0 < q < p \le \infty$. If $\chi(x) = x$, then $c_2 \le e^2(2\pi)^{-1}$.

In the proof of the second part of Theorem A.4.4 we will need the following Schur-type inequality, which is interesting in its own right.

Theorem A.4.5 (Schur-Type Inequality for GAP_N). *The inequality*

$$\|f\|^q_{[-1,1]} \le e(1+qN)\big\|\sqrt{1-x^2}f^q(x)\big\|_{[-1,1]}$$

holds for every $f \in \mathrm{GAP}_N$ and $q > 0$.

According to Theorem 5.1.9 (Schur's inequality), if $N \in \mathbb{N}$, $f \in \mathcal{P}_N$, and $q \in \mathbb{N}$, then the constant e in the above inequality can be replaced by 1.

It is sufficient to prove Theorems A.4.3 and A.4.4 when $p = \infty$, and then a simple argument gives the required results for arbitrary exponents $0 < q < p < \infty$. To see this, say, in the trigonometric case, assume that there is a constant C_N such that

$$\|\chi(f)\|_K \le C_N^{1/q}\|\chi(f)\|_{L_q(K)}$$

for every $f \in \mathrm{GTP}_N$ and $0 < q < \infty$. Then

$$\begin{aligned}\|\chi(f)\|^p_{L_p(K)} &= \|(\chi(f))^{p-q+q}\|_{L_1(K)}\\ &\le \|\chi(f)\|^{p-q}_K\|\chi(f)\|^q_{L_q(K)}\\ &\le C_N^{p/q-1}\|\chi(f)\|^{p-q}_{L_q(K)}\|\chi(f)\|^q_{L_q(K)},\end{aligned}$$

and therefore

$$\|\chi(f)\|_{L_p(K)} \le C_N^{1/q-1/p}\|\chi(f)\|_{L_q(K)}$$

for every $f \in \mathrm{GTP}_N$ and $0 < q < p \le \infty$.

Proof of Theorem A.4.3 (when $p = \infty$). Since χ is nonnegative and nondecreasing and $(\chi(x)/x)^q$ is nonincreasing on $[0,\infty)$, we have

$$(\chi(f(\theta)))^q \ge \exp(-qNs)\|\chi(f)\|^q_K$$

whenever

$$f(\theta) \ge \exp(-Ns)\|f\|_K.$$

Hence, by E.7 b], we can deduce that

$$m\left(\{\theta \in [-\pi,\pi) : (\chi(f(\theta)))^q \geq \exp(-qNs)\|\chi(f)\|_K^q\}\right) \geq \frac{s}{4}$$

for every $f \in \mathrm{GTP}_N$ and $s \in (0, 2\pi)$. Choosing $s := (1+qN)^{-1}$, we get

$$m\left(\left\{\theta \in [-\pi,\pi) : (\chi(f(\theta)))^q \geq e^{-1}\|\chi(f)\|_K^q\right\}\right) \geq (4(1+qN))^{-1}\,.$$

Hence, integrating only on the subset I of K where

$$(\chi(f(\theta)))^q \geq e^{-1}\|\chi(f)\|_K^q\,,$$

we conclude that

$$\begin{aligned}\|\chi(f)\|_K^q &\leq 4e(1+qN)\int_I (\chi(f(\theta)))^q\,d\theta \\ &\leq 4e(1+qN)\|\chi(f)\|_{L_q(K)}^q\,,\end{aligned}$$

and the first part of the theorem is proved.

Now we turn to the second statement. Let

$$D := \{z \in \mathbb{C} : |z| < 1\} \qquad \text{and} \qquad \partial D := \{z \in \mathbb{C} : |z| = 1\}\,.$$

If h is analytic in the open unit disk D and continuous on the closed unit disk $\overline{D}$, then by Cauchy's integral formula we have

$$(1-|rz|^2)h(rz) = \frac{1}{2\pi i}\int_{\partial D} h(u)\frac{1-r\overline{z}u}{u-rz}\,du$$

whenever $z \in \overline{D}$ and $r \in [0,1)$. Note that $u \in \partial D$ and $z \in \partial D$ imply $|1-r\overline{z}u| = |u-rz|$ for all $r \in [0,1)$. Hence, if $P \in \mathcal{P}_n^c$ and $0 < q < \infty$, then

$$(1-r^2)|P^*(rz)|^q \leq \frac{1}{2\pi}\int_{\partial D}|P^*(u)|^q\,|du|$$

whenever $z \in \partial D$ and $r \in [0,1]$, where P^* is obtained from the factorization of P by replacing each factor $(z-\alpha)$ of P with $|\alpha| < 1$ by $(1-\overline{\alpha}z)$. Since

$$\tfrac{1}{2}(1+r)|z-\sigma| \leq |rz-\sigma|\,, \qquad |\sigma| > 1\,, \quad z \in \partial D\,, \quad r \in [0,1]\,,$$

we have

$$(1-r^2)\left(\tfrac{1}{2}(1+r)\right)^{q\deg(P)}|P(z)|^q \leq \frac{1}{2\pi}\int_{\partial D}|P^*(u)|^q\,|du|$$

whenever $z \in \partial D$ and $r \in [0,1]$. Maximizing the left-hand side for $r \in [0,1]$ and using the fact that $|P^*(z)| = |P(z)|$ for $z \in \partial D$, we conclude that

$$|P(z)|^q \le \frac{(2+q\deg(P))e}{8\pi}\int_{-\pi}^{\pi}|P(e^{i\theta})|^q\,d\theta\,,\qquad z\in\partial D\,.$$

Hence, by E.8,

$$\|R\|_K^q \le \frac{(1+qn)e}{4\pi}\int_{-\pi}^{\pi}|R(\theta)|^q\,d\theta$$

for every $R\in\mathcal{T}_n$. If $f\in \mathrm{GTP}_N$ is of the form (A.4.3) with rational exponents $r_j=\alpha_j/\alpha$, where $\alpha_j,\alpha\in\mathbb{N}$, then on applying the above inequality to $R:=f^{2\alpha}\in\mathcal{T}_{2\alpha N}$ with q replaced by $q/(2\alpha)$, we conclude that

$$\|f\|_K^q \le \frac{1+qN}{4\pi}\int_{-\pi}^{\pi}f(\theta)^q\,d\theta$$

and the second statement of the theorem is proved. The case when the exponents $r_j>0$, $j=1,2,\dots,m$, are arbitrary real numbers can be reduced to the already proved rational case by a straightforward density argument. □

Proof of Theorem A.4.5. Let $P\in\mathcal{P}_n$ and

$$M:=\big\|\sqrt{1-x^2}\,|P(x)|^q\big\|_{[-1,1]}\,.$$

By E.8 c], there exists an $R\in\mathcal{P}_{2n}$ such that $|R(e^{i\theta})|=|P(\cos\theta)|$, $\theta\in\mathbb{R}$. We define $R^*\in\mathcal{P}_{2n}^c$ from the factorization of R by replacing all the factors $(z-\alpha)$ of R with $|\alpha|<1$ by $(1-\overline{\alpha}z)$. Note that $|1-e^{2i\theta}|=2|\sin\theta|$ and $|R(e^{i\theta})|=|R^*(e^{i\theta})|$ for all $\theta\in\mathbb{R}$. Hence the maximum principle yields that

$$\begin{aligned}|1-(rz)^2||R^*(rz)|^q &\le \max_{z\in\partial D}|1-z^2||R^*(z)|^q\\ &= \max_{\theta\in\mathbb{R}}2\,|\sin\theta||P(\cos\theta)|^q = 2M\,.\end{aligned}$$

By E.9 we have

$$|R^*(z)|\le\left(\frac{2}{1+r}\right)^{2n}|R^*(rz)|\,,\qquad z\in\partial D\,,\ \ r\in[0,1]\,.$$

