Reduction of systems of nonlinear partial differential equations to simplified involutive forms

Gregory J. Reid, Allan D. Wittkopf and Alan Boulton

Department of Mathematics
University of British Columbia
Vancouver, B.C., Canada V6T 1Z2
email: reid@math.ubc.ca

Short title: Simplifying nonlinear systems of pdes
AMS classification scheme numbers: 35-04, 35N10, 13P10, 58G35.

Abstract
We describe the *rif* algorithm which uses a finite number of differentiations and algebraic operations to simplify analytic *nonlinear* systems of partial differential equations to what we call reduced involutive form. This form includes the integrability conditions of the system and satisfies a constant rank condition.

The algorithm is useful for classifying initial value problems for determined *pde* systems and can yield dramatic simplifications of complex overdetermined nonlinear *pde* systems. Such overdetermined systems arise in analysis of physical *pde*s for reductions to *ode*s using the Nonclassical Method, the search for exact solutions of Einstein's field equations and the determination of discrete symmetries of differential equations. Application of the algorithm to the associated nonlinear overdetermined system of 856 *pde*s arising when the Nonclassical Method is applied to a cubic nonlinear Schrödinger system yields new results.

Our algorithm combines features of geometric involutive form algorithms and our previously developed standard form algorithm. An algebraic realization of the *rif* algorithm is given for polynomially nonlinear *pde* systems which uses Buchberger's algorithm for its nonlinear simplifier and algorithms for constructing the radical of a polynomial ideal.

1. Introduction

Considerable recent effort has been devoted to developing algorithms which unify and extend analytical methods of applied mathematics for systems of *pde*s. Our paper is a contribution to this program.

The aim of this *Algorithmization Program* is to provide methods which: are geometrical - they do not depend on a clever choice of coordinates; uncover links between seemingly unrelated methods and improve our understanding of these methods; consist of steps which in the most desirable cases can be realized on a computer so that they become available to nonspecialists.

Examples of this program include: Lie symmetry methods for *pde* and their generalizations (to nonclassical [18, 3], potential [4], Lie-Bäcklund and variational symmetries [41, 4]) and Cartan's method for determining whether *pde* systems are related by a change of variables [14].

A common feature of the above methods is that for a given determined physical system of *pde*s they require the analysis of an associated complex overdetermined system of *pde*s. In the analysis of *pde*s for Lie symmetries this associated system is linear and computer algebra packages have been developed which have often succeeded in explicitly solving these systems [26]. Generalizations of Lie's method yield associated systems which are often nonlinear. For example, analysis of differential equations for nonclassical reductions using the methods of Clarkson and Mansfield [18, 17] and the search for discrete symmetries [47] often requires analysis of large associated overdetermined systems of nonlinear *pde*s. Determination of Bäcklund transformations [53] for differential equations and exact solutions of Einstein's field equations [33, 12] also require analysis of nonlinear overdetermined systems.

In this paper we present a finite step algorithm which simplifies analytic systems of nonlinear *pde*s to what we call *reduced involutive form* (abbreviated *rif*-form). The steps of our *rif* algorithm consist of a finite number of differentiations and eliminations and do not depend on the heuristic process of integrating such systems. Although the reduced involutive form of a *pde* system is coordinate-dependent it can be simply transformed into a system which has the geometric property of involutivity [42]. Thus we can gain access to the useful geometric properties of involutive systems. The theory of systems in involution has its origins in Cauchy and Kovaleski's [32] work on existence theorems for *pde*s and the pioneering works of Riquier [49], Tresse [64] and Janet [29]. It was developed into a geometrical tool by Cartan [15] and has undergone many developments [34, 23, 59, 42, 5].

