A5

Inequalities for Polynomials with Constraints

Overview

This appendix deals with inequalities for constrained polynomials. Typically the constraints are on the location of the zeros, though various coefficient constraints are also considered.

Inequalities for Polynomials with Constraints

For integers $0 \le k \le n$, let

$$\mathcal{P}_{n,k} := \{p \in \mathcal{P}_n : p \text{ has at most } k \text{ zeros in } D\}$$

where, as before, $D := \{z \in \mathbb{C} : |z| < 1\}$. For $a < b$, let

$$\mathcal{B}_n(a,b) := \left\{p \in \mathcal{P}_n : p(x) = \pm \sum_{j=0}^{n} \alpha_j (b-x)^j (x-a)^{n-j}, \quad \alpha_j \ge 0\right\}.$$

For integers $0 \le k \le n$, let

$$\widetilde{\mathcal{P}}_{n,k}(a,b) := \{p = hq : h \in \mathcal{B}_{n-k}(a,b), \ q \in \mathcal{P}_k\}.$$

Two useful relations, given in E.1, are

$$\mathcal{P}_{n,0} \subset \mathcal{B}_n(-1,1) \qquad \text{and} \qquad \mathcal{P}_{n,k} \subset \widetilde{\mathcal{P}}_{n,k}(-1,1).$$

Theorem A.5.1 (Markov-Type Inequality for $\widetilde{\mathcal{P}}_{n,k}$). *The inequality*

$$\|p^{(m)}\|_{[-1,1]} \le (18n(k+2m+1))^m \|p\|_{[-1,1]}$$

holds for every $p \in \widetilde{\mathcal{P}}_{n,k}(-1,1)$. (When $m = 1$, the constant 18 can be replaced by 9.)

Proof. First we study the case $m = 1$. Applying Theorem 6.1.1 (Newman's inequality) with

$$(\lambda_0, \lambda_1, \dots, \lambda_k) = (n-k, n-k+1, \dots, n)$$

and using a linear shift from $[0,1]$ to $[-1,1]$, we obtain that

$$|p'(1)| \le \tfrac{9}{2} n(k+1) \|p\|_{[-1,1]} \tag{A.5.1}$$

for every $p \in \widetilde{\mathcal{P}}_{n,k}(-1,1)$ of the form

$$p(x) = (x+1)^{n-k} q(x)\,, \qquad q \in \mathcal{P}_k\,. \tag{A.5.2}$$

Now let $p \in \widetilde{\mathcal{P}}_{n,k}(-1,1)$ be of the form $p = hq$ with $q \in \mathcal{P}_k$ and

$$h(x) = \sum_{j=0}^{n-k} \alpha_j (1-x)^j (x+1)^{n-k-j} \qquad \text{with each } \alpha_j \ge 0\,.$$

Without loss of generality, we may assume that $n - k \ge 1$, otherwise Theorem 5.1.8 (Markov's inequality) gives the theorem. Using (A.5.1) and the fact that each $\alpha_j \ge 0$, we get

$$\begin{aligned} &|p'(1)| \\ &\quad = \left| \sum_{j=0}^{n-k} \left(\alpha_j (1-x)^j (x+1)^{n-k-j} q(x)\right)'(1) \right| \\ &\quad = \left| \sum_{j=0}^{1} (\alpha_j (1-x)^j (x+1)^{n-k-j} q(x))'(1) \right| \\ &\quad \le \tfrac{9}{2} n(k+2) \left\| (x+1)^{n-k-1} (\alpha_0 (x+1) + \alpha_1 (1-x)) q(x) \right\|_{[-1,1]} \\ &\quad \le \tfrac{9}{2} n(k+2) \left\| \sum_{j=0}^{n-k} \alpha_j (1-x)^j (x+1)^{n-k-j} q(x) \right\|_{[-1,1]} \\ &\quad \le \tfrac{9}{2} n(k+2) \|p\|_{[-1,1]}\,. \end{aligned} \tag{A.5.3}$$

Now let $y \in [-1,1]$ be arbitrary. To estimate $|p'(y)|$ we distinguish two cases.

If $y \in [0,1]$, then by a linear shift from $[-1,1]$ to $[-1,y]$, we obtain from (A.5.3) that

$$(A.5.4) \qquad |p'(y)| \le \frac{9}{y+1}\, n(k+2)\|p\|_{[-1,y]} \le 9n(k+2)\|p\|_{[-1,1]}$$

for every $p \in \widetilde{\mathcal{P}}_{n,k}(-1,y)$. It follows from E.1 d] that

$$\widetilde{\mathcal{P}}_{n,k}(-1,1) \subset \widetilde{\mathcal{P}}_{n,k}(-1,y)\,.$$

So (A.5.4) holds for every $p \in \widetilde{\mathcal{P}}_{n,k}(-1,1)$.

If $y \in [-1,0]$, then by a linear shift from $[-1,1]$ to $[y,1]$, we obtain from (A.5.3) that

$$(A.5.5) \qquad |p'(y)| \le \frac{9}{1-y}\, n(k+2)\|p\|_{[y,1]} \le 9n(k+2)\|p\|_{[-1,1]}$$

for every $p \in \widetilde{\mathcal{P}}_{n,k}(y,1)$. By E.1 d] again,

$$\widetilde{\mathcal{P}}_{n,k}(-1,1) \subset \widetilde{\mathcal{P}}_{n,k}(y,1)\,.$$

So (A.5.5) holds for every $p \in \widetilde{\mathcal{P}}_{n,k}(-1,1)$, which finishes the case when $m = 1$.

Now we turn to the case when $m \ge 2$. Note that an induction on m does not work directly for an arbitrary $p \in \widetilde{\mathcal{P}}_{n,k}(-1,1)$. However, it follows by induction on m that

$$(A.5.6) \qquad \|p^{(m)}\|_{[-1,1]} \le (9n(k+m+1))^m \|p\|_{[-1,1]}$$

for every $p \in \widetilde{\mathcal{P}}_{n,k}(-1,1)$ of the form (A.5.2). Now let $p \in \widetilde{\mathcal{P}}_{n,k}(-1,1)$ be of the form $p = hq$, where $q \in \mathcal{P}_k$ and

$$h(x) = \sum_{j=0}^{n-k} \alpha_j (1-x)^j (x+1)^{n-j} \quad \text{with each } \alpha_j \ge 0.$$

For notational convenience let $s := \min\{n-k, m\}$. Using (A.5.6) and the fact that each $\alpha_j \ge 0$, we get

$$\begin{aligned}
(A.5.7) \quad & |p^{(m)}(1)| \\
&= \left| \sum_{j=0}^{n-k} (\alpha_j (1-x)^j (1+x)^{n-k-j} q(x))^{(m)}(1) \right| \\
&= \left| \sum_{j=0}^{s} (\alpha_j (1-x)^j (1+x)^{n-k-j} q(x))^{(m)}(1) \right| \\
&\le (9n(k+s+m+1))^m \left\| \sum_{j=0}^{s} \alpha_j (1-x)^j (x+1)^{n-k-j} q(x) \right\|_{[-1,1]} \\
&\le (9n(k+2m+1))^m \left\| \sum_{j=0}^{n-k} \alpha_j (1-x)^j (x+1)^{n-k-j} q(x) \right\|_{[-1,1]} \\
&= (9n(k+2m+1))^m \|p\|_{[-1,1]}\,.
\end{aligned}$$

From (A.5.7), in a similar fashion to the case when $m = 1$, we conclude that

$$\|p^{(m)}\|_{[-1,1]} \le (18n(k+2m+1))^m \|p\|_{[-1,1]}$$

for every $p \in \widetilde{\mathcal{P}}_{n,k}(-1,1)$, which finishes the proof. □

Theorem A.5.1 is essentially sharp as is shown in E.2 and E.4 f]. However, a better upper bound can be given for $|p^{(m)}(y)|$ when $y \in (-1,1)$ is away from the endpoints -1 and 1.

Theorem A.5.2 (Markov-Bernstein Type Inequality for $\mathcal{P}_{n,k}$). *There exists a constant $c(m)$ depending only on m such that*

$$|p^{(m)}(y)| \le c(m) \left(\min \left\{ n(k+1)\,,\, \frac{\sqrt{n(k+1)}}{\sqrt{1-y^2}} \right\} \right)^m \|p\|_{[-1,1]}$$

for every $p \in \mathcal{P}_{n,k}$ and $y \in [-1,1]$.

Theorem A.5.2 has been proved in Borwein and Erdélyi [94]. Its proof is long, and we do not reproduce it here. However, the proof of a less sharp version, where $\sqrt{1-y^2}$ is replaced by $1-y^2$, is outlined in E.4 and E.5. The factor $\sqrt{n(k+1)}$ in Theorem A.5.2 is essentially sharp in the case that $m = 1$; see E.7.

Theorem A.5.3 (Markov-Bernstein Type Inequality for $\mathcal{B}_n(-1,1)$). *There exists a constant $c(m)$ depending only on m such that*

$$|p^{(m)}(y)| \le c(m) \left(\min \left\{ n\,,\, \frac{\sqrt{n}}{\sqrt{1-y^2}} \right\} \right)^m \|p\|_{[-1,1]}$$

for every $p \in \mathcal{B}_n(-1,1)$ (hence for every $p \in \mathcal{P}_{n,0}$) and $y \in [-1,1]$.

Note that the uniform (Markov-type) upper bound of the above theorem is a special case ($k = 0$) of Theorem A.5.1. Our proof of Theorem A.5.3 offers another way to prove the case of Theorem A.5.1 when $k = 0$. First we need a lemma.

Lemma A.5.4. *For $n \in \mathbb{N}$ and $y \in \mathbb{R}$, let*

$$\Delta_{n,y} := \frac{1}{4} \left(\frac{\sqrt{(1-y^2)^+}}{\sqrt{n}} + \frac{1}{n} \right),$$

where $x^+ := \max\{x, 0\}$. Then

$$|p(y + i\gamma\Delta_{n,y})| \le \sqrt{2}e \left| p\left(y \pm \tfrac{1}{2n}\right) \right|$$

for every $p \in \mathcal{B}_n(-1,1)$, $y \in \left[-1 - \frac{1}{8n}, 1 + \frac{1}{8n}\right]$, and $\gamma \in [0,1]$, where i is the imaginary unit. The $+$ sign is taken if $y \in \left[-1 - \frac{1}{8n}, 0\right)$, and the $-$ sign is taken if $y \in \left[0, 1 + \frac{1}{8n}\right]$.