Hence

$$|R^*(z)|^q\le\frac{2^{2nq}}{(1+r)^{2nq}}\frac{1}{1-r^2}2M\,,\qquad z\in\partial D\,,\ \ r\in[0,1]\,,$$

where the minimum on $[0,1]$ of the right-hand side is taken at

$$r:=\frac{qn}{1+qn}\,.$$

Estimating the right-hand side at this value of r, we get

$$\|P\|^q_{[-1,1]} = \max_{z\in\partial D} |R^*(z)|^q \le e(1+qn)\big\|\sqrt{1-x^2}\,|P(x)|^q\big\|_{[-1,1]}\,.$$

If $f \in \mathrm{GAP}_N$ is of the form (A.4.1) with rational exponents $r_j = \alpha_j/\alpha$, $\alpha_j, \alpha \in \mathbb{N}$, then applying the above inequality to $P = f^{2\alpha} \in \mathcal{P}_{2\alpha N}$ with q replaced by $q/(2\alpha)$, we get

$$\|f\|^q_{[-1,1]} \le e(1+qN)\big\|\sqrt{1-x^2}\,f^q(x)\big\|_{[-1,1]}\,,$$

and the theorem is proved. The case when the exponents $r_j > 0$ are arbitrary real numbers can be easily reduced to the already proved rational case once again by a standard approximation. □

Proof of Theorem A.4.4 (when $p = \infty$). Since χ is nonnegative and nondecreasing and $(\chi(x)/x)^q$ is nonincreasing on $[0,\infty)$, we have

$$(\chi(f(x)))^q \ge \exp(-qN\sqrt{s})\|\chi(f)\|^q_{[-1,1]}$$

whenever

$$f(x) \ge \exp(-N\sqrt{s})\|f\|_{[-1,1]}\,.$$

So by E.7 a] we can deduce that

$$m\left(\left\{x\in[-1,1] : (\chi(f(x)))^q \ge \exp(-qN\sqrt{s})\|\chi(f)\|^q_{[-1,1]}\right\}\right) \ge \frac{s}{8}$$

for all $s \in (0,2)$. Choosing $s := (1+qN)^{-2}$, we obtain

$$m\left(\left\{x\in[-1,1] : (\chi(f(x)))^q \ge e^{-1}\|\chi(f)\|_{[-1,1]}\right\}\right) \ge \frac{1}{8(1+qN)^2}\,.$$

Integrating on the subset I of $[-1,1]$ where

$$(\chi(f(x)))^q \ge e^{-1}\|\chi(f)\|_{[-1,1]}\,,$$

we conclude that

$$\begin{aligned}\|\chi(f)\|^q_{[-1,1]} &\le 8e(1+qN)^2\int_I (\chi(f(x)))^q\,dx\\ &\le 8e(1+qN)^2\|\chi(f)\|^q_{L_q[-1,1]}\,.\end{aligned}$$

Thus the first part of the theorem is proved.

To show that the given constant works in the case that $\chi(x) = x$ we use another method. Let $h \in \mathrm{GAP}_M$. Then by E.10, $g(\theta) = h(\cos\theta) \in \mathrm{GTP}_M$. On using the substitution $x = \cos\theta$, from Theorem A.4.3 we get

$$\|h\|_{[-1,1]} \le e(2\pi)^{-1}(1+qM)\int_{-1}^{1}|h(x)|^q(1-x^2)^{-1/2}\,dx\,.$$

If $f \in \mathrm{GAP}_N$, then $h(x) = f(x)(1-x^2)^{1/(2q)} \in \mathrm{GAP}_M$ with $M = N + q^{-1}$, so an application of the above inequality yields

$$(A.4.7) \qquad \left\| \sqrt{1-x^2} f^q(x) \right\|_{[-1,1]} \le e(2\pi)^{-1}(2+qN)\int_{-1}^{1} f^q(x)\,dx\,.$$

(Note that the weaker assumption $h \in \mathrm{GAP}_M$ instead of $f \in \mathrm{GAP}_N$ already implies (A.4.7).)

Now a combination of Theorem A.4.5 and inequality (A.4.7) gives that if $\chi(x) = x$, then the inequality of Theorem A.4.4 holds with $c_2 := e^2(2\pi)^{-1}$. □

Now we prove extensions (up to multiplicative absolute constants) of Markov's and Bernstein's inequalities for generalized nonnegative polynomials.

Theorem A.4.6 (Bernstein-Type Inequality for GTP_N). *There exists an absolute constant $c_3 > 0$ such that*

$$\|f'\|_K \le c_3 N \|f\|_K$$

for every $f \in \mathrm{GTP}_N$ of the form (A.4.3) with each $r_j \ge 1$.

Theorem A.4.7 (Bernstein-Type Inequality for GAP_N). *The inequality*

$$|f'(x)| \le \frac{c_3 N}{\sqrt{1-x^2}} \|f\|_{[-1,1]}\,, \qquad x \in (-1,1)\,,$$

holds for every $f \in \mathrm{GAP}_N$ of the form (A.4.1) with each $r_j \ge 1$, where c_3 is as in Theorem A.4.6.

Theorem A.4.8 (Markov-Type Inequality for GAP_N). *There exists an absolute constant $c_4 > 0$ such that*

$$\|f'\|_{[-1,1]} \le c_4 N^2 \|f\|_{[-1,1]}$$

for every $f \in \mathrm{GAP}_N$ of the form (A.4.1) with each $r_j \ge 1$.

To prove Theorem A.4.6 we need the following lemma.

Lemma A.4.9. *Suppose $g \in \mathrm{GTP}_N$ is of the form (A.4.3) with each $z_j \in \mathbb{R}$, and suppose at least one of any two adjacent (in K) zeros has multiplicity at least 1. Then there exists an absolute constant $c_5 > 0$ such that for every such g there is an interval $I \subset K$ of length at least $c_5 N^{-1}$ for which*

$$\min_{\theta \in I} g(\theta) \ge e^{-1}\|g\|_K\,.$$

Proof. Take a $g \in \mathrm{GTP}_N$ satisfying the hypothesis of the lemma. Because of the periodicity of g we may assume that

$$g(\pi) = \|g\|_K \,. \tag{A.4.8}$$

Define

$$Q_{n,\omega}(\theta) := T_{2n}\left(\frac{\sin(\theta/2)}{\sin(\omega/2)}\right) \tag{A.4.9}$$

with

$$n := \lfloor N \rfloor \qquad \text{and} \qquad \omega := \pi - (3N)^{-1}\,, \tag{A.4.10}$$

where T_{2n} is the Chebyshev polynomial of degree $2n$ defined by (2.1.1). By E.11 there exists an absolute constant $c_6 > 1$ such that $Q_{n,\omega}(\pi) \geq c_6$. Introduce the set

$$A := \left\{\theta \in [\pi - (3N)^{-1}, \pi + (3N)^{-1}] : g(\theta) \geq e^{-1} g(\pi)\right\} .$$

We study $h := g|Q_{n,\omega}| \in \mathrm{GTP}_{2N}$. The inequality $Q_{n,\omega}(\pi) \geq c_6$ and assumption (A.4.8) yield

$$h(\theta) \leq g(\theta) \leq c_6^{-1} Q_{n,\omega}(\pi) g(\pi) = c_6^{-1} \|h\|_K$$

for all $\theta \in [-\omega, \omega] = [-\pi + (3N)^{-1},\ \pi - (3N)^{-1}]$. Further, the definition of the set A, the fact that $\|Q_{n,\omega}\|_K = Q_{n,\omega}(\pi)$, and (A.4.8) imply that

$$h(\theta) \leq e^{-1} g(\pi) Q_{n,\omega}(\pi) = e^{-1} \|h\|_K$$

for all $\theta \in [\pi - (3N)^{-1}, \pi + (3N)^{-1}] \setminus A$. From the last two inequalities we conclude that

$$h(\theta) \leq c_7^{-1} \|h\|_{[-1,1]} \quad \text{for all } \theta \in [-\pi, \pi] \setminus A\,,$$

where $c_7 := \min\{c_6, e\} > 1$ is an absolute constant. Therefore, by E.7 b]

$$m(A) \geq c_8 N^{-1} \quad \text{with } c_8 := \tfrac{1}{17} \log c_7 > 0\,.$$

Since $g \in \mathrm{GTP}_N$ is of the form (A.4.3) with each $z_j \in \mathbb{R}$, and at least one of any two adjacent zeros of g has multiplicity at least 1, E.12 and assumption (8.1.8) imply that g cannot have two or more distinct zeros in $[\pi - (3N)^{-1},\ \pi + (3N)^{-1}]$. Hence A is the union of at most two intervals. Therefore there exists an interval $I \subset A$ such that $m(I) \geq c_8 (2N)^{-1}$, and the lemma is proved. □

