Notation

Definitions of the more commonly used spaces are given. The equation numbers here are the same as the equation numbers in the text.

Throughout the book, unless otherwise stated, spans should be assumed to be real. Likewise, in function spaces, unless otherwise stated, the functions should be assumed to be real valued.

The Basic Spaces.

$$\mathcal{P}_n^c := \left\{ p : p(z) = \sum_{k=0}^{n} a_k z^k, \quad a_k \in \mathbb{C} \right\} . \tag{1.1.1}$$

$$\mathcal{P}_n := \left\{ p : p(z) = \sum_{k=0}^{n} a_k z^k, \quad a_k \in \mathbb{R} \right\} . \tag{1.1.2}$$

$$\mathcal{R}_{m,n}^c := \left\{ \frac{p}{q} : p \in \mathcal{P}_m^c, q \in \mathcal{P}_n^c \right\} . \tag{1.1.3}$$

$$\mathcal{R}_{m,n} := \left\{ \frac{p}{q} : p \in \mathcal{P}_m, q \in \mathcal{P}_n \right\} . \tag{1.1.4}$$

$$\mathcal{T}_n^c := \left\{ t : t(\theta) := \sum_{k=-n}^{n} a_k e^{ik\theta}, \quad a_k \in \mathbb{C} \right\} \tag{1.1.5}$$

$$= \left\{ t : t(z) = a_0 + \sum_{k=1}^{n} (a_k \cos kz + b_k \sin kz), \quad a_k, b_k \in \mathbb{C} \right\}.$$

$$\mathcal{T}_n := \left\{ t : t(z) = a_0 + \sum_{k=1}^{n} (a_k \cos kz + b_k \sin kz), \quad a_k, b_k \in \mathbb{R} \right\}.$$

Müntz Spaces.

$$M_n(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}. \tag{3.4.2}$$

$$M(\Lambda) := \bigcup_{n=0}^{\infty} M_n(\Lambda) = \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}. \tag{3.4.3}$$

Associated with a sequence $(\lambda_i)_{i=0}^{\infty}$ with $\operatorname{Re}(\lambda_i) > -1/2$ for each i, the nth Müntz-Legendre polynomial is:

$$L_n(x) := L_n\{\lambda_0, \dots, \lambda_n\}(x) \tag{3.4.5}$$

$$:= \frac{1}{2\pi i} \int_{\Gamma} \prod_{k=0}^{n-1} \frac{t + \overline{\lambda}_k + 1}{t - \lambda_k} \frac{x^t \, dt}{t - \lambda_n}.$$

For distinct λ_i,

$$L_n\{\lambda_0, \dots, \lambda_n\}(x) = \sum_{k=0}^{n} c_{k,n} x^{\lambda_k}, \qquad x \in (0, \infty) \tag{3.4.6}$$

with

$$c_{k,n} := \frac{\prod_{j=0}^{n-1} (\lambda_k + \overline{\lambda}_j + 1)}{\prod_{j=0, j \neq k}^{n} (\lambda_k - \lambda_j)}.$$

Also,

$$L_n^* := (1 + \lambda_n + \overline{\lambda}_n)^{1/2} L_n \tag{3.4.8}$$

is the nth orthonormal Müntz-Legendre polynomial.

Rational Spaces.

$K := \mathbb{R} \pmod{2\pi}, \quad D := \{z \in \mathbb{C} : |z| < 1\}, \quad \partial D := \{z \in \mathbb{C} : |z| = 1\}.$

$$\mathcal{P}_n(a_1, a_2, \dots, a_n) := \left\{ \frac{p(x)}{\prod_{k=1}^n |x - a_k|} : p \in \mathcal{P}_n \right\}. \tag{3.5.3}$$

$$\mathcal{T}_n(a_1, a_2, \dots, a_n) := \left\{ \frac{t(\theta)}{\prod_{k=1}^n |\cos\theta - a_k|} : t \in \mathcal{T}_n \right\}. \tag{3.5.4}$$

Also

$$\begin{aligned} \mathcal{T}_n(a_1, a_2, \dots, a_{2n}; K) &:= \mathcal{T}_n(a_1, a_2, \dots, a_{2n}) \\ &:= \left\{ \frac{t(\theta)}{\prod_{k=1}^{2n} |\sin((\theta - a_k)/2)|} : t \in \mathcal{T}_n \right\} \end{aligned}$$

and

$$\mathcal{T}_n^c(a_1, a_2, \dots, a_{2n}; K) := \left\{ \frac{t(\theta)}{\prod_{k=1}^{2n} \sin((\theta - a_k)/2)} : t \in \mathcal{T}_n^c \right\}$$

on K with $a_1, a_2, \dots, a_{2n} \in \mathbb{C} \setminus \mathbb{R}$;

$$\mathcal{P}_n(a_1, a_2, \dots, a_n; [-1, 1]) := \left\{ \frac{p(x)}{\prod_{k=1}^n |x - a_k|} : p \in \mathcal{P}_n \right\}$$