The *rif* algorithm contains aspects of our previously developed standard form algorithm for linear *pde* systems [43] and geometric involutive form algorithms [42, 23, 34] for nonlinear *pde* systems. When applied to a linear system the *rif* algorithm yields the standard form of the system as defined in [43]. An essential aspect of the *rif* algorithm is that it uses a ranking defined on the infinite set of partial derivatives of dependent variables of the system. The *rif*-form of a nonlinear system with respect to an orderly ranking (i.e. one graded first by total order of derivative) can be simply transformed into involutive form. In particular an integer M can be determined from the *rif*-form of a system such that its prolongation (i.e. its differential consequences) to order M is in involutive form. Hence we can compute the valuable analytic and geometric information given by the involutive form of a system such as its symbol, Cartan characters and Hilbert polynomial. From the involutive form of a system it is possible to predict local initial value problems and implicitly determine Taylor series expansions of the solutions satisfying these initial value problems. Application of the *rif* algorithm with rankings which are not first graded by total order of derivative enables us to set up physically important initial value problems for *pde* systems which do not appear to be covered by the approaches given in [42, 15]. The *rif* algorithm can also yield valuable simplifications of *pde* systems.

1.1 Approaches to linear systems

Early references to computer implementation of formal methods for systems of *pde*s are given in [60, 1, 20]. Algorithms for simplifying linear systems by including integrability conditions have been extensively developed and have been useful in applications [26]. Schwarz [53], with his implementation of the Riquier-Janet method in the symbolic language Reduce, demonstrated the utility of such formal methods in the determination of Bäcklund transformations. Such algorithms have been used as simplifiers of infinitesimal determining systems for Lie symmetries of differential equations [56, 63, 7] and are becoming a standard part of computer packages designed for that purpose [57, 26]. The standard form algorithm which we developed in earlier work [43] has its roots in the classical Riquier-Janet method. It uses an equivalence class approach instead of the monomials of Riquier-Janet theory to avoid creating redundant equations and to provide a standard form of the systems to which it is applied. Integration-free algorithms based on the reduced forms produced by the standard form algorithm and other elimination algorithms have been developed which: determine the dimension of the solution space of linear systems [43, 55], isolate *ode*s [46, 19] from *pde* systems, determine the structure of Lie symmetry algebras of differential equations [44, 45], and determine the structure of the solutions of linear *pde* systems [39], amongst many other applications.

Methods such as those above can be applied to nonlinear systems of *pde*s in the restricted situations where the *pde*s generated in the reduction process can be solved for their highest order derivatives.

1.2 Differential-algebraic approaches to nonlinear systems

Differential algebra was initiated by Ritt [50] who was inspired by the works of Janet [29], Hilbert and Noether. It is concerned with differential counterparts of objects in algebra (such as differential rings, ideals and fields). Ritt was followed by Kolchin [31] who developed this theory extensively. The differential completion-elimination algorithms of this theory apply, in principle, to polynomially nonlinear systems of *pde*s. However difficulties arise in that no constructive test has been given for the termination of these algorithms. A major advance in polynomial ideal theory was given by Buchberger's algorithm which completes systems of polynomially nonlinear algebraic equations to the form of a Gröbner basis, and solves the *ideal membership question* [8]. One of the current objectives of differential-algebraic approaches is to obtain polynomially nonlinear *pde* systems in the form of a differential analogue of a Gröbner basis [9, 40, 36, 38]. Carrà-Ferro [9] defined a differential analogue of Gröbner bases and showed that such differential Gröbner bases can be infinite (unlike the case for polynomial equations). Ollivier [40] has given methods which construct the differential Gröbner bases of Carrà-Ferro up to a finite order. Mansfield [36, 38] circumvented the nontermination problem in algorithms to construct differential Gröbner bases by replacing Carrà-Ferro's reduction with pseudo-reduction. The resulting finite step algorithm always terminates but not necessarily with a differential Gröbner basis. The Maple implementation of Mansfield's algorithm has proved very useful in the search for nonclassical and classical reductions of systems of differential equations [17]. Wittkopf's algorithm *diff_reduce* [47] is similar to Mansfield's except that it uses reduction instead of pseudo-reduction and thus does not always terminate. Providing a finite-step algorithm to solve the differential ideal membership question remains an important unsolved problem.