Proof of Theorem A.4.6. Let $f \in \mathrm{GTP}_N$ be of the form (A.4.3) with each $r_j \geq 1$. Without loss of generality we may assume that $|f'(\pi)| = \|f'\|_K$, and it is sufficient to prove only that

$$|f'(\pi)| \leq c_3 N \|f\|_K \,.$$

By E.13 we may assume that each z_j is real in (A.4.3). Hence, by E.2, $g := |f'|$ satisfies the assumption of Lemma A.4.9. Denote the endpoints of the interval I coming from Lemma A.4.9 by $a < b$. We can now deduce that

$$\begin{aligned}\|f'\|_K = |f'(\pi)| &\leq \frac{e}{b-a}\int_a^b |f'(\theta)|\,d\theta \\ &\leq \frac{eN}{c_5}\int_a^b |f'(\theta)|\,d\theta = \frac{eN}{c_5}|f(b)-f(a)| \leq c_3 N\|f\|_K\end{aligned}$$

with $c_3 := ec_5^{-1}$, and the proof is finished. □

Proof of Theorem A.4.7. The theorem follows from Theorem A.4.6 by using the substitution $x = \cos\theta$ and E.10 b]. □

Proof of Theorem A.4.8. Let $\alpha := 1-(1+N)^{-2}$. Using Theorem A.4.7 and then E.14, we obtain

$$\begin{aligned}\|f'\|_{[-\alpha,\alpha]} &\leq c_3 N(N+1)\|f\|_{[-1,1]} \\ &\leq c_3 N(N+1)c_9\|f\|_{[-\alpha,\alpha]} \leq c_4 N^2\|f\|_{[-\alpha,\alpha]}\,,\end{aligned}$$

and then the theorem follows by a linear transformation. □

Now we establish Remez-, Bernstein-, and Markov-type inequalities for generalized nonnegative polynomials in L_p. In the proofs we use the inequalities proved in this appendix so far, and the methods illustrate how one can combine the "basic" inequalities in the proofs of various other inequalities for generalized nonnegative polynomials. First we state the main results.

Theorem A.4.10 (L_p Remez-Type Inequality for GAP_N). *Let χ be a nonnegative nondecreasing function defined on $[0,\infty)$ such that $\chi(x)/x$ is nonincreasing on $[0,\infty)$. There exists an absolute constant $c \leq 5\sqrt{2}$ such that*

$$\int_{-1}^{1}(\chi(f(x)))^p\,dx \leq \left(1+\exp\left(cpN\sqrt{s}\right)\right)\int_A (\chi(f(x)))^p\,dx$$

for every $f \in \mathrm{GAP}_N$, $A \subset [-1,1]$ with $m([-1,1]\setminus A) \leq s \leq 1/2$, and for every $p \in (0,\infty)$.

Theorem A.4.11 (L_p Remez-Type Inequality for GTP_N). *Let χ be a nonnegative nondecreasing function defined on $[0,\infty)$ such that $\chi(x)/x$ is nonincreasing on $[0,\infty)$. There exists an absolute constant $c \le 8$ such that*

$$\int_{-\pi}^{\pi} (\chi(f(\theta)))^p \, d\theta \le (1+\exp(cpNs)) \int_A (\chi(f(\theta)))^p \, d\theta$$

for every $f \in \mathrm{GTP}_N$, $A \subset [-\pi,\pi]$ with $m([-\pi,\pi] \setminus A) \le s \le \pi/2$, and for every $p \in (0,\infty)$.

Theorem A.4.12 (L_q Bernstein-Type Inequality for GTP_N). *Let χ be a nonnegative, nondecreasing, convex function defined on $[0,\infty)$. There exists an absolute constant c such that*

$$\int_{-\pi}^{\pi} \chi\big(N^{-q}|f'(\theta)|^q\big) \, d\theta \le \int_{-\pi}^{\pi} \chi(cf(\theta)^q) \, d\theta$$

for every $f \in \mathrm{GTP}_N$ of the form (A.4.3) with each $r_j \ge 1$, and for every $q \in (0,1]$.

Corollary A.4.13 (L_p Bernstein-Type Inequality for GTP_N). *The inequality*

$$\int_{-\pi}^{\pi} |f'(\theta)|^p \, d\theta \le c^{p+1} N^p \int_{-\pi}^{\pi} |f(\theta)|^p \, d\theta$$

holds for every $f \in \mathrm{GTP}_N$ of the form (A.4.3) with each $r_j \ge 1$, and for every $p \in (0,\infty)$, where c is as in Theorem A.4.12.

Theorem A.4.14 (L_p Markov-Type Inequality for GAP_N). *There exists an absolute constant c such that*

$$\int_{-1}^{1} |f'(x)|^p \, dx \le c^{p+1} N^{2p} \int_{-1}^{1} |f(x)|^p \, dx$$

for every $f \in \mathrm{GAP}_N$ of the form (A.4.1) with each $r_j \ge 1$, and for every $p \in (0,\infty)$.

Theorems A.4.1 and A.4.2 can be easily obtained from their L_∞ analogs, Theorems A.4.1 and A.4.2, respectively; see E.15 and E.16, where hints are given.

Proof of Theorem A.4.12. For $n := \lfloor N \rfloor$ let

$$D_n(\theta) := \left| \sum_{j=-n}^{n} e^{ij\theta} \right|$$

be the modulus of the nth Dirichlet kernel. Choose $q \in (0,1]$, and set $m := 2q^{-1} \ge 2$. Let $g \in \mathrm{GTP}_N$ be of the form (A.4.3) with each $r_j \ge 1$. On applying the Nikolskii-type inequality of Theorem A.4.3 to

$$G := gD_n^m \in \mathrm{GTP}_{N+2nq^{-1}} ,$$

we obtain

$$(A.4.11) \qquad \begin{aligned} \|gD_n^m\|_K^q &\le c_1 \left(1 + q\left(N + 2nq^{-1}\right)\right) \|gD_n^m\|_{L_q(K)}^q \\ &\le c_1(1+3N) \int_{-\pi}^{\pi} (g(\theta)D_n^m(\theta))^q \, d\theta \\ &= c_1(1+3N) \int_{-\pi}^{\pi} g^q(\theta) D_n^2(\theta) \, d\theta . \end{aligned}$$