Proof. It is sufficient to prove the lemma for the polynomials

$$p_{n,j}(x) := (1-x)^j(x+1)^{n-j}\,, \qquad n \in \mathbb{N}\,, \quad j = 0, 1, \ldots, n\,.$$

The general case when

$$p = \sum_{j=0}^{n} \alpha_j p_{n,j} \qquad \text{with each } \alpha_j \geq 0 \text{ or each } \alpha_j \leq 0$$

can then be obtained by taking linear combinations. Without loss of generality, we may assume that $y \in \left[0, 1 + \frac{1}{8n}\right]$. Then

$$\Delta_{n,y}^2 \leq \frac{1}{8}\left(\frac{(1-y^2)^+}{n} + \frac{1}{n^2}\right) \leq \frac{(1-y)^+}{4n} + \frac{1}{8n^2}$$

from which it follows that

$$\begin{aligned}
|p_{n,j}(y + i\gamma\Delta_{n,y})| &\leq |p_{n,j}(y + i\Delta_{n,y})| \\
&= \left((1-y)^2 + \Delta_{n,y}^2\right)^{j/2} \left((1+y)^2 + \Delta_{n,y}^2\right)^{(n-j)/2} \\
&\leq \left((1-y) + \tfrac{1}{2n}\right)^j \left((1+y) + \tfrac{1}{2n}\right)^{n-j} \\
&= \left(1 - \left(y - \tfrac{1}{2n}\right)\right)^j \left(1 + \left(y - \tfrac{1}{2n}\right)\right)^{n-j} \left(\frac{1 + y + \frac{1}{2n}}{1 + y - \frac{1}{2n}}\right)^{n-j} \\
&\leq \left(\frac{2n+1}{2n-1}\right)^n p_{n,j}\left(y - \tfrac{1}{2n}\right) \leq \sqrt{2}e\, p_{n,j}\left(y - \tfrac{1}{2n}\right)\,,
\end{aligned}$$

and the lemma is proved. □

Proof of Theorem A.5.3. For $n \in \mathbb{N}$ and $y \in [-1,1]$, let $B_{n,y}$ denote the circle of the complex plane with center y and radius $\frac{1}{4}\Delta_{n,y}$. Using E.4, Lemma A.5.4, and the maximum principle, we obtain

$$|p(z)| \leq \sqrt{2}e\,\|p\|_{[-1,1]}$$

for every $p \in \mathcal{B}_n(-1,1)$, $z \in B_{n,y}$, $n \in \mathbb{N}$, and $y \in [-1,1]$. Hence, by Cauchy's integral formula,

$$\begin{aligned}
|p^{(m)}(y)| &= \left|\frac{m!}{2\pi i}\int_{B_{n,y}} \frac{p(\xi)}{(\xi - y)^{m+1}}\, d\xi\right| \\
&\leq \frac{m!}{2\pi}\int_{B_{n,y}} \left|\frac{p(\xi)}{(\xi - y)^{m+1}}\right| |\, d\xi| \\
&\leq \frac{m!}{2\pi}\, 2\pi\, \tfrac{1}{4}\Delta_{n,y}\left(\tfrac{1}{4}\Delta_{n,y}\right)^{-(m+1)} \sqrt{2}e\|p\|_{[-1,1]} \\
&\leq c(m)\left(\min\left\{n\,,\ \frac{\sqrt{n}}{\sqrt{1-y^2}}\right\}\right)^m \|p\|_{[-1,1]}
\end{aligned}$$

for every $p \in \mathcal{B}_n(-1,1)$ and $y \in [-1,1]$, which proves the theorem. □

For $r \in (0,1]$, let

$$D_r^{\pm} := \{z \in \mathbb{C} : |z \pm (1-r)| < r\}.$$

For $n \in \mathbb{N}$, $k = 0, 1, \dots, n$, and $r \in (0,1]$, let $\mathcal{P}_{n,k,r}^{\pm}$ be the set of all $p \in \mathcal{P}_n$ that have at most k zeros (counting multiplicities) in $D_r^{\pm}$, and let $\mathcal{P}_{n,k,r} := \mathcal{P}_{n,k,r}^{+} \cap \mathcal{P}_{n,k,r}^{-}$. The following result is proved in Erdélyi [89a]. The proof in a special case is outlined in E.9.

Theorem A.5.5 (Markov-Type Inequality for $\mathcal{P}_{n,k,r}$). *There exists a constant $c(m)$ depending only on m such that*

$$\|p^{(m)}\|_{[-1,1]} \le c(m) \left(\frac{n(k+1)^2}{\sqrt{r}}\right)^m \|p\|_{[-1,1]}$$

for every $p \in \mathcal{P}_{n,k,r}$, $m \in \mathbb{N}$, and $r \in (0,1]$.

We state, without proof, the L_q analogs of Theorem A.5.2 for $m = 1$, established in Borwein and Erdélyi [to appear 2].

Theorem A.5.6 (L_q Markov-Type Inequality for $\mathcal{P}_{n,k}$). *There exists an absolute constant c such that*

$$\int_{-1}^{1} |p'(x)|^q \, dx \le c^{q+1}(n(k+1))^q \int_{-1}^{1} |p(x)|^q \, dx$$

for every $p \in \mathcal{P}_{n,k}$ and $q \in (0,\infty)$.

Theorem A.5.7 (L_q Bernstein-Type Inequality for $\mathcal{P}_{n,k}$). *There exists an absolute constant c such that*

$$\int_{-\pi}^{\pi} |p'(\cos t) \sin t|^q \, dt \le c^{q+1}(n(k+1))^{q/2} \int_{-\pi}^{\pi} |p(\cos t)|^q \, dt$$

for every $p \in \mathcal{P}_{n,k}$ and $q \in (0,\infty)$.

Both of the above inequalities are sharp up to the absolute constant $c > 0$.

A Markov-type inequality for polynomials having at most k zeros in the disk

$$D_r := \{z \in \mathbb{C} : \ |z| < r\}$$

is given by the following theorem; see Erdélyi [90a].

Theorem A.5.8 (Markov-Type Inequality for Polynomials with At Most k Zeros in D_r). *Let $k \in \mathbb{N}$ and $r \in (0,1]$. Then there exist constants $c_1(k) > 0$ and $c_2(k) > 0$ depending only on k so that*

$$c_1(k)\big(n + (1-r)n^2\big) \le \sup_p \frac{\|p'\|_{[-1,1]}}{\|p\|_{[-1,1]}} \le c_2(k)\big(n + (1-r)n^2\big),$$

where the supremum is taken for all $p \in \mathcal{P}_n$ that have at most k zeros in D_r.

Comments, Exercises, and Examples.

Erdős [40] proved that

$$\|p'\|_{[-1,1]} \le \frac{n}{2}\left(\frac{n}{n-1}\right)^{n-1}\|p\|_{[-1,1]} < \frac{e}{2}n\,\|p\|_{[-1,1]}$$

for every $p \in \mathcal{P}_{n,0}$ $n \ge 2$, having only real zeros. In this result, the assumption that p has only real zeros can be dropped. In E.11 we outline the proof of the above inequality for every $p \in \mathcal{P}_{n,0}$, $n \ge 2$. The polynomials

$$p_n(x) := (x+1)^{n-1}(1-x)\,, \qquad n = 1,2,\dots\,,$$

show that Erdős's result is the best possible for $\mathcal{P}_{n,0}$. Erdős [40] also showed that there exists an absolute constant c such that

$$|p'(y)| \le \frac{c\sqrt{n}}{(1-y^2)^2}\,\|p\|_{[-1,1]}$$

for every $p \in \mathcal{P}_{n,0}$ having only real zeros. Markov- and Bernstein-type inequalities for $\mathcal{B}_n(-1,1)$ were first established by Lorentz [63], who proved Theorem A.5.3. Lorentz's approach is outlined for $m = 1$ in E.8. The proof presented in the text follows Erdélyi [91c]. Up to the constant $c(m) > 0$ Theorem A.5.3 is sharp; see E.12. Scheick [72] found the best possible asymptotic constant in Lorentz's Markov-type inequality for $m = 1$ and $m = 2$. He proved the inequalities

$$\|p'\|_{[-1,1]} \le \frac{e}{2}n\,\|p\|_{[-1,1]} \quad \text{and} \quad \|p''\|_{[-1,1]} \le \frac{e}{2}n(n-1)\|p\|_{[-1,1]}$$

for every $p \in \mathcal{B}_n(-1,1)$ (and hence for every $p \in \mathcal{P}_{n,0}$). Note that with $p_n(x) := (x+1)^{n-1}(1-x)$,

$$\frac{\|p_n'\|_{[-1,1]}}{n\|p_n\|_{[-1,1]}} \to \frac{e}{2} \quad \text{and} \quad \frac{\|p_n''\|_{[-1,1]}}{n(n-1)\|p\|_{[-1,1]}} \to \frac{e}{2} \quad \text{as } n \to \infty\,.$$

A slightly weaker version of Theorem A.5.1 was conjectured by Szabados [81], who gave polynomials $p_{n,k} \in \mathcal{P}_{n,k}$ with only real zeros so that

$$|p_{n,k}'(1)| \ge \tfrac{1}{3}n(k+1)\|p_{n,k}\|_{[-1,1]}$$

for all integers $0 \le k \le n$. After some results of Szabados and Varma [80] and Máté [81], Szabados's conjecture has been settled in Borwein [85], where it is shown that

$$\|p'\|_{[-1,1]} \le 9n(k+1)\|p\|_{[-1,1]}$$

for every $p \in \mathcal{P}_{n,k}$ having $n-k$ zeros in $\mathbb{R} \setminus (-1,1)$. The crucial part of this proof is outlined in E.4 d] and e]. The above inequality is extended to all $p \in \widetilde{\mathcal{P}}_{n,k}$ (and hence to all $p \in \mathcal{P}_{n,k}$) and to higher derivatives in Erdélyi [91b], whose approach is followed in our proof of Theorem A.5.1.

After partial results of Erdélyi and Szabados [88, 89b] and Erdélyi [91b], the "right" Markov-Bernstein-type analogs of Theorem A.5.2 for the classes $\mathcal{P}_{n,k}$ has been proved in Borwein and Erdélyi [94]. Note that $\mathcal{P}_{n,n} = \mathcal{P}_n$, and hence, up to the constant $c(m)$, Theorem A.5.2 contains the classical inequalities of Markov and Bernstein, and of course the case $k = 0$ gives back Lorentz's inequalities for the classes $\mathcal{P}_{n,0} \subset \mathcal{B}_n(-1,1)$. The "right" Markov- and Bernstein-type inequalities of Theorems A.5.8 and A.5.9 for all classes $\mathcal{P}_{n,k}$ in L_q, $0 < q < \infty$, are established in Borwein and Erdélyi [to appear 2].