1.3 Geometric approaches to nonlinear systems

In these approaches the objective is often to obtain the system in involutive form [15, 42]. Cartan [15] developed a geometrical definition of involution in terms of his exterior calculus. He was interested in using it to derive structural features of infinite-dimensional Lie pseudo-groups from their nonlinear defining equations [13] and it formed an important part of his method for determining whether two differential systems were equivalent by a change of coordinates [14]. His approach also yielded the Cartan-Kähler Theorem [30] which is a local existence uniqueness theorem for analytic nonlinear systems of *pde*s which are in involutive form. This theorem is a geometric analogue of Riquier's Existence Theorem [49] for orthonomic systems. Janet [29] also examined the geometrical implications of involution in the operational framework of Riquier's theory. In the

1920's Vessiot [65] developed a geometrical method, based on vector fields, to determine whether a system of *pde*s was involutive. Significant progress was made by Kuranishi [34] who proved the Cartan-Kuranishi Theorem: that under certain constant rank assumptions any analytic nonlinear system of *pde*s becomes involutive after a finite number of prolongations and projections. In the 1960's the Americans Goldschmidt [23] and Spencer [59] used methods from cohomological algebra to study formal properties of systems of *pde*s. Goldschmidt [23] obtained integrability criteria for analytic nonlinear systems. Pommaret [42] used the methods of Spencer, Goldschmidt and Janet to develop an involutive form algorithm which does not use exterior calculus.

Pommaret's involutive form algorithm has been implemented in the symbolic language Axiom [52]. The computer automation of Cartan's method was investigated in [1] and Hartley and Tucker [25] gave an algorithm implemented in Reduce which uses the exterior calculus of Cartan to determine whether a system of *pde*s is involutive, and, if so, determines the Cartan characters of the system. Hudson [28] gave a Macsyma program for completing systems of *pde*s to involutive form using the techniques of Vessiot [65].

1.4 Brief outline of the reduced involutive form algorithm

The objective of the *rif* algorithm described in this article is to obtain general analytic systems of *pde* in a form which can be simply transformed into the form of an involutive system. The objective of this article is *not*, even for polynomially nonlinear *pde* systems, to obtain such systems in the form of a differential Gröbner basis. We do however employ some aspects of differential algebra: rankings on derivatives and elimination. We believe that both paths (geometric and differential-algebraic) are worthwhile and that the obstacles to effective algorithms in both are related. Valuable work in exploring the relationship between these approaches has been given by Mansfield [36, 37] and Carrà-Ferro and Duzhin [10].

A basic step of the *rif* algorithm is one of detecting whether an equation is new relative to a given system of equations. The meaning of new here is geometric: an equation is (geometrically) new in relation to a given system if it restricts the locus of the solutions of the system. For polynomially nonlinear equations, in the context of Polynomial Ideal Theory, an equation is (algebraically) new if it does not lie in the polynomial ideal generated by the system.

A crucial aspect of our *rif* algorithm is the use of rankings, on the set of derivatives of a given *pde* system, which are preserved under differentiation. Every *pde* in the system has a unique *leading* derivative which is highest in such a ranking. The *pde*s of the system can be split into two sets with respect to such a ranking: those that are linear in their leading derivative (the *leading linear pde*s), and the remaining (*leading nonlinear*) *pde*s. The *rif* algorithm depends critically on the fact that differentiation of any leading nonlinear *pde* will yield a leading linear *pde*.

The *rif* algorithm consists of three nested loops: the integrability conditions loop is contained in the spawning loop, and the spawning loop is contained in the constant rank loop. The inner-most loop (the integrability conditions loop) consists of the steps: Gauss-reducing the leading linear *pde*s and taking integrability conditions between them; appending geometrically new integrability conditions to the system (i.e. appending those that restrict the locus defined by the *pde*s in the jet bundle of the system). These steps are repeated until there are no further new integrability conditions and no further leading linear *pde*s which can be solved for their leading derivative. The spawning loop consists of two major steps: the integrability conditions loop, followed by a step in which the algorithm spawns more leading linear *pde*s by forming all first order partial derivatives of the leading nonlinear equations of the system. The integrability conditions loop and the spawning step are repeated until all integrability conditions and spawned linear *pde*s are consequences of the current system. The outer (constant rank) loop of the algorithm consists of the spawning loop followed by a constant rank step. In the constant rank step the algorithm determines whether the system is of constant rank and if not transforms the system into constant rank form. These steps are repeated until the system passes the constant rank test and there are no new spawned linear equations.