If $g \in \mathrm{GTP}_N$ is of the form (A.4.3) with each $r_j \ge 1$, then $m \ge 2$ implies that $G \in \mathrm{GTP}_N$ is of the form (A.4.3) with each $r_j \ge 1$ as well. If we apply the Bernstein-type inequality of Theorem A.4.6 to G and use (A.4.11), we can deduce that

$$\begin{aligned} &\left| g'(\theta) D_n^m(\theta) + m D_n^{m-1}(\theta) D_n'(\theta) g(\theta) \right|_K^q \\ &\qquad \le \left(c_3 \left(N + 2nq^{-1}\right)\right)^q \|gD_n^m\|_K^q \\ &\qquad \le c_3^q N^q \left(1 + 2q^{-1}\right)^q c_1(1+3N) \int_{-\pi}^{\pi} g^q(\theta) D_n^2(\theta) \, d\theta \end{aligned}$$

for every $\theta \in K$ (we take one-sided derivatives everywhere). By putting $\theta = 0$, and noticing that

$$D_n'(0) = 0 \quad \text{and } D_n(0)^m = (2n+1)^{2/q} \ge N^{2/q} ,$$

we get

$$(A.4.12) \qquad |g'(0)|^q \le cN^q \int_{-\pi}^{\pi} g^q(\theta)(2\pi)^{-1}(2n+1)^{-1} D_n^2(\theta) \, d\theta$$

with an absolute constant c. Now let $f \in \mathrm{GTP}_n$ be of the form (A.4.3) with each $r_j \ge 1$. Let $\tau \in K$ be fixed. On applying (A.4.12) to $g(\theta) := f(\theta + \tau)$, we conclude that

$$|f'(\tau)|^q \le cN^q \int_{-\pi}^{\pi} f^q(\theta)(2\pi)^{-1}(2n+1)^{-1} D_n^2(\theta - \tau) \, d\theta .$$

Hence

$$(A.4.13) \qquad N^{-q}|f'(\tau)|^q \le \int_{-\pi}^{\pi} cf^q(\theta)(2\pi)^{-1}(2n+1)^{-1} D_n^2(\theta - \tau) \, d\theta .$$

Since

$$(A.4.14) \qquad \int_{-\pi}^{\pi} (2\pi)^{-1}(2n+1)^{-1} D_n^2(\theta - \tau) \, d\theta = 1 ,$$

Jensen's inequality (see E.7) and (A.4.3) imply that

$$\chi\big(N^{-q}|f'(\tau)|^q\big) \le \int_{-\pi}^{\pi} \chi(cf^q(\theta))(2\pi)^{-1}(2n+1)^{-1}D_n^2(\theta-\tau)\,d\theta\,.$$

If we integrate both sides with respect to τ, Fubini's theorem and (A.4.14) (on interchanging the role of θ and τ) yield the inequality of the theorem. □

Proof of Corollary A.4.13. If $0 < p \le 1$, then Theorem A.4.12 yields the corollary with $q = p$ and $\chi(x) = x$. If $1 \le p < \infty$, then the corollary follows from Theorem A.4.12 again with $q = 1$ and $\chi(x) = x^p$. □

Proof of Theorem A.4.14. We distinguish two cases.

Case 1: $p \ge 1$. Let $f \in \mathrm{GAP}_N$ be of the form (A.4.1) with each $r_j \ge 1$. Then by E.10 b], $g(\theta) := f(\cos\theta) \in \mathrm{GTP}_N$ is of the form (A.4.3) with each $r_j \ge 1$. With the substitution $x = \cos\theta$, Corollary A.4.13 and Theorem A.4.11 imply that

$$\begin{aligned}(A.4.15)\qquad &\int_{-1}^{1} |f'(x)|^p (1-x^2)^{(p-1)/2}\,dx\\ &\qquad\le c_1^p N^p \int_{-1}^{1} f^p(x)(1-x^2)^{-1/2}\,dx\\ &\qquad\le c_1^p N^p \exp(c_2 pNN^{-1}) \int_{-\delta}^{\delta} f^p(x)(1-x^2)^{-1/2}\,dx\,,\end{aligned}$$

where $\delta := \max\{1-N^{-2}, \cos(\pi/16)\}$ and c_1 and c_2 are appropriate absolute constants. Since $p-1 \ge 0$, it follows from (A.4.15) that

$$\begin{aligned}(A.4.16)\qquad &\int_{-\delta}^{\delta} |f'(x)|^p\,dx\\ &\qquad\le (1-\delta^2)^{(1-p)/2} \int_{-\delta}^{\delta} |f'(x)|^p (1-x^2)^{(p-1)/2}\,dx\\ &\qquad\le (1-\delta^2)^{(1-p)/2} c_1^p N^p \exp(c_2 p)(1-\delta^2)^{-1/2} \int_{-\delta}^{\delta} f^p(x)\,dx\\ &\qquad\le c_3^p N^{p-1} N^p N \int_{-\delta}^{\delta} f^p(x)\,dx\\ &\qquad= c_3^p N^{2p} \int_{-\delta}^{\delta} f^p(x)\,dx\,,\end{aligned}$$

where c_3 is also an absolute constant. Since (A.4.16) is valid for every $f \in \mathrm{GAP}_N$ of the form (A.4.1) with each $r_j \ge 1$, the theorem follows by a linear shift from $[-\delta, \delta]$ to $[-1, 1]$.

Case 2: $0 < p \le 1$. Let $f \in \mathrm{GAP}_N$ be of the form (A.4.1) with each $r_j \ge 1$. Using the inequality $|a+b|^p \le |a|^p + |b|^p$ for $p \in (0,1]$, we can deduce that

$$\begin{aligned}(A.4.17)\qquad &\int_A \left(|f'(\cos\theta)||\sin\theta|^{1/p+1}\right)^p d\theta \\ &\qquad \le \int_A \left|\left(f(\cos\theta)|\sin\theta|^{1/p}\right)'\right|^p d\theta \\ &\qquad\quad + \int_A \left(f(\cos\theta)p^{-1}|\sin\theta|^{1/p-1}|\cos\theta|\right)^p d\theta\end{aligned}$$

for every measurable subset A of $[-\pi,\pi)$. Applying Theorem A.4.12 (with $\chi(x) = x$) to

$$g(\theta) := f(\cos\theta)|\sin\theta|^{1/p} \in \mathrm{GTP}_{N+1/p}\,,$$

then using (A.4.17) with $A := [-\delta,\delta]$, $\delta := 1-(N+1)^{-2}$, we conclude, by the substitution $x = \cos\theta$, that

$$\begin{aligned}(A.4.18)\qquad &\int_{-\delta}^{\delta} |f'(x)|^p (1-x^2)^{p/2}\,dx \\ &\qquad \le c_1(N+1/p)^p \int_{-1}^{1} f^p(x)\,dx + p^{-p}\int_{-\delta}^{\delta} f^p(x)(1-x^2)^{-p/2}\,dx\,,\end{aligned}$$

where c_1 is an absolute constant. Note that Theorem A.4.10, $0 < p \le 1$, and the choice of δ imply that

$$(A.4.19)\qquad \int_{-1}^{1} f^p(x)\,dx \le c_2 \int_{-\delta}^{\delta} f^p(x)\,dx$$

with an absolute constant c_2. A combination of (A.4.18) and (A.4.19) yields

$$\begin{aligned}(A.4.20)\qquad &\int_{-\delta}^{\delta} |f'(x)|^p\,dx \\ &\le (1-\delta^2)^{-p/2}\int_{-\delta}^{\delta} |f'(x)|^p (1-x^2)^{p/2}\,dx \\ &\le (1-\delta^2)^{-p/2}(c_1c_2(N+1/p)^p + p^{-p}(1-\delta^2)^{-p/2})\int_{-\delta}^{\delta} f^p(x)\,dx \\ &\le 2^{p/2}(N+1)^p(c_1c_2(N+1/p)^p + p^{-p}2^{p/2}(N+1)^p)\int_{-\delta}^{\delta} f^p(x)\,dx \\ &\le c_3N^{2p}\int_{-\delta}^{\delta} f^p(x)\,dx\,,\end{aligned}$$

where c_3 is an absolute constant. Since (A.4.20) is valid for every $f \in \mathrm{GAP}_N$ of the form (A.4.1) with each $r_j \ge 1$, the theorem follows by a linear shift from $[-\delta,\delta]$ to $[-1,1]$. □

Comments, Exercises, and Examples.