The Markov-type inequality for the classes $\mathcal{P}_{n,k,r}$ given by Theorem A.5.5 is proved in Erdélyi [89a]. A weaker version, when $k = 0$ and the factor $r^{-1/2}$ is replaced by a constant $c(r)$ depending on r, is established in Rahman and Labelle [68]. When $k = 0$, Theorem A.5.5 is sharp up to the constant $c(m)$ depending only on m; see E.10.

E.1 Relation Between Classes of Constrained Polynomials.

a] Show that $\mathcal{P}_{n,0}(-1,1) \subset \mathcal{B}(-1,1)$. (This is an observation of G. G. Lorentz.)

Hint: Use the identities

$$(x-\alpha)(x-\overline{\alpha}) = \tfrac{1}{4}|1+\alpha|^2(1-x)^2 + \tfrac{1}{2}(|\alpha|^2-1)(1-x^2) + \tfrac{1}{4}|1-\alpha|^2(1+x)^2$$

and

$$x - \alpha = \tfrac{1}{2}(1-\alpha)(x+1) - \tfrac{1}{2}(\alpha+1)(1-x)\,.$$

□

b] Show that $\mathcal{P}_{n,k} \subset \widetilde{\mathcal{P}}_{n,k}(-1,1)$.

c] Show that $\mathcal{B}_n(a,b) \subset \mathcal{B}_n(c,d)$ whenever $[c,d] \subset [a,b]$.

Hint: First show that

$$x - a = \alpha_1(x-c) + \alpha_2(d-x) \quad \text{and} \quad b - x = \alpha_3(x-c) + \alpha_4(d-x)$$

with some nonnegative coefficients. □

d] Show that $\widetilde{\mathcal{P}}_{n,k}(a,b) \subset \widetilde{\mathcal{P}}_{n,k}(c,d)$ whenever $[c,d] \subset [a,b]$.

E.2 Sharpness of Theorem A.5.1. Show that there exist polynomials $p_{n,k} \in \mathcal{P}_{n,k}$ of the form

$$p_{n,k}(x) = (x+1)^{n-k} q_{n,k}(x)\,, \qquad q_{n,k} \in \mathcal{P}_k$$

such that

$$|p'_{n,k}(1)| \geq \tfrac{1}{6} n(k+1) \|p_{n,k}\|_{[-1,1]}$$

for every $n \in \mathbb{N}$, $k = 0, 1, \ldots, n$.

Hint: Use the lower bound in Theorem 6.1.1 (Newman's inequality) with

$$(\lambda_0, \lambda_1, \dots, \lambda_k) = (n-k, n-k+1, \dots, n\}\,.$$

□

E.3 A Technical Detail in the Proof of Lemma A.5.4. For $n \in \mathbb{N}$ and $y \in [-1,1]$, let

$$F_n := \left\{ z = a + ib : \ a \in \left[-1 - \tfrac{1}{8n}, 1 + \tfrac{1}{8n}\right] \ , \ b \in (-\Delta_{n,a}, \Delta_{n,a}) \right\} \ ,$$

and

$$B_{n,y} := \left\{ z \in \mathbb{C} : |z - y| = \tfrac{1}{4}\Delta_{n,y} \right\} \ .$$

Show that $B_{n,y} \subset F_n$ for every $y \in [-1,1]$.

E.4 Bernstein-Type Inequality for $\mathcal{P}_{n,k}$. Prove that there exists an absolute constant c such that

$$|p'(y)| \le \frac{c\sqrt{n(k+1)}}{1-y^2} \, \|p\|_{[-1,1]}$$

for every $p \in \mathcal{P}_{n,k}$, and $y \in (-1,1)$. Proceed as follows:

a] Show that for every $n \in \mathbb{N}$ and $k = 0, 1, \dots, n$, there exists a polynomial $Q \in \mathcal{P}_{n,k}$ such that

$$\frac{|Q'(0)|}{\|Q\|_{[-1,1]}} = \sup_{p \in \mathcal{P}_{n,k}} \frac{|p'(0)|}{\|p\|_{[-1,1]}} \, .$$

Hint: Use a compactness argument. Use Rouché's theorem to show that the uniform limit of a sequence of polynomials from $\mathcal{P}_{n,k}$ on $[-1,1]$ is also in $\mathcal{P}_{n,k}$. □

b] Show that Q has only real zeros, and at most $k+1$ of them (counting multiplicities) are different from ± 1.

Hint: Use a variational method. For example, if $z_0 \in \mathbb{C} \setminus \mathbb{R}$ is a zero of Q, then the polynomial

$$Q_\epsilon(x) := Q(x) \left(1 - \frac{\epsilon x^2}{(x - z_0)(x - \overline{z}_0)} \right)$$

with sufficiently small $\epsilon > 0$ is in $\mathcal{P}_{n,k}$ and contradicts the maximality of Q. □

c] Let $\delta := (36n(k+3))^{-1}$. Show that

$$\|p\|_{[-\delta,1]} \le 2 \, \|p\|_{[0,1]}$$

for every polynomial $\mathcal{P}_{n,k}$ having all its zeros in $[0,\infty)$ with at most k of them (counting multiplicities) in $(0,1)$.

Hint: Use the Mean Value Theorem and the result of Theorem A.5.1 transformed linearly from $[-1,1]$ to $[0,1]$. □

d] Let $z_0 := i\gamma(36n(k+3))^{-1/2}$, where i is the imaginary unit and $\gamma \in [0,1]$. Show that

$$|p(z_0)| \le \sqrt{2}\,\|p\|_{[-1,1]}$$

for every polynomial $p \in \mathcal{P}_n$ having only real zeros with at most k of them (counting multiplicities) in $(-1,1)$.

Outline. Let $p(x) = \prod_{j=1}^{s}(x-u_j)$ with some $s \le n$. Applying part c] to

$$q(x) := \prod_{j=1}^{s}(x-u_j^2)\,,$$

we obtain

$$\begin{aligned}|q(-(36n(k+3))^{-1})| &\le 2\|q\|_{[0,1]} = 2\|q(x^2)\|_{[0,1]} = 2\|p(x)p(-x)\|_{[-1,1]}\\ &\le 2\|p\|^2_{[-1,1]}\,.\end{aligned}$$

Observe that

$$\begin{aligned}|p(z_0)|^2 \le |p(i(36n(k+3))^{-1/2})|^2 &= \prod_{j=1}^{s}(u_j^2 + (36n(k+3))^{-1})\\ &= |q(-(36n(k+3))^{-1})|\,,\end{aligned}$$

which, together with the previous inequality, yields the desired result. □

e] Let $Q \in \mathcal{P}_{n,k}$ be the extremal polynomial of part a], and let

$$F_{n,k} := \left\{z = a+ib:\ \ |a| \le 1\,,\ \ |b| \le (36n(k+3))^{-1/2}(1-|a|)\right\}\,.$$

Show that

$$|Q(z)| \le \sqrt{2}\,\|Q\|_{[-1,1]}$$

for every $z \in F_{n,k}$.

Hint: Use parts b] and d] and a linear shift from the interval $[-1,1]$ to $[2\,\mathrm{Re}(z)-1,1]$ if $\mathrm{Re}(z) \ge 0$, or to $[-1,2\,\mathrm{Re}(z)+1]$ if $\mathrm{Re}(z) < 0$. □

f] Show that there exists an absolute constant $c > 0$ such that

$$|p'(0)| \le c\sqrt{n(k+1)}\,\|p\|_{[-1,1]}$$

for every $p \in \mathcal{P}_{n,k}$.

Hint: By part a] it is sufficient to prove the inequality when $p = Q$, in which case use part e] and Cauchy's integral formula. □

g] Prove the main result stated in the beginning of the exercise.

Hint: In order to estimate $|p'(y)|$ when, for example, $y \in [0,1]$ use a linear shift from $[-1,1]$ to $[2y-1,1]$ and apply part f]. □

E.5 Bernstein-Type Inequality for $\mathcal{P}_{n,k}$ for Higher Derivatives. Prove that there exists a constant $c(m)$ depending only on m so that

$$|p^{(m)}(y)| \le c(m)\left(\frac{\sqrt{n(k+1)}}{1-y^2}\right)^m \|p\|_{[-1,1]}$$

for every $p \in \mathcal{P}_{n,k}$ and $y \in (-1,1)$.

Hint: First show that for every $n, m \in \mathbb{N}$, $k = 0,1,\dots,n$, and $\delta \in (0,1]$, there exists a polynomial $Q_\delta \in \mathcal{P}_{n,k}$ such that

$$\frac{|Q_\delta^{(m)}(0)|}{\|Q_\delta\|_{[-1,1]\setminus[-\delta,\delta]}} = \sup_{p\in\mathcal{P}_{n,k}} \frac{|p^{(m)}(0)|}{\|p\|_{[-1,1]\setminus[-\delta,\delta]}}.$$

Show that Q_δ has at most $k+m$ zeros different from ± 1. Show that there is a polynomial $Q \in \mathcal{P}_{n,k}$ having at most $k+m$ zeros (by counting multiplicities) different from ± 1 such that

$$\frac{|Q^{(m)}(0)|}{\|Q\|_{[-1,1]}} = \sup_{p\in\mathcal{P}_{n,k}} \frac{|p^{(m)}(0)|}{\|p\|_{[-1,1]}}.$$

If $y = 0$, then use E.5 and induction on m to prove the inequality of the exercise for all $p \in \mathcal{P}_{n,k}$ having at most $k+m$ zeros different from ± 1. For an arbitrary $y \in (-1,1)$ use a linear shift as is suggested in the hint to E.5 g]. □

E.6 Sharpness of Theorem A.5.2. Show that there exist polynomials $p_{n,k} \in \mathcal{P}_{n,k}$ and an absolute constant $c > 0$ such that

$$|p'_{n,k}(0)| \ge c\sqrt{n(k+1)}\|p_{n,k}\|_{[-1,1]}$$

for every $n \in \mathbb{N}$ and $k = 0,1,\dots,n$.