In this paper we have presented the algorithms in a simplified manner. The crucial issue of efficiency deserves a separate treatment. For systems of nonlinear *pde*s the difficulties due to expression explosion in such algorithms are even worse than those for nonlinear algebraic systems since new variables are continually being introduced by differentiation. We do mention however that the *rif* algorithm has a criterion which enables it to avoid calculating many integrability conditions and by Gaussian elimination the algorithm attempts to minimize the size, order and nonlinearity of intermediate systems arising during its execution.

1.5 Contents of the paper

In §2 jet notation, rankings, pivots and other preliminaries for the paper are discussed. In §3 the *rif* algorithm is described and a detailed example is given in §4. The *rif* algorithm was designed for general analytic nonlinear *pde* systems and can be given effective realizations for special types of systems. For example to give a realization for polynomially nonlinear systems the algorithm's geometric tester for new equations is replaced by Buchberger's normal form function and the constant rank condition is replaced by the step of calculating the radical of the polynomial ideal generated by the equations of the system. The resulting algorithm, which we call the *gröbner_rif* algorithm, is discussed in §5. In §6 *gröbner_rif* is applied to the nonclassical nonlinear determining system of the 3+1 dimensional cubic Schrödinger system and new results are obtained. We give a discussion in §7 and present details for the *rif* subroutine *orthsimp* in Appendix A. In Appendix B we give transformations of the *rif*-form of a *pde* system into involutive form and also into an equivalent system of exterior forms. Pseudo-code for a geometric involutive form algorithm is given in Appendix C.

2. Preliminaries

2.1 Multi-index jet notation

We consider systems of *pde*s which are local analytic (or real-analytic) functions of their independent variables $x = (x_1, x_2, ..., x_n)$, dependent variables $u = (u^1, ..., u^m)$ and derivatives of dependent variables. We employ the geometric picture [42] of a system of *pde*s as a variety in the jet bundle defined by its equations, in which the graphs of solutions of the *pde* system lie. In particular we associate the partial derivative

$$(\partial_{x_1})^{c_1} ... (\partial_{x_n})^{c_n} u^k(x_1, ..., x_n) \tag{2.1}$$

with the corresponding jet variable in multi-index notation

$$u^k_c, \tag{2.2}$$

where $c = (c_1, ..., c_n) \in \mathbf{W}^n$, $\mathbf{W} = \{0, 1, 2, ...\}$. Here the order of the derivative u^k_c is $\mathrm{ord}(c) = c_1 + \cdots + c_n$ and the order zero derivative $u^k_{\mathbf{0}}$ (the trivial derivative) corresponds to $u^k(x)$. The term *derivative* will be used to include dependent variables (i.e. to include zeroth order or trivial derivatives). For simplicity we will sometimes write $u^k_{\mathbf{0}}$ as u^k and $(u^1_c, \ldots, u^m_c)$ as u_c.

Let $J^q(\mathbb{F}^n, \mathbb{F}^m)$ ($\mathbb{F} = \mathbf{R}$ or $\mathbf{C}$) denote the q-th order jet bundle, whose coordinates (x, u_c) represent the n independent variables and the $m\binom{q+n}{q}$ derivatives u_c, $\mathrm{ord}(c) \le q$, $c \in \mathbf{W}^n$. Then the prolonged graphs of solutions of a system of order q, containing p *pde*s, will lie in the variety in the q-th order jet bundle [42] defined by the equations

$$\phi^r(x, u_c) = 0, \qquad r = 1, \ldots, p, \tag{2.3}$$

where (2.3) has been obtained by replacing the partial derivatives (2.1) in the *pde*s by their corresponding jet variables (2.2). The variety defined by (2.3) will be referred to as the locus of the system (2.3).

The operator of formal total differentiation with respect to x_i is defined by

$$D_{x_i} \equiv \partial_{x_i} + \sum_{\mathrm{ord}(c) \ge 0} u^k_{c+1_i} \partial_{u^k_c}, \tag{2.4}$$

where summation on $k = 1, ..., m$ is assumed and 1_i is a multi-index having its i-th entry 1 and the remaining $n-1$ entries 0. This operator has the same action as the usual total derivative operator on solutions of the given system. Higher order formal total derivatives are defined by

$$D^c \equiv (D_{x_1})^{c_1} \dots (D_{x_n})^{c_n}, \quad c \in \mathbf{W}^n. \tag{2.5}$$

The prolongation to order r of a *single pde* $\{\psi(x, u_c) = 0\}$ of order q, consists of the set of equations $\{D^s\psi(x, u_c) = 0 : 0 \leq \mathrm{ord}(s) \leq r - q\}$. The prolongation to order r of a *system* S of *pde*s of order q is the union of the prolongations to order r of each of its *pde*s. We denote this prolongation by $\rho^r(S)$.