Most of the results of this section can be found in Erdélyi [91a] and [92a]; Erdélyi, Máté, and Nevai [92]; and Erdélyi, Li, and Saff [94]; however, the proofs are somewhat simplified here. For polynomials $f \in \mathcal{P}_n$ and for arbitrary $q \in (0, \infty)$, Theorem A.4.5 was also proved by Kemperman and Lorentz [79]. An early version of Markov's inequality in L_p for ordinary polynomials is proven in Hille, Szegő, and Tamarkin [37]. Weighted Markov- and Bernstein-type analogs of Theorems A.4.6 to A.4.8 are obtained in Erdélyi [92b]. Applications of the inequalities of this section are given in Erdélyi, Magnus, and Nevai [94] and in Erdélyi and Nevai [92], where bounds are established for orthonormal polynomials and related functions associated with (generalized) Jacobi weight functions. Further applications in the theory of orthogonal polynomials may be found in Freud [71] and Erdélyi [91d].

L_p extensions of Theorem 5.1.4 (Bernstein's inequality) and Theorem 5.1.8 (Markov's inequality) have been studied by a number of authors. The sharp L_p version of Bernstein's inequality for trigonometric polynomials was first established by Zygmund [77] for $p \geq 1$. Using an interpolation formula of M. Riesz, he proved that

$$\int_{-\pi}^{\pi} |t'(\theta)|^p \, d\theta \leq n^p \int_{-\pi}^{\pi} |t(\theta)|^p \, d\theta \tag{A.4.21}$$

for every $t \in \mathcal{T}_n$ (see E.5 h] of Appendix 4). For $0 < p < 1$, first Klein [51] and later Osval'd [76] proved (A.4.21) with a multiplicative constant $c(p)$. Nevai [79a] showed that $c(p) \leq 8p^{-1}$. Subsequently, Máté and Nevai [80] showed the validity of (A.4.21) with a multiplicative absolute constant, and then Arestov [81] proved (A.4.21) (with the best possible constant 1) for every $0 < p < 1$. Golitschek and Lorentz [89] gave a very elegant proof of Arestov's theorem.

The L_p analog of Markov's inequality states that

$$\int_{-1}^{1} |Q'(x)|^p \, dx \leq c^{p+1} n^{2p} \int_{-1}^{1} |Q(x)|^p \, dx \tag{A.4.22}$$

for every $Q \in \mathcal{P}_n$ and $0 < p < \infty$, where c is an absolute constant. This can be obtained from Arestov's theorem similarly to the way that Theorem A.4.14 is proven from Corollary A.4.13. To find the best possible constant in (A.4.22) is still an open problem even for $p = 2$ or $p = 1$.

The magnitude of

$$\frac{\|f'w\|_{[-1,1]}}{\|fw\|_{[-1,1]}}, \tag{A.4.23}$$

$$(A.4.24) \qquad \frac{|f'(y)w(y)|}{\|fw\|_{[-1,1]}}, \qquad -1 \le y \le 1,$$

and their corresponding L_p analogs for $f \in \mathcal{P}_n$ and generalized Jacobi weight functions

$$(A.4.25) \qquad w(z) = \prod_{j=1}^{k} |z - z_j|^{r_j}, \qquad z_j \in \mathbb{C}, \quad r_j \in (-1, \infty)$$

have been examined by several people. See, for example, Ditzian and Totik [88], Khalilova [74], Konjagin [78], Lubinsky and Nevai [87], Nevai [79a] and [79b], and Protapov [60], but a multiplicative constant depending on the weight function appears in these estimates. The magnitude of (A.4.23), (A.4.24), and their L_p analogs are examined in Erdélyi [92b] and [93], when both f and w are generalized nonnegative polynomials. In these inequalities only the degree of f, the degree of w, and a multiplicative absolute constant appear. The most general results are the following:

Theorem A.4.15. *There exists an absolute constant c such that*

$$\int_{-\pi}^{\pi} |f'(\theta)|^p w(\theta)\, d\theta \le c^{p+1}(N+M)^p(Mp^{-1}+1)^p \int_{-\pi}^{\pi} |f(\theta)|^p w(\theta)\, d\theta$$

and

$$\|f'w\|_{[-\pi,\pi]} \le c(N+M)(M+1)\|fw\|_{[-\pi,\pi]}$$

for every $f \in \mathrm{GTP}_N$ of the form (A.4.3) with each $r_j \ge 1$, for every $w \in \mathrm{GTP}_M$, and for every $p \in (0,\infty)$.

Theorem A.4.16. *There exists an absolute constant c such that*

$$\int_{-1}^{1} |f'(x)|^p w(x)\, dx \le c^{p+1}(N+M)^{2p}(Mp^{-1}+1)^{2p} \int_{-1}^{1} |f(x)|^p w(x)\, dx$$

and

$$\|f'w\|_{[-1,1]} \le c(N+M)^2\|fw\|_{[-1,1]}$$

for every $f \in \mathrm{GAP}_N$ of the form (A.4.1) with each $r_j \ge 1$, for every $w \in \mathrm{GAP}_M$, and for every $p \in (0,\infty)$.

E.1 Another Representation of Generalized Nonnegative Polynomials.

a] Show that if $f = \prod_{j=1}^{k} |Q_j|^{r_j}$ with each $Q_j \in \mathcal{P}_{n_j}$ and $r_j > 0$, then $f \in \mathrm{GAP}_N$ with $N \le \sum_{j=1}^{k} r_j n_j$. Similarly, if $f = \prod_{j=1}^{k} |Q_j|^{r_j}$ with each $Q_j \in \mathcal{T}_{n_j}$ and $r_j > 0$, then $f \in \mathrm{GTP}_N$ with $N \le \sum_{j=1}^{k} r_j n_j$.

b] Show that if $f \in \mathrm{GAP}_N$ is of the form (A.4.1), then $f = \prod_{j=1}^{m} |Q_j|^{r_j/2}$ with each $Q_j \in \mathcal{P}_2$ and $0 \le Q_j$ on $\mathbb{R}$. Similarly, if $f \in \mathrm{GTP}_N$ is of the form (A.4.3), then $f = \prod_{j=1}^{m} |Q_j|^{r_j/2}$ with each $Q_j \in \mathcal{T}_1$ and $0 \le Q_j$ on $\mathbb{R}$.

E.2 The Derivative of an $f \in \mathrm{GTP}_N$ with Real Zeros. Show that if $f \in \mathrm{GTP}_N$ is of the form (A.4.3) with each $r_j \geq 1$ and $z_j \in \mathbb{R}$, then $|f'| \in \mathrm{GTP}_N$ is of the form (A.4.3) with each $r_j > 0$ and $z_j \in \mathbb{R}$, and at least one of any two adjacent (in K) zeros of $|f'|$ has multiplicity exactly 1.

E.3 Generalized Nonnegative Polynomials with Rational Exponents. Show that if $f \in \mathrm{GAP}_N$ is of the form (A.4.1) with rational exponents $r_j = q_j/q$, where $q_j, q \in \mathbb{N}$, then $f^{2q} \in \mathcal{P}_{2qN}$, while if $f \in \mathrm{GTP}_N$ is of the form (A.4.3) with rational exponents r_j of the above form then $f^{2q} \in \mathcal{T}_{2qN}$.

E.4 Proof of (A.4.6). Prove (A.4.6) from the explicit formula (2.1.1) for the Chebyshev polynomial T_n.

E.5 Sharpness of the Remez-Type Inequality for GAP_N. Let

$$f_n(x) := \left| T_n\left(\frac{2x+s}{2-s}\right)\right|^{N/n} \in \mathrm{GAP}_N\,, \qquad n = 1, 2, \ldots\,.$$

Show that

$$m(\{x \in [-1,1] : f_n(x) \leq 1\}) = m([-1, 1-s]) = 2 - s$$

and

$$\lim_{n\to\infty} f_n(1) = \left(\frac{\sqrt{2}+\sqrt{s}}{\sqrt{2}-\sqrt{s}}\right)^N\,.$$

The upper bound in Theorem A.4.1 is actually not achieved by an element of GAP_N; see Erdélyi, Li, and Saff [94].