Hint: If $k = 0$, then let $m := \lfloor \frac{1}{3}(n-1) \rfloor$ and

$$p_{n,k}(x) := (x-1)^m (x+1)^{m+1+\lfloor \sqrt{m} \rfloor} .$$

If $1 \le k \le \frac{1}{3}n$, then let $m := \lfloor \frac{1}{3}n \rfloor$, $s := \lfloor \frac{1}{3}(k-1) \rfloor$, and

$$p_{n,k}(x) := (x^2-1)^m \, T_{2s+1}\left(\sqrt{\frac{m}{2s+1}}\, x \right) .$$

If $\frac{1}{3}n < k \le n$, then let $m := \lfloor \frac{1}{3}n \rfloor$ and $p_{n,k} := p_{n,m}$. □

E.7 Some Inequalities of Lorentz. (See Lorentz [63].)

a] Show that

$$|p'(x)| \le \left(\frac{n}{n-1} \right)^n n \left| p\left(x \pm \tfrac{1}{n} \right) \right| \le 4n \left| p\left(x \pm \tfrac{1}{n} \right) \right|$$

for every $p \in \mathcal{B}_n(-1,1)$ (hence for every $p \in \mathcal{P}_{n,0}$), $n \ge 2$, and $x \in [-1,1]$, where in $x \pm \frac{1}{n}$ the $+$ sign is taken if $x \in [-1,0)$, while the $-$ sign is taken if $x \in [0,1]$.

Hint: Observe that it is sufficient to prove the inequality only for

$$p_{n,j}(x) := (1-x)^j (x+1)^{n-j} , \qquad n \in \mathbb{N}, \quad j = 0, 1, \ldots , n .$$

□

b] Let

$$\delta_n(x) := \sqrt{\frac{1-x^2}{n}} , \qquad n \in \mathbb{N}, \quad x \in [-1,1] .$$

Show that there exists an absolute constant $c > 0$ such that

$$|p'(x)| \le (\delta_n(x))^{-1} \max\left\{ |p(x)|, \left| p\left(x \pm \tfrac{1}{2}\delta_n(x) \right) \right| \right\}$$

for every $p \in \mathcal{B}_n(-1,1)$ (hence for every $p \in \mathcal{P}_{n,0}$) and $x \in \left[-1 + \frac{c}{n}, 1 - \frac{c}{n} \right]$.

Hint: Note that it is sufficient to prove the inequality only for

$$p_{n,j}(x) := (1-x)^j (x+1)^{n-j} , \qquad n \in \mathbb{N}, \quad j = 0, 1, \ldots , n .$$

□

c] Show that

$$\int_{-1}^{1} |p'(x)|^q \, dx \le 2 \cdot 4^q n^q \int_{-1}^{1} |p(x)|^q \, dx$$

for every $p \in \mathcal{B}_n(-1,1)$ (hence for every $p \in \mathcal{P}_{n,0}$) and $q \in (0, \infty)$.

Hint: Use part a]. □

d] Show that there is an absolute constant $c > 0$ such that

$$\int_{-1}^{1} \left|p'(x)\sqrt{1-x^2}\right|^q dx \le 5c^q n^{q/2} \int_{-1}^{1} |p(x)|^q dx$$

for every $p \in \mathcal{B}_n(-1,1)$ (hence for every $p \in \mathcal{P}_{n,0}$) and $q \in (0,\infty)$.

Hint: Use parts b] and c]. □

Analogs of parts a] and b] for higher derivatives are established by Lorentz [63]. From these, analogs of parts c] and d] for higher derivatives can be proven.

E.8 Theorem A.5.5 in a Special Case. Show that there is an absolute constant c_1 such that

$$|p'(1)| \le \frac{c_1 n}{\sqrt{r}} \|p\|_{[-1,1]}$$

for every $p \in \mathcal{P}_n$ having all its zeros in $(-\infty, 1-2r]$, $r \in (0,1]$.

Proof. First assume that

$$|p(1)| = \|p\|_{[-1,1]}. \tag{A.5.8}$$

Without loss of generality, we may assume that $p \in \mathcal{P}_n \setminus \mathcal{P}_{n-1}$. Denote the zeros of p by $(-\infty <)x_1 \le x_2 \le \cdots \le x_n (\le 1-2r)$. Let

$$I_\nu := (1-2(\nu+1)^4 r, 1-2\nu^4 r], \qquad \nu = 1,2,\ldots .$$

Using E.12 of Section 5.1, we obtain

$$\begin{aligned}
\frac{|p'(1)|}{|p(1)|} &= \sum_{j=1}^{n} \frac{1}{1-x_j} = \sum_{\nu=1}^{\infty} \sum_{x_j \in I_\nu} \frac{1}{1-x_j} \\
&\le \sum_{v=1}^{\infty} \frac{e}{\sqrt{2}} n \sqrt{2(\nu+1)^4 r}\, \frac{1}{2\nu^4 r} \\
&\le \frac{e}{2} \sum_{v=1}^{\infty} \frac{(\nu+1)^2}{\nu^4}\, \frac{n}{\sqrt{r}} \le \frac{c_1 n}{\sqrt{r}}
\end{aligned}$$

with an absolute constant c_1.

Now we can drop assumption (A.5.8) as follows: Since p has all its zeros in $(-\infty, 1-2r]$, $|p|$ and $|p'|$ are increasing on $[1-2r,\infty)$. Pick the unique $y \in [1,\infty)$ satisfying $|p(y)| = \|p\|_{[-1,1]} = \|p\|_{[-1,y]}$. Using a linear transformation, from the already proved case we easily obtain

$$\begin{aligned}
|p'(1)| \le |p'(y)| &\le c_1 \frac{2n}{y+1} \left(\frac{2r+y-1}{y+1}\right)^{-1/2} \|p\|_{[-1,y]} \\
&\le \frac{c_1 n}{\sqrt{r}} \|p\|_{[-1,1]}.
\end{aligned}$$

□

E.9 Sharpness of Theorem A.5.5. For $n \in \mathbb{N}$ and $r \in (0,\infty)$, let $S_{n,r}$ be the family of all $p \in \mathcal{P}_n$ that have no zeros in the strip

$$\{z \in \mathbb{C} : |\mathrm{Im}(z)| < r\}\,.$$

a] Show that there exist polynomials $p_{n,m,r} \in S_{n,r}$ and a constant $c(m) > 0$ depending only on m such that

$$|p^{(m)}_{n,m,r}(1)| \geq c(m) \left(\min\left\{\frac{n}{\sqrt{r}}\,,\, n^2\right\}\right)^m \|p_{n,m,r}\|_{[-1,1]}$$

for every $n \in \mathbb{N}$, $r \in (0,1]$, and $m = 1,2,\ldots,n$.

Outline. If $0 < r \leq \frac{\pi}{4m}$, then with

$$x_j := \left(1 - \tfrac{4}{\pi} mr\right) \cos\left(\tfrac{2j-1}{2n}\pi\right)\,, \qquad j = 1,2,\ldots,n$$

and

$$z_j := x_j + ir\,, \qquad j = 1,2,\ldots,n$$

let

$$p_{n,m,r}(x) := \prod_{j=1}^{n} (x - z_j)(x - \overline{z}_j) \in S_{n,r}\,.$$

By E.1 of Section 5.2, $|p_{n,m,r}(1)| = \|p_{n,m,r}\|_{[-1,1]}$. Prove that

$$\begin{aligned}
\frac{|p^{(m)}_{n,m,r}(1)|}{\|p_{n,m,r}\|_{[-1,1]}} &= \frac{|p^{(m)}_{n,m,r}(1)|}{|p_{n,m,r}(1)|} \\
&\geq 2\left(1 + \frac{\pi}{4m}\right)^{-m} \frac{1}{\sqrt{2}} \frac{|q^{(m)}_{n,m,r}(1)|}{|q_{n,m,r}(1)|} \\
&\geq \frac{\sqrt{2}}{e} \left(\sum_{j=m}^{n} \frac{1}{1 - x_j}\right)^m \\
&\geq c(m) \left(\min\left\{\frac{n}{\sqrt{r}}\,,\, n^2\right\}\right)^m\,,
\end{aligned}$$

where $q_{n,m,r}(x) := \prod_{j=1}^{n} (x - x_j)$ and $c(m) > 0$ depends only on m. If $\frac{\pi}{4m} < r \leq 1$, then let $p_{n,m,r} := p_{n,m,\widetilde{r}}$, where $\widetilde{r} := \frac{\pi}{4m}$. □

b] Conclude from Theorem A.5.5 and part a] that there exist two constants $c_1(m) > 0$ and $c_2(m) > 0$ depending only on m such that

$$\begin{aligned}
c_1(m) \left(\min\left\{\frac{n}{\sqrt{r}}\,,\, n^2\right\}\right)^m &\leq \sup \frac{\|p^{(m)}\|_{[-1,1]}}{\|p\|_{[-1,1]}} \\
&\leq c_2(m) \left(\min\left\{\frac{n}{\sqrt{r}}\,,\, n^2\right\}\right)^m\,,
\end{aligned}$$

where the supremum is taken either for all $p \in S_{n,r}$ or for all $p \in \mathcal{P}_{n,0,r}$.

E.10 An Inequality of Erdős. (See Erdős [40].) Prove that

$$\|p'\|_{[-1,1]} \le \frac{n}{2}\left(\frac{n}{n-1}\right)^{n-1}\|p\|_{[-1,1]}$$

for every $p \in \mathcal{P}_{n,0}$, $n \ge 2$. This extends a result of Erdős [40] where the above inequality is proven under the additional assumption that p has only real zeros.

a] Suppose $p \in \mathcal{P}_{n,0}$, all the zeros of p are real, $p(1) = p(-1) = 0$, and $\|p\|_{[-1,1]} = 1$. Show that

$$p(x) \le \left(\frac{n}{n-1}\right)^{n-1}\frac{1+x}{1+x_0}$$

for every $x \in [-1,1]$, where x_0 is the only point in $(-1,1)$ with $p'(x_0) = 0$.