Throughout this paper we only consider what we call *admissible* systems. Such systems, denoted by (S, Λ), consist of a system of *pdes* S together with a *pivot list* Λ. For (S, Λ) to be admissible on $\mathcal{B}$ we require S to be a system of finite order (q say) of the form $\phi^1(x, u_c) = 0, \dots, \phi^p(x, u_c) = 0$ where $\mathcal{B}$ is a connected open subset of $J^q(\mathbb{F}^n, \mathbb{F}^m)$ and the $\phi^i: \mathcal{B} \to \mathbb{F}$ are analytic ($\mathbb{F} = \mathbf{C}$) or real analytic ($\mathbb{F} = \mathbf{R}$) functions. The pivot list Λ is an ordered tuple of the form ($\lambda^1(x, u_c) \neq 0$, $\lambda^2(x, u_c) \neq 0, \dots, \lambda^r(x, u_c) \neq 0$) where the $\lambda^i : \mathcal{B} \to \mathbb{F}$ are analytic (or real analytic) functions on $\mathcal{B}$. We restrict our attention to local analytic solutions of such systems at points of $\mathcal{B}$ where none of the pivot functions λ^i vanish.

The requirement that the functions above are analytic can be replaced by the requirement that they are meromorphic functions (i.e. quotients of analytic functions). However Lewy's famous example [35] of a C^∞ differential equation without a solution illustrates the difficulty of extending our methods from analytic to C^∞ systems of *pde*.

2.2 Rankings of derivatives

Central to our algorithm will be rankings $\succ$ defined on the set of a given system's dependent variables and their derivatives

$$\{u_c^k : \quad k = 1, ..., m, \ \ c \in \mathbf{W}^n\}. \tag{2.6}$$

In addition to the usual properties of total orders (transitivity, comparability and trichotomy), we also require that $\succ$ satisfies the two conditions:

$$u_c^k \succ u_b^l \Rightarrow u_{a+c}^k \succ u_{b+c}^l \qquad \text{and} \qquad u_c^k \succ u_{\mathbf{0}}^k \tag{2.7}$$

for all $a, b, c \in \mathbf{W}^n$ and $k, l \in \{1, ..., n\}$. Such rankings are preserved under differentiation.

A large class of rankings satisfying (2.7) was given by Riquier [49] and can be described by invertible matrices with non-negative integer entries. The class of rankings has been recently considerably extended [11, 66] and a characterization of the classification of all such rankings is given in [51].

Formal approaches to *pde*s have concentrated on the use of *orderly rankings*, that is rankings which are first graded by total order of derivative. Such rankings satisfy $\mathrm{ord}(a) > \mathrm{ord}(b) \Rightarrow u_a^k \succ u_b^l$. We also mention that rankings which are not graded first by total order (e.g. lexicographic rankings) have important applications [19, 36, 46].

Given an admissible q-th order *pde* $\phi(x, u_c) = 0$ on $\mathcal{B} \subseteq J^q(\mathbb{F}^n, \mathbb{F}^m)$, the (unique) largest derivative which appears in $\phi(x, u_c)$ with respect to a given ranking $\succ$ is called the *leading derivative* of the *pde* in $\mathcal{B}$ with respect to $\succ$. Note that there may be subsets of $\mathcal{B}$ on which ϕ has smaller leading derivatives. If the *pde* has the form

$$\phi(x, u_c) := f(x, u_c)u_b^k - g(x, u_c) = 0, \tag{2.8}$$

where $f(x, u_c)$ and $g(x, u_c)$ do not depend on the leading derivative u_b^k and $f(x, u_c)$ is nonzero on some nonempty subset of $\mathcal{B}$, then we will say that the *pde* is *leading linear* with respect to the given ranking. Otherwise a *pde* will be called *leading nonlinear* with respect to the given ranking. We will call $f(x,u_c)$ the *leading coefficient* of the *pde*. The set of leading linear *pde*s of an admissible system S in $\mathcal{B}$ with respect to $\succ$ will be denoted by $L_\succ(S)$. The complementary set of leading nonlinear *pde*s of S with respect to $\succ$ will be denoted by $N_\succ(S)$. This set includes degenerate *pde*s - those without derivatives.