E.6 Proof of Theorem A.4.2.

Hint: First assume that $f \in \mathrm{GTP}_N$ is of the form (A.4.3) with rational exponents $r_j = q_j/q$, where $q_j, q \in \mathbb{N}$. Then $p := f^{2q} \in \mathcal{T}_{2qN}$, and the desired inequality can be obtained from Theorem 5.1.2 as in the proof of Theorem A.4.1. □

E.7 Corollaries of Theorems A.4.1 and A.4.2.

a] *The inequality*

$$m\left(\left\{x \in [-1,1] : f(x) \geq \exp(-N\sqrt{s})\|f\|_{[-1,1]}\right\}\right) \geq \frac{s}{8}$$

holds for every $f \in \mathrm{GAP}_N$ and $0 < s < 2$. In particular,

$$m\left(\left\{x \in [-1,1] : f(x) \geq e^{-1}\|f\|_{[-1,1]}\right\}\right) \geq (8N^2+4)^{-1}$$

holds for every $f \in \mathrm{GAP}_N$.

b] *The inequality*

$$m\left(\{\theta \in [-\pi,\pi) : f(\theta) \geq \exp(-Ns)\|f\|_K\}\right) \geq \frac{s}{4}$$

holds for every $f \in \mathrm{GTP}_N$ *and* $0 < s < 2\pi$. *In particular,*

$$m\left(\left\{\theta \in [-\pi,\pi) : f(\theta) \geq e^{-1}\|f\|_K\right\}\right) \geq (4(N+1))^{-1}$$

holds for every $f \in \mathrm{GTP}_N$.

E.8 Nonnegative Trigonometric Polynomials. Part a] restates E.3 c] of Section 2.4. Parts b] and c] discuss simple related observations.

a] Let $t \in \mathcal{T}_n$ be nonnegative on $\mathbb{R}$. Show that there is a $p \in \mathcal{P}_n^c$ such that $t(\theta) = |p(e^{i\theta})|^2$ for every $\theta \in \mathbb{R}$.

b] Let $p \in \mathcal{P}_n^c$ and $t(\theta) := |p(e^{i\theta})|^2$ for every $\theta \in \mathbb{R}$. Then $t \in \mathcal{T}_n$ and t is nonnegative on $\mathbb{R}$.

c] Show that if $t \in \mathcal{T}_n^c$, then there is a $p \in \mathcal{P}_{2n}^c$ such that $|t(\theta)| = |p(e^{i\theta})|$ for every $\theta \in \mathbb{R}$.

E.9 A Crucial Inequality in the Proof of Theorem A.4.5. Show that

$$|P(z)| \leq \left(\frac{2}{1+r}\right)^n |P(rz)|, \qquad z \in \partial D, \ \ r \in [0,1]$$

for every $P \in \mathcal{P}_n^c$ having all its zeros outside the open unit disk D.

Hint: Let $P(z) = c\prod_{j=1}^m (z - z_j)$, where $c \in \mathbb{C}$, $z_j \in \mathbb{C} \setminus D$, and $m \leq n$. Show that

$$|z - z_j| \leq \frac{2}{1+r}|rz - z_j|, \qquad j = 1,2,\ldots,m, \quad r \in [0,1].$$

□

E.10 $f \in \mathrm{GAP}_N$ **Implies** $f(\cos\theta) \in \mathrm{GTP}_N$.

a] Show that if $f \in \mathrm{GAP}_N$, then $g(\theta) = f(\cos\theta) \in \mathrm{GTP}_N$.

b] Show that if $f \in \mathrm{GAP}_N$ is of the form (8.1.1) with each $r_j \geq 1$, then $g(\theta) = f(\cos\theta) \in \mathrm{GTP}_N$ is of the form (A.4.3) with each $r_j \geq 1$.

E.11 A Property of $Q_{n,\omega}$. Let $Q_{n,\omega}$ be defined by (A.4.9) and (A.4.10). Show that there is an absolute constant $c_6 > 1$ such that $Q_{n,\omega}(\pi) \geq c_6$.

Hint: Use the explicit formula (2.1.1) for the Chebyshev polynomial T_n. □

E.12 A Property of the Zeros of a $g \in \mathrm{GTP}_N$. Assume that $g \in \mathrm{GTP}_N$ is of the form (A.4.3) and let

$$M := \sum_{\substack{j=1 \\ z_j \in [\pi-\rho,\pi+\rho]}}^{m} r_j \,.$$

Show that $M \le 3N\rho \|g\|^{-1}_{[-1,1]}$.

Hint: First assume that each r_j is rational with common denominator $q \in \mathbb{N}$, and apply E.12 of Section 5.1 to $p := g^{2q} \in \mathcal{T}_{2qN}$. □

E.13 Extremal Functions for the Bernstein-Type Inequality.

a] Let $r_j \ge 1$, $j = 1, 2, \dots, m$, be fixed real numbers. Show that there exists an $\widetilde{f} \in \mathrm{GTP}_N$ of the form

$$(A.4.26) \qquad \widetilde{f}(z) = \prod_{j=1}^{m} |\sin((z-\widetilde{\zeta}_j)/2)|^{r_j} \,, \qquad \widetilde{\zeta}_j \in \mathbb{C}$$

such that

$$\frac{|\widetilde{f}'(\pi)|}{\|\widetilde{f}\|_K} = \sup_f \frac{|f'(\pi)|}{\|f\|_K} \,,$$

where the supremum in the right-hand side is taken for all $f \in \mathrm{GTP}_N$ of the form

$$(A.4.27) \qquad f(z) = |\omega| \prod_{j=1}^{m} |\sin((z-z_j)/2)|^{r_j} \,, \qquad z_j \in \mathbb{C}, \quad 0 \ne \omega \in \mathbb{C}.$$

Hint: Write each f of the form (A.4.27) for which the supremum is taken as

$$f(z) = |\omega_0| \prod_{j=1}^{m} |\omega_j \sin((z-z_j)/2) \sin((z-\overline{z}_j)/2)|^{r_j/2} \,,$$

where the numbers $\omega_j > 0$ are defined by

$$\|\omega_j \sin((z-z_j)/2) \sin((z-\overline{z}_j)/2)\|_K = 1 \,, \qquad j = 1, 2, \dots, m \,,$$

and then use a compactness argument for each factor separately. □

b] Let $\widetilde{f}$ be as in part a]. Show that each zero of $\widetilde{f}$ is real, that is, $\widetilde{\zeta}_j \in \mathbb{R}$ for each j in (A.4.26).

Hint: Assume that there is an index $1 \le j \le m$ such that $\zeta_j \in \mathbb{C} \setminus \mathbb{R}$. Then

$$\widetilde{f}_\epsilon(z) := \widetilde{f}(z) \left(1 - \frac{\epsilon \sin^2((z-\pi)/2)}{\sin((z-\zeta_j)/2) \sin((z-\overline{\zeta}_j)/2)} \right)^{r_j} \in \mathrm{GTP}_N$$

with sufficiently small $\epsilon > 0$ contradicts the maximality of $\widetilde{f}$. □

E.14 A Corollary of the Remez-Type Inequality for GAP_N. *Suppose that $\alpha := 1 - (1+N)^{-2}$. Show that there is an absolute constant $c_9 > 0$ such that*

$$\|f\|_{[-1,1]} \le c_9 \|f\|_{[-\alpha,\alpha]}$$

for every $f \in \mathrm{GAP}_N$.

Hint: This is a corollary of Theorem A.4.1. □

E.15 Proof of Theorem A.4.10.