Proof. Without loss of generality, we may assume that $\deg(p) = n$ and $-1 < x < x_0$. Let $d := x_0 - x$. Let $x_1 := -1, x_2, \ldots, x_n$ denote the zeros of p. Then

$$p(x) = \frac{p(x)}{p(x_0)} = \frac{1+x}{1+x_0}\prod_{j=2}^{n}\left(1 - \frac{d}{x_0 - x_j}\right).$$

Since the geometric mean of $n-1$ nonnegative numbers is not greater than their arithmetic mean, we have

$$\begin{aligned}\prod_{j=2}^{n}\left(1 - \frac{d}{x_0 - x_j}\right) &\le \left(\frac{1}{n-1}\left(n - 1 - \sum_{j=2}^{n}\frac{d}{x_0 - x_j}\right)\right)^{n-1} \\ &\le \left(\frac{1}{n-1}\left(n - 1 + \frac{d}{x_0 + 1}\right)\right)^{n-1} \le \left(\frac{n}{n-1}\right)^{n-1}.\end{aligned}$$

□

b] Under the assumptions of part a] show that

$$\|p'\|_{[-1,1]} \le \frac{n}{2}\left(\frac{n}{n-1}\right)^{n-1}\|p\|_{[-1,1]}.$$

Proof. Note that

$$\sum_{x_j \ge 1}\frac{1}{x_j - x_0} = \sum_{x_j \le -1}\frac{1}{x_0 - x_j} \le \min\left\{\frac{k}{1 - x_0}, \frac{n-k}{x_0 + 1}\right\} \le \frac{n}{2},$$

where k denotes the number of zeros of p in $[1, \infty)$. Hence, by part a],

$$\begin{aligned} p'(x) &= p(x) \sum_{j=1}^{n} \frac{1}{x - x_j} \\ &\le \left(\frac{n}{n-1} \right)^{n-1} \frac{1+x}{1+x_0} \sum_{x_j \le -1} \frac{1}{x - x_j} \\ &\le \left(\frac{n}{n-1} \right)^{n} \frac{1+x}{1+x_0} \frac{1+x_0}{1+x} \sum_{x_j \le -1} \frac{1}{x_0 - x_j} \\ &\le \frac{n}{2} \left(\frac{n}{n-1} \right)^{n-1} \end{aligned}$$

for every $x \in [-1,1]$. Similarly

$$p'(x) \ge -\frac{n}{2} \left(\frac{n}{n-1} \right)^{n-1}$$

for every $x \in [-1,1]$. □

c] Suppose $p \in \mathcal{P}_{n,0}$ has only real zeros, p' does not vanish in $[-1,1]$, $p(-1) = 0$, and $p(1) = 1$. Show that

$$\|p'\|_{[-1,1]} \le \frac{n}{2} \|p\|_{[-1,1]} .$$

Hint: Use the relation $\mathcal{P}_{n,0} \subset \mathcal{B}_n(-1,1)$ to show that

$$p(x) \le \left(\frac{x+1}{2} \right)^{n}, \quad x \in [-1,1] .$$

Denote the zeros of p by $x_1 = -1$, $x_2, x_3, \ldots, x_n$. Then

$$0 \le p'(x) = p(x) \sum_{j=1}^{n} \frac{1}{x - x_j} \le \left(\frac{x+1}{2} \right)^{n} \frac{n}{x+1} \le \frac{n}{2} = \frac{n}{2} \|p\|_{[-1,1]}$$

for every $x \in [-1,1]$. □

d] Prove Erdős's inequality for every $p \in \mathcal{P}_{n,0}$ having only real zeros.

Hint: Reduce the general case to either part b] or part c] by a linear transformation. □

e] Prove Erdős's inequality for every $p \in \mathcal{P}_{n,0}$.

Hint: Show that for every $n \in \mathbb{N}$ and $y \in [-1,1]$, there exists a polynomial $Q \in \mathcal{P}_{n,0}$ such that

$$\frac{|Q'(y)|}{\|Q\|_{[-1,1]}} = \sup_{p \in \mathcal{P}_{n,0}} \frac{|p'(y)|}{\|p\|_{[-1,1]}} .$$

Show by a variational method that if $y \in (-1,1)$, then Q has only real zeros. Now part d] finishes the proof. □

f] Show that

$$\|p'\|_{[-1,1]} = \frac{n}{2}\left(\frac{n}{n-1}\right)^{n-1}\|p\|_{[-1,1]}$$

holds for a $p \in \mathcal{P}_{n,0}$ having only real zeros if and only if either

$$p(x) = c(x+1)^{n-1}(1-x) \qquad \text{or} \qquad p(x) = c(1-x)^{n-1}(x+1)$$

with some $0 \neq c \in \mathbb{R}$.

E.11 Sharpness of Theorem A.5.3. For every $n \in \mathbb{N}$, $m = 1, 2, \ldots, n$, and $y \in [-1,1]$, there are polynomials $p_{n,m,y} \in \mathcal{B}_n(-1,1)$ having zeros only in $\mathbb{R} \setminus (-1,1)$ such that

$$|p^{(m)}_{n,m,y}(y)| \geq c(m)\left(\min\left\{n\,,\ \frac{\sqrt{n}}{\sqrt{1-y^2}}\right\}\right)^m \|p_{n,m,y}\|_{[-1,1]}$$

with a constant $c(m) > 0$ depending only on m.

Outline. If

$$y \in [-1,1] \setminus \left[-1+\tfrac{2m}{n}\,,\ 1-\tfrac{2m}{n}\right]\,,$$

then let

$$p_{n,m,y}(x) := (x+1)^n\,.$$

In what follows, assume that

$$y \in [-1+\tfrac{2m}{n}\,,\ 1-\tfrac{2m}{n}]\,.$$

Let $q_{n,j}(x) := (1-x)^j(x+1)^{n-j}$, $n \in \mathbb{N}$, $j = 0, 1, \ldots, n$. Show that

$$\frac{q^{(m)}_{n,j}(x)}{q_{n,j}(x)} = \frac{Q_{n,j,m}(x)}{(x^2-1)^m}\,, \qquad m \leq j \leq n-m\,,$$

where $Q_{n,j,m}$ is a polynomial of degree m with only real zeros and with leading coefficient $\frac{n!}{(n-m)!}$. Let

$$\Delta_{n,y} := \max\left\{\frac{1}{n}\,,\ \frac{\sqrt{1-y^2}}{\sqrt{n}}\right\}\,, \qquad n \in \mathbb{N}, \quad y \in [-1,1]\,.$$

Use the Mean Value Theorem and Theorem A.5.3 to show that there exists an absolute constant $c \in (0,1)$ such that

$$q_{n,j}(x) \geq \tfrac{1}{2}\|q_{n,j}\|_{[-1,1]}\,, \qquad m \leq j \leq n-m$$

for every

$$x \in I_y := [y - c\Delta_{n,y}, y + c\Delta_{n,y}] \cap [-1,1]\,, \qquad y = 1 - \tfrac{2j}{n}\,.$$

Let $m \le j \le n-m$ and $y = 1 - \frac{2j}{n}$ be fixed. Choose a point $\xi \in I_y$ such that

$$|\xi - \alpha_i| \ge \frac{c\Delta_{n,y}}{2(m+1)}, \qquad i = 1, 2, \dots, m,$$

where the numbers α_i are the zeros of $Q_{n,j,m}$. Now show that there exists a constant $c_1(m) > 0$ depending only on m such that

$$(A.5.9) \qquad |q_{n,j}^{(m)}(\xi)| \ge c_1(m) \left(\min \left\{ n, \frac{\sqrt{n}}{\sqrt{1-y^2}} \right\} \right)^m \|q_{n,j}\|_{[-1,1]}.$$

Next show that if

$$y \in \left[-1 + \tfrac{2m}{n}, 1 - \tfrac{2m}{n} \right],$$

then there exists a point

$$\xi \in \left[y - \tfrac{1}{2}(1-|y|), y + \tfrac{1}{2}(1-|y|) \right]$$

and a value of j, $m \le j \le n-m$, such that (A.5.9) holds. Polynomials $p_{n,m,y}$ with the desired properties can now be easily defined by using linear transformations. □

E.12 An Inequality of Turán. (See Turán [39].) Show that

$$\frac{\|p'\|_{[-1,1]}}{\|p\|_{[-1,1]}} > \frac{1}{6}\sqrt{n}$$

for every $p \in \mathcal{P}_n \setminus \mathcal{P}_{n-1}$ having all its zeros in $[-1,1]$.

Outline. Assume that $p \in \mathcal{P}_n$ has all its zeros in $[-1,1]$, and $\|p\|_{[-1,1]} = 1$. Choose an $a \in [-1,1]$ such that $|p(a)| = 1$. Without loss of generality, assume that $p(a) = 1$.

a] Show that if $a = \pm 1$, then $|p'(1)| \ge \frac{1}{2}n > \frac{1}{6}\sqrt{n}$.

If $a \in (-1,1)$, then $p'(a) = 0$. Without loss of generality, assume that $a \in [-1,0]$. Let

$$I := \left[a, a + 2n^{-1/2} \right] \subset [-1,1].$$

If $n \le 3$, then the result follows by the Mean Value Theorem; let $n \ge 4$.

b] Use the Mean Value Theorem to show that if $|p'(x)| \le \frac{1}{6}\sqrt{n}$ on I, then $p(x) \ge \frac{2}{3}$ on I.

c] Show that if $|p''(x)| > \frac{1}{12}n$ on I, then $|p'(a + 2n^{-1/2})| > \frac{1}{6}\sqrt{n}$.

d] The proof of Turán's inequality can now be finished as follows. Suppose $p(x) > \frac{2}{3}$ on I, and there exists a $\xi \in I$ such that $p''(\xi) \le \frac{1}{12}n$. Note that

$$p'(x)^2 - p(x)p''(x) = p(x)^2 \sum_{k=1}^{n} \frac{1}{(x - x_k)^2},$$

where $x_1, x_2, \ldots, x_n$ denote the zeros of p. Since each x_k lies in $[-1,1]$, the inequality $\sum_{k=1}^{n}(x-x_k)^{-2} \geq \frac{1}{4}n$ holds for every $x \in I$. Since $p(x) \geq \frac{2}{3}$ on I, this implies that

$$p'(x)^2 - p(x)p''(x) \geq \frac{n}{9}, \qquad x \in I$$

and hence

$$p'(\xi)^2 \geq \frac{n}{9} - |p(\xi)||p''(\xi)| > \frac{n}{9} - \frac{n}{12} = \frac{n}{36}.$$

□

An extension of Turán's inequality to L_p norms is given by Zhou [92b].