and

$$\mathcal{P}_n^c(a_1, a_2, \dots, a_n; [-1, 1]) := \left\{ \frac{p(x)}{\prod_{k=1}^n (x - a_k)} : p \in \mathcal{P}_n^c \right\}$$

on $[-1, 1]$ with $a_1, a_2, \dots, a_n \in \mathbb{C} \setminus [-1, 1]$;

$$\mathcal{P}_n^c(a_1, a_2, \dots, a_n; \partial D) := \left\{ \frac{p(z)}{\prod_{k=1}^n (z - a_k)} : p \in \mathcal{P}_n^c \right\}$$

on ∂D with $a_1, a_2, \dots, a_n \in \mathbb{C} \setminus \partial D$; and

$$\mathcal{P}_n(a_1, a_2, \dots, a_n; \mathbb{R}) := \left\{ \frac{p(x)}{\prod_{k=1}^n |x - a_k|} : p \in \mathcal{P}_n \right\}$$

and

$$\mathcal{P}_n^c(a_1, a_2, \dots, a_n; \mathbb{R}) := \left\{ \frac{p(z)}{\prod_{k=1}^n (z - a_k)} : p \in \mathcal{P}_n^c \right\}$$

on $\mathbb{R}$ with $a_1, a_2, \dots, a_n \in \mathbb{C} \setminus \mathbb{R}$.

Generalized Polynomials.

The function

$$f(z) = |\omega| \prod_{j=1}^{m} |z - z_j|^{r_j} \tag{A.4.1}$$

with $0 < r_j \in \mathbb{R}$, $z_j \in \mathbb{C}$, and $0 \neq \omega \in \mathbb{C}$ is called a generalized nonnegative (algebraic) polynomial of (generalized) degree

$$N := \sum_{j=1}^{m} r_j \,. \tag{A.4.2}$$

The set of all generalized nonnegative algebraic polynomials of degree at most N is denoted by GAP_N.

The function

$$f(z) = |\omega| \prod_{j=1}^{m} |\sin((z - z_j)/2)|^{r_j} \tag{A.4.3}$$

with $0 < r_j \in \mathbb{R}$, $z_j \in \mathbb{C}$, and $0 \neq \omega \in \mathbb{C}$ is called a generalized nonnegative trigonometric polynomial of degree

$$N := \frac{1}{2} \sum_{j=1}^{m} r_j \,. \tag{A.4.4}$$

The set of all generalized nonnegative trigonometric polynomials of degree at most N is denoted by GTP_N.

Constrained Polynomials.

The following classes of polynomials with constraints appear in Appendix 5:

$$\mathcal{P}_{n,k} := \{p \in \mathcal{P}_n : p \text{ has at most } k \text{ zeros in } D\} \,, \qquad 0 \leq k \leq n \,,$$

$$\mathcal{B}_n(a,b) := \left\{p \in \mathcal{P}_n : p(x) = \pm \sum_{j=0}^{n} \alpha_j (b-x)^j (x-a)^{n-j} \,,\ \alpha_j \geq 0\right\}, \quad a < b \,,$$

$$\widetilde{\mathcal{P}}_{n,k}(a,b) := \{p = hq : h \in \mathcal{B}_{n-k}(a,b) \,,\ q \in \mathcal{P}_k\} \,, \qquad 0 \leq k \leq n \,.$$

Function Spaces.

The *uniform* or *supremum norm* of a complex-valued function f defined on a set A is defined by

$$\|f\|_A := \sup_{x \in A} |f(x)| .$$

The space of all real-valued continuous functions on a topological space A equipped with the uniform norm is denoted by $C(A)$. If $A := [a, b]$ is equipped with the usual metric topology, then the notation $C[a, b] := C(A)$ is used.

Let (X, μ) be a measure space (μ nonnegative) and $p \in (0, \infty]$. The space $L_p(\mu)$ is defined as the collection of equivalence classes of real-valued measurable functions for which $\|f\|_{L_p(\mu)} < \infty$, where

$$\|f\|_{L_p(\mu)} := \left(\int_X |f|^p \, d\mu \right)^{1/p}, \qquad p \in (0, \infty)$$

and

$$\|f\|_{L_\infty(\mu)} := \sup\{\alpha \in \mathbb{R} : \mu(\{x \in X : |f(x)| > \alpha\}) > 0\} < \infty .$$

The equivalence classes are defined by the equivalence relation $f \sim g$ if $f = g$ μ-almost everywhere on X. When we write $L_p[a, b]$ we always mean $L_p(\mu)$, where μ is the Lebesgue measure on $X = [a, b]$. The notations $L_p(a, b), L_p[a, b)$, and $L_p(a, b]$ are also used analogously to $L_p[a, b]$. Again, it is always our understanding that the space $L_p(\mu)$ is equipped with the $L_p(\mu)$ norm.

Sometimes $C(A)$ denotes the space of all *complex-valued* continuous functions defined on A equipped with the uniform norm. Similarly, $L_p(\mu)$ may denote the space all *complex-valued* continuous functions for which $\|f\|_{L_p(\mu)} < \infty$. This should always be clear from the context; many times, but not always, the reader is reminded if $C[a, b]$ or $L_p(\mu)$ is complex.