Outline. For $f \in \mathrm{GAP}_N$, let

$$I(f) := \left\{x \in [-1,1] : (\chi(f(x)))^p \ge \exp(-5pN\sqrt{2s})\|\chi(f)\|^p_{[-1,1]}\right\}.$$

From Theorem 5.1.1, $0 < s \le 1/2$, and the assumptions prescribed for χ it follows that $m(I(f)) \ge 2s$. Let $I := A \cap I(f)$. Since $m([-1,1] \setminus A) \le s$, $m(I) \ge s$. Therefore

$$\begin{aligned}\int_{[-1,1]\setminus A} (\chi(f(x)))^p dx &\le \int_{[-1,1]\setminus A} \|\chi(f)\|^p_{[-1,1]}\, dx \\ &\le \exp\bigl(5pN\sqrt{2s}\bigr) \int_I (\chi(f(x)))^p\, dx \\ &\le \exp\bigl(5pN\sqrt{2s}\bigr) \int_A (\chi(f(x)))^p\, dx\,.\end{aligned}$$

□

E.16 Proof of Theorem A.4.11.

Hint: For $f \in \mathrm{GTP}_N$, let

$$I(f) := \left\{\theta \in [-\pi,\pi] : (\chi(f(\theta)))^p \ge \exp(-8pNs)\|\chi(f)\|^p_K\right\}.$$

From Theorem 5.2.2, $0 < s \le \pi/4$, and the assumptions for χ, it follows that $m(I(f)) \ge 2s$. Now finish the proof as in the hint for E.1. □

E.17 Sharpness of Theorem A.4.10.

a] Show that there exists a sequence of polynomials $Q_n \in \mathcal{P}_n$ and an absolute constant $c > 0$ such that

$$\int_{-1}^{1} |Q_n(x)|^p\, dx \ge s\bigl(1 + \exp\bigl(cpn\sqrt{s}\bigr)\bigr) \int_{-1}^{1-s} |Q_n(x)|^p\, dx$$

for every $n \in \mathbb{N}$, $s \in (0,1]$, and $p \in (0,\infty)$.

Hint: Study the Chebyshev polynomials T_n transformed linearly from $[-1,1]$ to $[-1,1-s]$. □

b] Show that there exist a sequence of trigonometric polynomials $t_n \in \mathcal{T}_n$ and an absolute constant $c > 0$ such that

$$\int_{-\pi}^{\pi} |t_n(\theta)|^p \, d\theta \geq s(1 + \exp(cpns)) \int_{-\pi+s/2}^{\pi-s/2} |t_n(\theta)|^p \, d\theta$$

for every $p \in \mathbb{N}$, $s \in (0, \pi]$, and $p \in (0, \infty)$.

Hint: Study $Q_{n,\omega}$ defined by (A.4.9) with $\omega := \pi - s/2$. □

E.18 Sharpness of Corollary A.4.13 and Theorem A.4.14.

a] Find a sequence of trigonometric polynomials $t_n \in \mathcal{T}_n$ that shows the sharpness of Corollary A.4.13 up to the constant $c > 0$ for every $p \in (0, \infty)$ simultaneously.

Hint: Take $t_n(\theta) := \cos n\theta$. □

b] For every $p \in (0, \infty)$, find a sequence of polynomials $Q_{n,p} \in \mathcal{P}_n$ which shows the sharpness of Theorem A.4.14 up to the constant $c > 0$.

Outline. Let $L_k \in \mathcal{P}_k$ be the kth orthonormal Legendre polynomial on $[-1, 1]$ (see E.5 of Section 2.3), and let

$$G_m(x) := c \sum_{k=0}^{m} L_k'(1) L_k(x) \,,$$

where c is chosen so that

$$\int_{-1}^{1} G_m^2(x) \, dx = 1 \,. \tag{A.4.28}$$

Show that there exist absolute constants $c_1 > 0$ and $c_2 > 0$ (independent of m) such that

$$|G_m(1)| \geq c_1 m \qquad \text{and} \qquad |G_m'(1)| \geq c_2 m^3 \,. \tag{A.4.29}$$

For a fixed $n \in \mathbb{N}$, let $u := \lfloor 2/p \rfloor + 1$ and $m := \lfloor n/u \rfloor$. If $m \geq 1$, then let $Q_{n,p} := G_m^u \in \mathcal{P}_n$, otherwise let $Q_{n,p}(x) := x \in \mathcal{P}_n$. If $m = 0$, then the calculation is trivial, so let $m \geq 1$. Using (A.4.29) and the Nikolskii-type inequality of Theorem A.4.4, show that there exists an absolute constant $c_3 > 0$ such that

$$\int_{-1}^{1} |Q_{n,p}'(x)|^p \, dx \geq c_3^{p+1} m^{(u+2)p} (1 + pn)^{-2} \,. \tag{A.4.30}$$

Use the inequality

$$\int_{-1}^{1} |Q_{n,p}(x)|^p \, dx = \int_{-1}^{1} |G_m(x)|^{up} \, dx \leq \|G_m\|_{[-1,1]}^{up-2} \int_{-1}^{1} G_m^2(x) \, dx \,,$$

the Nikolskii-type inequality of Theorem A.4.4, and (A.4.28) to show that there is an absolute constant $c_4 > 0$ such that

$$\int_{-1}^{1} |Q_{n,p}(x)|^p \, dx \le c_4^{p+1} m^{up-2} . \tag{A.4.31}$$

Finally, combine (A.4.30) and (A.4.31). Note that if $p > 2$, then $m = \lfloor n \rfloor$, while if $p \le 2$, then $\frac{1}{4}p(n-1) \le m \le 2pn$. Note also that p^p is between two positive bounds for $p \in (0, 2]$. □

E.19 Sharpness of Theorems A.4.3 and A.4.4.

a] Let $q \in (0, \infty)$ be fixed. Show that there exists a sequence of polynomials $Q_{n,q} \in \mathcal{P}_n$ and an absolute constant $c > 0$ such that

$$\|Q_{n,q}\|_{[-1,1]} \ge c^{1+1/q}(1+qn)^{2/q}\|Q_{n,q}\|_{L_q[-1,1]}$$

for every $n \in \mathbb{N}$.

Hint: Study $Q_{n,p}$ with $p = q$ in the hint to the previous exercise. □

b] Let $Q_{n,q} \in \mathcal{P}_n$ be the same as in part a]. Show that there exist absolute constants $c_1 > 0$ and $c_2 > 0$ such that

$$\|Q_{n,q}\|_{L_p[-1,1]} \ge c_1^{1+1/q} c_2^{1/p}(1+qn)^{2/q}(1+pn)^{-2/p}\|Q_{n,q}\|_{L_p[-1,1]}$$

for every $n \in \mathbb{N}$ and $0 < q < p \le \infty$.

Hint: Combine part a] and the Nikolskii-type inequality of Theorem A.4.4. □

c] Let $q \in (0, \infty)$ be fixed. Show that there exists a sequence of trigonometric polynomials $t_{n,q} \in \mathcal{T}_n$ and an absolute constant $c > 0$ such that

$$\|t_{n,q}\|_K \ge c^{1+1/q}(1+qn)^{1/q}\|t_{n,q}\|_{L_q(K)}$$

for every $n \in \mathbb{N}$.

Hint: Let $t_{n,q}(\theta) := Q_{n,q}(\cos\theta)$, where $Q_{n,q}$ are the same as in part a]. Use part a] and the Schur-type inequality of Theorem A.4.5. □

d] Let $t_{n,q} \in \mathcal{T}_n$ be the same as in part c]. Show that there exist absolute constants $c_1 > 0$ and $c_2 > 0$ such that

$$\|t_{n,q}\|_{L_p(K)} \ge c_1^{1+1/q} c_2^{1/p}(1+qn)^{1/q}(1+pn)^{1/p}\|t_{n,q}\|_{L_q(K)}$$

for every $n \in \mathbb{N}$ and $0 < q < p \le \infty$.