E.13 An Inequality of Erdős and Turán. Let $p \in \mathcal{P}_n$ be of the form

$$p(x) = \pm \prod_{j=1}^{n}(x - x_j), \qquad -1 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq 1.$$

Suppose p is convex on $[x_{k-1}, x_k]$ for some index k. Then

$$x_k - x_{k-1} \leq \frac{16}{\sqrt{n}}.$$

Proceed as follows: Let a be the only point in $[x_{k-1}, x_k]$ for which $p'(a) = 0$.

a] Show that there exist $\xi_1, \xi_2 \in \mathbb{R}$ such that

$$a - 2n^{-1/2} \leq \xi_1 < a < \xi_2 \leq a + 2n^{-1/2},$$
$$|p'(\xi_1)| \geq \tfrac{1}{6}\sqrt{n}\,|p(a)|, \qquad \text{and} \qquad |p'(\xi_2)| \geq \tfrac{1}{6}\sqrt{n}\,|p(a)|.$$

Hint: Modify the outline of the proof of E.12. □

b] Show that

$$\xi_1 - x_{k-1} \leq \frac{6}{\sqrt{n}} \qquad \text{and} \qquad x_k - \xi_2 \leq \frac{6}{\sqrt{n}}.$$

Outline. To prove, say, the first inequality, we may assume that $x_{k-1} < \xi_1$, otherwise the inequality is trivial. Using the convexity of p on $[x_{k-1}, x_k]$, we get

$$|p'(x)| \geq |p'(\xi_1)|, \qquad x \in [x_{k-1}, \xi_1].$$

Hence

$$|p(a)| \geq |p(\xi_1)| = |p(\xi_1) - p(x_{k-1})| = \left| \int_{x_{k-1}}^{\xi_1} p'(x)\,dx \right| = \int_{x_{k-1}}^{\xi_1} |p'(x)|\,dx$$
$$\geq (\xi_1 - x_{k-1})|p'(\xi_1)| \geq (\xi_1 - x_{k-1})\tfrac{1}{6}\sqrt{n}\,|p(a)|,$$

and the result follows. □

c] Conclude that

$$a - x_{k-1} \le (a - \xi_1) + (\xi_1 - x_{k-1}) \le \frac{2}{\sqrt{n}} + \frac{6}{\sqrt{n}} = \frac{8}{\sqrt{n}}\,,$$

and similarly

$$x_k - a \le (\xi_2 - a) + (x_k - \xi_2) \le \frac{2}{\sqrt{n}} + \frac{6}{\sqrt{n}} = \frac{8}{\sqrt{n}}\,.$$

□

d] Erőd [39] establishes the sharp inequalities

$$x_k - x_{k-1} \le \frac{2}{\sqrt{2n-3}} \qquad \text{if } n \text{ is even,}$$

$$x_k - x_{k-1} \le \frac{2}{\sqrt{2n-3}} \frac{\sqrt{n^2-2n}}{n-1} \qquad \text{if } n \ge 3 \text{ is odd.}$$

E.14 Schur-Type Inequality for $\mathcal{B}_n(-1,1)$. Let α be an arbitrary positive real number. Show that

$$\sup_{0 \ne p \in \mathcal{B}_n(-1,1)} \frac{\|p\|_{[-1,1]}}{\|p(x)(1-x^2)^\alpha\|_{[-1,1]}} = \frac{(n+2\alpha)^{n+2\alpha}}{(4\alpha)^\alpha (n+\alpha)^{n+\alpha}} < \left(\frac{e}{4\alpha}(n+2\alpha)\right)^\alpha .$$

The supremum is attained if and only if $p(x) = c(1 \pm x)^n$, $0 \ne c \in \mathbb{R}$.

Hint: Let $x_1 := \frac{n}{n+2\alpha}$. If $|y| \le x_1$, then

$$\frac{|p(y)|}{\|p(x)(1-x^2)^\alpha\|_{[-1,1]}} \le \frac{1}{(1-y^2)^\alpha} \le \frac{1}{(1-x_1^2)^\alpha} = \frac{(n+2\alpha)^{2\alpha}}{(4\alpha)^\alpha (n+\alpha)^\alpha} < \frac{(n+2\alpha)^{n+2\alpha}}{(4\alpha)^\alpha (n+\alpha)^{n+\alpha}}\,.$$

If $x_1 < |y| \le 1$, say $x_1 < y \le 1$, then

$$\begin{aligned}(1-y)^j (y+1)^{n-j} &\le \|(1-x)^j (x+1)^{n-j}\|_{[-1,1]} = \frac{2^n j^j (n-j)^{n-j}}{n^n} \\ &= \frac{j^j (n-j)^{n-j} (n+2\alpha)^n}{\alpha^j n^n (n+\alpha)^{n-j}} (1-x_1)^j (x_1+1)^{n-j} \\ &\le \left(\frac{j(n+\alpha)}{\alpha(n-j)}\right)^j \left(\frac{n+2\alpha}{n+\alpha}\right)^n (1-x_1)^j (x_1+1)^{n-j} \\ &\le \left(\frac{n+2\alpha}{n+\alpha}\right)^n (1-x_1)^j (x_1+1)^{n-j}\end{aligned}$$

whenever $0 \le j \le \frac{\alpha n}{n+2\alpha}$.

On the other hand, since the function $(1-x)^j(x+1)^{n-j}$ is monotone decreasing in $\left[1-\frac{2j}{n},1\right]$, the inequality

$$(1-y)^j(y+1)^{n-j} < (1-x_1)^j(x_1+1)^{n-j}$$

follows whenever $\frac{\alpha n}{n+2\alpha} < j \le n$. Finally, use the representation

$$p(x) = \sum_{j=0}^{n} a_j(1-x)^j(x+1)^{n-j}$$

with each $a_j \ge 0$ or each $a_j \le 0$ to show that

$$|p(y)| \le \frac{(n+2\alpha)^{n+2\alpha}}{(4\alpha)^\alpha(n+\alpha)^{n+\alpha}}\left|p(x_1)(1-x_1^2)^\alpha\right|$$

for every $p \in \mathcal{B}_n(-1,1)$. □

E.15 Schur-Type Inequality for $\mathcal{T}_n(-\omega,\omega)$. For $n \in \mathbb{N}$ and $\omega \in (0,\pi]$, let

$$\mathcal{T}_n(-\omega,\omega) := \{t \in \mathcal{T}_n : d_\omega(t) \le n\},$$

where the Lorentz degree $d_\omega(t)$ is defined in E.5 of Section 2.4. Let α be an arbitrary positive real number. Show that

$$\sup_{0\ne t\in\mathcal{T}_n(-\omega,\omega)} \frac{\|t\|_{[-\omega,\omega]}}{\left\|t(\theta)\left(\frac{1}{2}(\cos\theta-\cos\omega)\right)^\alpha\right\|_{[-\omega,\omega]}}$$
$$\sim \begin{cases} (n(\pi-2\omega)+\sqrt{\omega n})^\alpha, & 0<\omega\le\pi/2 \\ (2\omega-\pi+n^{-1/2})^{-\alpha}, & \pi/2\le\omega\le\pi, \end{cases}$$

and the supremum is attained if and only if

$$t(\theta) = c\sin^{2n}\frac{\omega\pm\theta}{2}, \qquad 0\ne c\in\mathbb{R}.$$

Here the $\sim$ symbol means that the ratio of the two sides is between two positive constants depending only on α (and independent of $n \in \mathbb{N}$ and $\omega \in (0,\pi]$).

E.16 Extensions and Variations of Lax's Inequality. Theorem 7.1.11 contains, as a limiting case, an inequality of Lax [44] conjectured by Erdős; see part a]. Various extensions of this inequality are given by Ankeny and Rivlin [55], Govil [73], Malik [69], and others. Parts b] to e] discuss some of these. As before, let

$$D := \{z\in\mathbb{C} : |z|<1\} \qquad \text{and} \qquad \partial D := \{z\in\mathbb{C} : |z|=1\}.$$

a] Lax's Inequality. The inequality

$$\|p'\|_D \le \frac{n}{2}\,\|p\|_D$$

holds for all $p \in \mathcal{P}_n^c$ that have no zeros in the open unit disk.

Proof. This follows from Theorem 7.1.11 as a limiting case. □

b] Malik's Extension. Associated with

$$p(z) = c\prod_{j=1}^{n}(z - z_j)\,, \qquad z_j \in \mathbb{C}, \quad 0 \ne c \in \mathbb{C},$$

let

$$p^*(z) := \overline{c}\prod_{j=1}^{n}(1 - z\overline{z}_j) = z^n\overline{p(1/\overline{z})}\,.$$

Then

$$\max_{z\in\partial D}\left(|p'(z)| + |p^{*\prime}(z)|\right) = n$$

for every $0 \ne p \in \mathcal{P}_n^c$.

Proof. See Malik [69]. □

c] An Observation of Kroó. Suppose $p \in \mathcal{P}_n^c$ satisfies that if $p(z) = 0$ for some $z \in D$, then $p(1/\overline{z}) = 0$ (there is no restriction for the zeros of p outside D). Then

$$\|p'\|_D \le \frac{n}{2}\,\|p\|_D\,.$$

Hint: Show that if $p \in \mathcal{P}_n^c$ satisfies the assumption of the lemma, then $|p'(z)| \le |p^{*\prime}(z)|$ for every $z \in \partial D$. Use part b] to finish the proof. □

d] An Inequality of Ankeny and Rivlin. Let $r \ge 1$. The inequality

$$\max_{|z|=r}|p(z)| \le \frac{r^n + 1}{2}\,\max_{|z|=1}|p(z)|$$

holds for all $p \in \mathcal{P}_n^c$ that have no zeros in the open unit disk.

Proof. See Ankeny and Rivlin [55]. □

e] An Inequality of Govil. Let $r \ge 1$. The inequality

$$\|p'\|_D \le \frac{n}{1 + r}\,\|p\|_D$$

holds for all $p \in \mathcal{P}_n^c$ that have no zeros in the disk $\{z \in \mathbb{C} : |z| < r\}$.

Proof. See Govil [73]. □

f] Let

$$p(z) := c \prod_{k=1}^{n} (z - z_k), \qquad 0 \neq c \in \mathbb{C}.$$

Show that

$$\|p'\|_D\| \geq \left(\sum_{k=1}^{n} \frac{1}{1 + |z_k|} \right) \|p\|_D .$$

Proof. If $z \in \partial D$, then

$$\operatorname{Re}\left(\frac{zp'(z)}{p(z)} \right) = \sum_{k=1}^{n} \operatorname{Re}\left(\frac{z}{z - z_k} \right) \geq \sum_{k=1}^{n} \frac{1}{1 + |z_k|},$$

and the result follows. □

g] Let $r \in (0, 1]$. The inequality

$$\|p'\|_D \geq \frac{n}{1 + r} \|p\|_D$$

holds for all $p \in \mathcal{P}_n^c$ that have all their zeros in the disk $\{z \in \mathbb{C} : |z| \leq r\}$.

Proof. Use part f]. □

h] Another Inequality of Govil. Let $r > 1$. The inequality

$$\|p'\|_D \geq \frac{n}{1 + r^n} \|p\|_D$$

holds for all $p \in \mathcal{P}_n^c$ which have no zeros in the disk $\{z \in \mathbb{C} : |z| < r\}$.

Proof. See Govil [73]. □

E.17 Markov-Type Inequality for Nonnegative Polynomials. Show that

$$\|p'\|_{[-1,1]} \leq \frac{n^2}{2} \|p\|_{[-1,1]}$$

for every $p \in \mathcal{P}_n$ positive on $[-1, 1]$.

Proof. Suppose $p \in \mathcal{P}_n$ is positive on $[-1, 1]$ and $\|p\|_{[-1,1]} = 2$. Apply Theorem 5.1.8 (Markov's inequality) to $q := p - 1$. □

E.18 Markov's Inequality for Monotone Polynomials. It has been observed by Bernstein that Markov's inequality for monotone polynomials is not essentially better than for arbitrary polynomials. He proved that if n is odd, then

$$\sup_{0 \neq p} \frac{\|p'\|_{[-1,1]}}{\|p\|_{[-1,1]}} = \left(\frac{n+1}{2} \right)^2,$$

where the supremum is taken for all $p \in \mathcal{P}_n$ that are monotone on $[-1, 1]$.

For even n, the inequality

$$\sup_{0\neq p} \frac{\|p'\|_{[-1,1]}}{\|p\|_{[-1,1]}} \leq \left(\frac{n+1}{2}\right)^2$$

still holds. Parts a] and b] of this exercise outline a proof.

a] Show that for every odd n, there is a $p \in \mathcal{P}_n$ monotone on $\mathbb{R}$ for which

$$\frac{|p'(1)|}{\|p\|_{[-1,1]}} = \left(\frac{n+1}{2}\right)^2 .$$

Proof. Let $m := \frac{1}{2}(n-1)$. Use E.2 e] of Section 6.1 to show that there is a $q \in \mathcal{P}_m$ for which

$$q^2(1) = \left(\sum_{k=0}^{m}(1+2k)\right)\int_0^1 q^2(t)\,dt = (m+1)^2\int_0^1 q^2(t)\,dt\,.$$

Now let

$$p(x) := \int_0^x q^2(t)\,dt - \frac{1}{2}\int_0^1 q^2(t)\,dt\,.$$

Obviously $p \in \mathcal{P}_n$, p is monotone on $\mathbb{R}$, and

$$\frac{|p'(1)|}{\|p\|_{[-1,1]}} = \frac{q^2(1)}{\frac{1}{2}\int_0^1 q^2(t)\,dt} = 2(m+1)^2 = 2\left(\frac{n+1}{2}\right)^2 .$$

Now the proof can be finished by a linear transformation mapping $[-1,1]$ to $[0,1]$. □

b] Let n be odd. Show that

$$\sup_{0\neq p} \frac{\|p'\|_{[-1,1]}}{\|p\|_{[-1,1]}} \leq \left(\frac{n+1}{2}\right)^2 ,$$

where the supremum is taken for all $p \in \mathcal{P}_n$ that are monotone on $[-1,1]$.

Hint: It is sufficient to prove that if $n := 2m$ is even, then

$$\|p\| \leq \frac{1}{2}\left(\frac{n+2}{2}\right)^2 \int_{-1}^{1} p(t)\,dt$$

for every $p \in \mathcal{P}_n$ nonnegative on $[-1,1]$. Show that there is an extremal polynomial $\widetilde{p}$ for the above inequality for which $\|\widetilde{p}\|_{[-1,1]}$ is achieved at 1. Show, by a variational method, that this $\widetilde{p}$ must have at least $2m$ zeros (counting multiplicities) in $[-1,1)$. Since $\widetilde{p}$ is nonnegative, it is of the form $\widetilde{p} = q^2$ with a $q \in \mathcal{P}_m$. Now use E.2 e] of Section 6.1 to show that

$$q^2(1) \le \frac{1}{2}(m+1)^2 \int_{-1}^{1} q^2(t)\,dt = \frac{1}{2}\left(\frac{n+2}{2}\right)^2 \int_{-1}^{1} q^2(t)\,dt\,,$$

and the result follows. □

c] Let $r = p/q$, where $p, q \in \mathcal{P}_n$ and q is positive and monotone non-decreasing on an interval $[a,b]$. Show that

$$\|r'\|_{[a+\epsilon,b]} \le \frac{4n^2}{\epsilon}\,\|r\|_{[a,b]}$$

for every $\epsilon \in (0, b-a)$.

Proof. The proof follows Borwein [80]. Let $c \in [a+\epsilon, b]$ be a point where $|r'(c)| = \|r'\|_{[a+\epsilon,b]}$. Let $d \in [a,c]$ be a point $|p(d)| = \|p\|_{[a,c]}$. Then

$$r'(c) = \frac{p'(c)}{q(c)} - \frac{q'(c)}{q(c)}\,r(c)\,.$$

Theorem 5.1.8 (Markov's inequality) and the monotonicity of q imply that

$$\frac{|p'(c)|}{|q(c)|} \le \frac{2n^2\|p\|_{[a,c]}}{(c-a)|q(c)|} = \frac{2n^2|p(d)|}{(c-a)|q(c)|} \le \frac{2n^2}{c-a}\frac{|p(d)|}{|q(d)|} \le \frac{2n^2}{\epsilon}\,|r(c)|$$

and

$$\frac{|q'(c)|}{|q(c)|}\,|r(c)| \le \frac{2n^2\|q\|_{[a,c]}}{|q(c)|}|r(c)| \le \frac{2n^2}{\epsilon}\,|r(c)|\,.$$

Thus

$$\|r'\|_{[a+\epsilon,b]} = |r'(c)| \le \frac{4n^2}{\epsilon}\,\|r\|_{[a,b]}\,.$$

□

d] Sharpness of Part c]. Let n be odd. By part a], there exists a $p \in \mathcal{P}_n$ such that p is monotone increasing on $\mathbb{R}$, $p(a) = -1$, $p(a+\epsilon) = 1$, and

$$|p'(a+\epsilon)| = \frac{1}{2\epsilon}\left(\frac{n+1}{2}\right)^2\,.$$

Let $q := p + 2$ and $r := 1/q$. Show that

$$\|r\|_{[a,\infty]} = 1 \qquad \text{and} \qquad |r'(a+\epsilon)| = \frac{1}{9}\cdot\frac{1}{2\epsilon}\left(\frac{n+1}{2}\right)^2\,.$$

Weighted polynomial inequalities and their applications are beyond the scope of this book. A thorough discussion requires serious potential theoretic background, and proofs are usually quite long. Some of the main results in this area include von Golitschek, Lorentz, and Makovoz [92]; He [91]; Ivanov and Totik [90]; Levin and Lubinsky [87a] and [87b]; Lubinsky [89], Lubinsky and Saff [88a], [88b], and [89]; Mhaskar and Saff [85]; Nevai and Totik [86] and [87]; Saff and Totik [to appear]; and Totik [94]. The following exercise treats a weighted Markov-type inequality with respect to the Laguerre weight on $[0, \infty)$. The method presented below works in more general cases; however, the technical details coming from the "right" polynomial approximation of the weight function is typically far more complicated.

E.19 A Weighted Markov-Type Inequality of Szegő. Part a] presents a simple weighted polynomial inequality of Szegő [64].

a] Show that

$$\|p'(x)e^{-x}\|_{[0,\infty)} \le (8n+2)\|p(x)e^{-x}\|_{[0,\infty)}$$

for every $p \in \mathcal{P}_n$.

Proof. Let $p \in \mathcal{P}_n$. First prove the inequality

$$|p'(0)| \le (8n+2)\|p(x)e^{-x}\|_{[0,\infty)}$$

as follows. Apply Theorem 5.1.8 (Markov's inequality) on $[0, n/2]$ (by a linear transformation) to

$$q(x) := p(x)\left(1 - \frac{x}{n}\right)^n$$

and note that

$$\left(1 - \frac{x}{n}\right)^n \le e^{-x} \quad \text{and} \quad \left(1 - \frac{x}{n}\right)^{-1} \le 2\,, \qquad x \in [0, n/2]\,.$$

The general case can be easily reduced to the case discussed above. If $y \in [0, \infty)$ is fixed, then let $q(x) := p(x - y) \in \mathcal{P}_n$. On applying the already proved inequality with q, we obtain

$$\begin{aligned} |p'(y)e^{-y}| &= |q'(0)e^{-y}| \\ &\le e^{-y}(8n+2) \max_{x\in[0,\infty)} |q(x)e^{-x}| \\ &\le (8n+2) \max_{x\in[0,\infty)} |p(x+y)e^{-(x+y)}| \\ &\le (8n+2) \max_{x\in[0,\infty)} |p(x)e^{-x}|\,, \end{aligned}$$

which finishes the proof. □

b] There is an absolute constant $c > 0$ such that

$$\|p'(x)e^{-x}\|_{[0,\infty)} \le c\sqrt{n}\,\|p(x)e^{-x}\|_{[0,\infty)}$$

holds for every $p \in \mathcal{P}_n$ having no zeros in the disk with diameter $[0,2]$.

Proof. See Erdélyi [89c], where this result is extended to various other general classes of weight functions and constrained polynomials. □

E.20 Inequalities for Generalized Polynomials with Constraints. For $s \in (0,2)$, let

$$\widetilde{\mathcal{P}}_{n,k}(s) := \left\{ p \in \widetilde{\mathcal{P}}_{n,k} : m(\{x \in [-1,1] : |p(x)| \le 1\}) \ge 2 - s \right\} .$$

The polynomials $T_{n,k} \in \mathcal{P}_{n,k}$ are defined by

$$T_{n,k}(x) := T_n\{n-k, n-k+1, \dots, n; [0,1]\}\left(\tfrac{1}{2}(x+1)\right) ,$$

where $0 \le k \le n$ are integers. (Note that the notation introduced in Section 3.3 defines $T_{n,k}$ only on $[-1,1]$, and, to be precise, $T_{n,k}$ denotes the polynomial defined above on $[-1,1]$.)

a] Remez-Type Inequality for $\widetilde{\mathcal{P}}_{n,k}$. The inequality

$$\|p\|_{[-1,1]} \le T_{n,k}\left(\frac{2+s}{2-s}\right)$$

holds for every $p \in \widetilde{\mathcal{P}}_{n,k}(s)$ and $s \in (0,2)$. Equality holds if and only if

$$p(x) \equiv \pm T_{n,k}\left(\frac{\pm 2x}{2-s} + \frac{s}{2-s}\right) .$$

Proof. The proof is quite similar to that of Theorem 5.1.1 (Remez Inequality); see Borwein and Erdélyi [92]. □

b] A Numerical Version of Part a]. Show that

$$T_{n,k}\left(\frac{2+s}{2-s}\right) \le \exp\left(5\left(\sqrt{nks} + ns\right)\right)$$

for every $s \in (0,1]$.