Hint: Combine part a] and the Nikolskii-type inequality of Theorem A.4.3. □

E.20 Jensen's Inequality. Let χ be a real-valued convex function defined on $[0,\infty)$, and let f and w be nonnegative Riemann integrable functions on the interval $[a,b]$, where $\int_a^b w = 1$. Show that

$$\chi\left(\int_a^b f(x)w(x)\,dx\right) \le \int_a^b \chi(f(x))w(x)\,dx\,.$$

Hint: First assume that w is a step function. Use the fact that the convexity of χ implies its continuity, and hence the functions $\chi(f)w$ and fw are Riemann integrable. □

E.21 A Pointwise Remez-Type Inequality for GAP_N.

a] Show that there exists an absolute constant $c > 0$ such that

$$|p(y)| \le \exp\left(cn \min\left\{\frac{s}{\sqrt{1-y^2}}\,,\,\sqrt{s}\right\}\right)\,, \qquad y \in [-1,1]$$

for every $p \in \mathcal{P}_n$ and $s \in (0,1]$ satisfying

$$m(\{x \in [-1,1] : |p(x)| \le 1\}) \ge 2 - s\,,$$

that is, with the notation of Theorem 5.1.1, for every $p \in \mathcal{P}_n(s)$ with $s \in (0,1]$.

Proof. Assume, without loss of generality, that $y \in [0,1]$. Let

$$a := y + \tfrac{1}{2}(1-y)\,, \qquad \alpha := \arccos a\,,$$

$$\beta := 2\arccos y - \arccos a\,, \qquad \omega := \pi - \tfrac{1}{2}(\beta - \alpha)\,,$$

$$Q_{n,\omega}(\theta) := T_{2n}\left(\frac{\sin(\theta/2)}{\sin(\omega/2)}\right)\,,$$

where T_{2n} is the Chebyshev polynomial of degree $2n$ defined by (2.1.1), and let

$$Q_{n,\alpha,\beta}(\theta) := Q_{n,\omega}\left(\tfrac{1}{2}(\theta - (2\pi - (\alpha+\beta)))\right)\,.$$

Associated with $p \in \mathcal{P}_n(s)$, we introduce

$$g(\theta) := p(\cos\theta)Q_{n,\alpha,\beta}(\theta) \in \mathcal{T}_{2n}\,.$$

Obviously

$$|Q_{n,\alpha,\beta}(\theta)| \le 1\,, \qquad \theta \in [0,2\pi) \setminus (\alpha,\beta)$$

and

$$\|Q_{n,\alpha,\beta}\|_{[0,2\pi]} = Q_{n,\alpha,\beta}\left(\tfrac{1}{2}(\alpha+\beta)\right) = Q_{n,\omega}(\pi)\,.$$

The definition of ω implies that there exist absolute constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1\sqrt{1-y^2} \le \pi - \omega \le c_2\sqrt{1-y^2}\,.$$

This, together with E.3 c], yields that there are absolute constants $c_3 > 0$ and $c_4 > 0$ such that

$$Q_{n,\omega}(\pi) \ge \exp\bigl(c_3 n\sqrt{1-y^2}\bigr) \ge \exp\bigl(5n\sqrt{s}\bigr)$$

whenever $y \in [0, 1 - c_4 s]$. It follows now from Theorem 5.1.1 that

$$|g(\theta)| \le \exp\bigl(5n\sqrt{s}\bigr) \le Q_{n,\omega}(\pi)$$

for every $\theta \in [0, 2\pi) \setminus (\alpha, \beta)$ and $y \in [0, 1 - c_4 s]$. Furthermore

$$|g(\theta)| \le Q_{n,\omega}(\pi)$$

for every $\theta \in (\alpha, \beta)$ for which $|p(\cos\theta)| \le 1$. Note that

$$|\cos\theta| \le 1 - \tfrac{1}{2}(1-y)\,, \qquad \theta \in (\alpha, \beta)$$

and hence $p \in \mathcal{P}_n(s)$ with $s \in (0, 1]$ implies that there exists an absolute constant $c_5 > 0$ such that

$$m(\{\theta \in (\alpha, \beta) : |p(\cos\theta)| \ge 1\}) \le \frac{c_5 s}{\sqrt{1-y^2}}\,.$$

Therefore

$$f := (Q_{n,\omega}(\pi))^{-1} g \in \mathcal{T}_{2n}$$

satisfies

$$m(\{\theta \in [0, 2\pi) : |f(\theta)| \le 1\}) \ge 2\pi - \widetilde{s}$$

with

$$\widetilde{s} := \frac{c_5 s}{\sqrt{1-y^2}}\,, \qquad y \in [0, 1 - c_4 s]\,.$$

Applying Theorem 5.1.2 with f and $\widetilde{s}$, we conclude that

$$\begin{aligned} |p(y)| &= \bigl|p\left(\cos\left(\tfrac{1}{2}(\alpha+\beta)\right)\right)\bigr| = (Q_{n,\omega}(\pi))^{-1} g\left(\tfrac{1}{2}(\alpha+\beta)\right) \\ &= f\left(\tfrac{1}{2}(\alpha+\beta)\right) \le \exp(4n\widetilde{s}) \le \exp\left(\frac{c_5 s}{\sqrt{1-y^2}}\right) \end{aligned}$$

whenever $\widetilde{s} \in (0, \pi/2]$ and $y \in [0, 1 - c_4 s]$. If $s \in (0, 1]$, but $\widetilde{s} > \pi/2$ or $y \in [1 - c_4 s, 1]$, then Theorem 5.1.1 yields the required inequality. □

b] Show that the inequality of part a] is sharp up to the absolute constant $c > 0$.

Hint: Assume, without loss, that $y \in [0,1]$. Show that there exists an absolute constant $c_1 > 0$ such that the polynomials

$$W_{n,y,s}(x) := T_n\left(\frac{2x}{2-x} + \frac{s}{2-s}\right) \in \mathcal{P}_n(s)\,,$$

where T_n is the Chebyshev polynomial of degree n defined by (2.1.1), satisfy

$$|W_{n,y,s}(y)| \ge \exp(c_1 n\sqrt{s})$$

for every $n \in \mathbb{N}$, $y \in [1-s/2, 1]$, and $s \in (0,1]$.

If $y \in [0, 1-s/2]$, then let

$$a := y + \frac{s}{4}\,, \qquad \alpha := \arccos a\,,$$

$$\beta := 2\arccos y - \arccos a\,, \qquad \omega := \pi - \frac{\beta-\alpha}{2}\,,$$

$$Q_{n,\alpha,\beta}(\theta) := T_{2n}\left(\frac{\sin((\theta+\pi-(\alpha+\beta)/2)/2)}{\sin(\omega/2)}\right)\,,$$
$$R_{n,\alpha,\beta}(\theta) := \tfrac{1}{2}(Q_{n,\alpha,\beta}(\theta) + Q_{n,\alpha,\beta}(-\theta))\,,$$

and we define $W_{n,y,s}$ for every $n \in \mathbb{N}$, $y \in [0, 1-s/2)$, and $s \in (0,1]$ by

$$R_{n,\alpha,\beta}(\theta) = W_{n,y,s}(\cos\theta)\,, \qquad W_{n,y,s} \in \mathcal{P}_n\,.$$

Show that $W_{n,y,s} \in \mathcal{P}_n(s)$ and that there exists an absolute constant $c > 0$ such that

$$|W_{n,y,s}(y)| \ge \exp\left(cn\,\frac{s}{\sqrt{1-y^2}}\right)$$

for every $n \in \mathbb{N}$, $y \in [0, 1-s/2)$, and $s \in (0,1]$. □

c] Extend the validity of part a] to the class GAP_N; that is, prove that there exists an absolute constant $c > 0$ such that

$$|f(y)| \le \exp\left(cN\min\left\{\frac{s}{\sqrt{1-y^2}}\,,\ \sqrt{s}\right\}\right), \qquad y \in [-1,1]$$

for every $f \in \mathrm{GAP}_N$ and $s \in (0,1]$ satisfying

$$m(\{x \in [-1,1] : f(x) \le 1\}) \ge 2 - s\,.$$