Proof. See Borwein and Erdélyi [92]. □

Let $\mathrm{GAP}_{N,K}$, $0 \le K \le N$, be the set of all $f \in \mathrm{GAP}_N$ of the form (A.4.1) for which

$$\sum_{\substack{j=1 \\ |z_j| \le 1}}^{n} r_j \le K\,.$$

Note that if $0 \le K \le N$ are integers and $p \in \mathcal{P}_{N,K}$, then $|p| \in \mathrm{GAP}_{N,K}$.

For $0 \le K \le N$ and $0 < s < 2$, let $\mathrm{GAP}_{N,K}(s)$ denote the collection of all $f \in GAP_{N,K}$ for which

$$m(\{x \in [-1,1] : f(x) \le 1\}) \ge 2 - s\,.$$

The next part of the exercise extends a numerical version of part a] to the classes $\mathrm{GAP}_{N,K}(s)$.

c] Remez-Type Inequality for $\mathrm{GAP}_{N,K}$. There exists a constant $c_1 \le 5$ such that

$$\|f\|_{[-1,1]} \le \exp\left(c_1\left(\sqrt{NKs} + Ns\right)\right)$$

for every $f \in \mathrm{GAP}_{N,K}(s)$ and $s \in (0,1]$.

Hint: Let $q \in \mathbb{N}$ be the common denominator of the numbers r_j. Apply part a] with $p := f^{2q} \in \widetilde{\mathcal{P}}_{2qN,2qK}(s)$, and then use part b]. □

d] Nikolskii-Type Inequality for $\mathrm{GAP}_{N,K}$. Let χ be a nonnegative nondecreasing function defined on $[0,\infty)$ such that $\chi(x)/x$ is nonincreasing on $[0,\infty)$. Then there exists an absolute constant $c_2 \le 25e^2$ such that

$$\|\chi(f)\|_{L_p[-1,1]} \le (c_2 \max\{1\,,\, q^2NK\,,\, qN\})^{1/q-1/p}\|\chi(f)\|_{L_q[-1,1]}$$

for every $f \in \mathrm{GAP}_{N,K}$ and $0 < q < p \le \infty$.

The case $K = N$ (when there is no restriction on the zeros) of part d] is the content of Theorem A.4.4. If $qK \ge 1$, then the Nikolskii factor in the unrestricted case ($K = N$) is like $(\sqrt{c_2}\,qN)^{2/q-2/p}$, while in our restricted cases it improves to $(\sqrt{c_2}\,q\sqrt{NK})^{2/q-2/p}$.

Proof. It is sufficient to prove the inequality when $p = \infty$, and then a similar argument, as in the proof of Theorems A.4.3 and A.4.4, gives the result for arbitrary $0 < q < p \le \infty$. Thus, in the sequel, let $0 < q < p = \infty$.

Using the inequality of part c] with

$$s := \min\{1\,,\,(c_1^2q^2NK)^{-1}\,,\,(c_1qN)^{-1}\}$$

and recalling the conditions prescribed for χ, we conclude that

$$\begin{aligned}
&m\big(\{x \in [-1,1] : (\chi(f(x)))^q \ge e^{-2}\|\chi(f)\|^q_{[-1,1]}\}\big)\\
&\qquad \ge m\big(\{x \in [-1,1] : f(x) \ge e^{-2/q}\|f\|_{[-1,1]}\}\big)\\
&\qquad \ge m\big(\{x \in [-1,1] : f(x) \ge \exp\big(-c_1\big(\sqrt{NKs} + Ns\big)\big)\|f\|_{[-1,1]}\}\big)\\
&\qquad \ge s
\end{aligned}$$

for every $f \in \mathrm{GAP}_{N,K}$. Now integrating only on the subset E of $[-1,1]$ where

$$(\chi(f(x)))^q \geq e^{-2}\|\chi(f)\|^q_{[-1,1]},$$

we obtain

$$\begin{aligned}\|\chi(f)\|^q_{[-1,1]} &\leq \frac{e^2}{m(E)}\int_E (\chi(f(x)))^q\,dx \\ &\leq c_2 \max\{1\,,\, q^2NK\,,\, qN\}\|\chi(f)\|^q_{L_q[-1,1]}\end{aligned}$$

for every $f \in \mathrm{GAP}_{N,K}$, where $c_2 := c_1^2e^2$ (assuming $c_1 \geq 1$). Since $c_1 \leq 5$, we have $c_2 \leq 25e^2$. □

e] Remez-Type Inequality for $\mathcal{B}_n(-1,1)$. Show that

$$\|p\|_{[-1,1]} \leq \left(\frac{2}{2-s}\right)^n$$

for every $p \in \mathcal{B}_n(-1,1)$ satisfying

$$m(\{x \in [-1,1] : |p(x)| \leq 1\}) \geq 2 - s\,.$$

Equality holds if and only if p is of the form

$$p(x) = \pm\left(\frac{1 \pm x}{2-s}\right)^n\,.$$

Proof. See Erdélyi [90b]. □

f] Markov-Type Inequality for $\mathrm{GAP}_{N,K}$. There exists an absolute constant $c > 0$ such that

$$\|f'\|_{[-1,1]} \leq cN(K+1)\|f\|_{[-1,1]}$$

for every $f \in \mathrm{GAP}_{N,K}$ of the form (A.4.1) with each $r_j \geq 1$.

Recall that $|f'(x)|$ is well-defined for every $f \in \mathrm{GAP}_{N,K}$ and $x \in \mathbb{R}$, as the modulus of the one-sided derivative of f at x. The condition that $r_j \geq 1$ for each j in (A.4.1) is needed to ensure that $|f'(z_j)| < \infty$ if $z_j \in \mathbb{R}$. The above result generalizes the corresponding polynomial inequality for the classes $\mathcal{P}_{n,k}$.

Proof. See Borwein and Erdélyi [92]. □

E.21 The Ilyeff-Sendov Conjecture. The following problem is known as the Ilyeff-Sendov conjecture. As before, let $D := \{z \in \mathbb{C} : |z| < 1\}$.

Suppose $P \in \mathcal{P}_n$ has all its zeros in the closed unit disk $\overline{D}$. Then each closed disk centered at a zero of P contains a zero of of P'.

Although the problem in this generality is still unanswered, several special cases have been settled. The case outlined in part a] was first proved in Rubinstein [68]. Vâjâitu and Zaharescu [93] contains stronger results; see also Miller [90]. Milovanović, Mitrinović, and Rassias [94] has a discussion about the recent status of the conjecture.

a] Suppose

$$P(z) = \prod_{k=1}^{n} (z - z_k)\,, \qquad |z_k| \le 1\,, \quad |z_1| = 1\,.$$

Show that P' has at least one zero in the closed disk centered at z_1.

Proof. Without loss of generality, we may assume that $z_1 := 1$. Then P is of the form $P(z) = (z-1)Q(z)$, where

$$Q(z) := \prod_{k=2}^{n} (z - z_k)\,, \qquad |z_k| \le 1\,.$$

Suppose the statement is false. Then $R(z) := P'(z+1)$ has no zero in the closed unit disk $\overline{D}$. Hence

$$\left|\frac{R'(0)}{R(0)}\right| < n - 1\,.$$

Observe that

$$R(0) = Q(1) \qquad \text{and} \qquad R'(0) = 2Q'(1)\,.$$

Hence

$$\left|\frac{Q'(1)}{Q(1)}\right| = \frac{1}{2}\left|\frac{R'(0)}{R(0)}\right| < \frac{n-1}{2}\,.$$

However,

$$\operatorname{Re}\left(\frac{Q'(1)}{Q(1)}\right) = \operatorname{Re}\left(\sum_{k=2}^{n} \frac{1}{1 - z_k}\right) \ge \frac{n-1}{2}\,.$$

This contradiction finishes the proof. □

b] The polynomial $P(z) := z^n - 1$ shows the sharpness of part a].

It follows from Theorem A.5.3 that there exists an absolute constant $c > 0$ such that

$$\|p'\|_{[-1,1]} \le cn \|p\|_{[-1,1]}$$

for every $p \in \mathcal{P}_n$ with no zeros in the open unit disk. The polynomials $p_n(x) := (x+1)^n$ show that up to the absolute constant $c > 0$ this inequality is the best possible. The next exercise shows that the "right" Markov factor on $[-1,1]$ for polynomials of degree at most n *with complex coefficients* and with no zeros in the open unit disk is $cn(1+\log n)$ rather than cn. This is an observation of Halász.

E.22 Markov Inequality for $\mathcal{P}_n^c$ with No Zeros in the Unit Disk. Show that there exists an absolute constant $c_1 > 0$ so that

$$\|p'\|_{[-1,1]} \le c_1 n(1+\log n)\|p\|_{[-1,1]}$$

for every $p \in \mathcal{P}_n^c$ with no zeros in the open unit disk.

Show also that there exist polynomials $p_n \in \mathcal{P}_n^c$ with no zeros in the open unit disk such that

$$\|p_n'\|_{[-1,1]} \ge c_2 n(1+\log n)\|p_n\|_{[-1,1]}\,, \qquad n = 1,2,\dots$$

with an absolute constant $c_2 > 0$.

Proceed as follows:

a] Show that if $z \in \mathbb{C}$ is outside the open unit disk, then

$$\frac{|x-z|}{|x^{-1}-z|} \le |x|$$

for every $x \in \mathbb{R} \setminus [-1,1]$.

b] Suppose $p \in \mathcal{P}_n^c$ has no zeros in the open unit disk. Let $x \in \mathbb{R} \setminus [-1,1]$. Show that

$$|p(x)| \le |x|^n |p(x^{-1})|\,.$$

c] Prove the upper bound of the exercise.

Hint: Use part b] and E.11 a]. □

d] Let

$$z_k := e^{2ik\pi/(2n+1)}\,, \qquad k = 1,2,\dots,n$$

be the $(2n+1)$th roots of unity in the open upper half-plane. Let

$$p_{2n+1}(z) := p_{2n+2}(z) := (z+1)\prod_{k=1}^{n}(z-z_k)^2\,.$$

Show that $|p_{2n+1}(x)| = |x^{2n+1}-1|$ for every $x \in \mathbb{R}$. Note that this implies

$$|p_{2n+1}(-1)| = \|p_{2n+1}\|_{[-1,1]} = 2\,.$$

Show also that

$$\left|\frac{p_{2n+1}'(-1)}{p_{2n+1}(-1)}\right| \ge cn(1+\log n)$$

with an absolute constant $c > 0$.