2.3 Pivots, solved and unsolved pdes

If the *rif* algorithm, when it is applied to a *pde* system, solves a leading linear *pde* for its leading derivative, the coefficient of the leading derivative (which we call a pivot) is appended to a list (the pivot list Λ) of the previous pivots. The pivot list is kept in the order in which the pivots occurred. For example suppose (2.8) is an equation of a system (currently of order q) and that the current pivot list is $\Lambda = (\lambda^1(x,u_c) \neq 0, \ldots, \lambda^r(x,u_c) \neq 0)$. When (2.8) is solved for its leading derivatives to yield the solved-form $u^k_b = g(x,u_c)/f(x,u_c)$ we must assume $f(x,u_c) \neq 0$, and this condition is appended to the pivot list: $\Lambda = (\lambda^1(x,u_c) \neq 0, \ldots, \lambda^r(x,u_c) \neq 0, f(x,u_c) \neq 0)$. When the algorithm has terminated we have to consider the cases when one or more of the pivots vanish. In general a tree of cases has to be considered.

During the application of the *rif* algorithm to a system its current form (S say) is divided into two types of *pde*s: $S = solved(S) \cup unsolved(S)$ where $solved(S)$ denotes the set of *pde*s in S which are in solved-form for their leading derivatives, and $unsolved(S)$ denotes the set of the remaining *pde*s in S. Note that $solved(S) \subseteq L_\succ(S)$ and at input $solved(S) = \emptyset$.

2.4 Simplified orthonomic form

We say that an admissible system of *pde*s (S,Λ) of order q is in *simplified orthonomic form* in $\mathcal{B} \subseteq J^q(\mathbb{F}^n, \mathbb{F}^m)$ if

(a) Each leading derivative of $L_\succ(S)$ appears only once in S.

(b) No nontrivial derivative of any leading derivative of $L_\succ(S)$ appears in S.

For linear systems S these conditions are equivalent to conditions (i), (ii), (iii) and (iv) for the standard form of a system [43, p. 299]. Any admissible nonlinear system of *pde*s can, by a finite number of linear eliminations, be reduced to simplified orthonomic form. A simple but inefficient algorithm, *orthsimp*, to accomplish these criteria is given in Appendix A.

2.5 Integrability conditions of a system in simplified orthonomic form

Let $L_\succ(S)$ be the leading linear *pde*s of a system S which we assume is in simplified orthonomic form. We define the set of integrability conditions of S as the set of minimal and supplementary integrability conditions of $L_\succ(S)$, denoted by $integ\_condns(L_\succ(S))$, generated when Algorithm 6 of Reid [43, p.300-301, p.312-317] is applied to $L_\succ(S)$.

The set $integ\_condns(L_\succ(S))$ is a subset of the set of all compatibility conditions of $L_\succ(S)$ defined below. Let $L_\succ(S) = \bigcup_{k=1}^m \mathcal{L}^k_\succ(S)$ where $\mathcal{L}^k_\succ(S)$ are the *pde*s of $L_\succ(S)$ which are leading linear in some derivative of u^k. Given two pdes ϕ, ψ in $\mathcal{L}^k_\succ(S)$

$$\phi = pu^k_a - r = 0, \quad \psi = qu^k_b - s = 0, \tag{2.9}$$

and $c \in \mathbf{W}^n$ with $c \geq a$ and $c \geq b$ (i.e. $c_i \geq a_i$ and $c_i \geq b_i$, $i = 1, ..., n$) we define the *compatibility condition* of ϕ and ψ at c by

$$comp(\phi,\psi,c) := qD^{c-a}\phi - pD^{c-b}\psi = 0. \tag{2.10}$$

This condition is the *pde* that results from eliminating the common derivative u^k_c obtained by differentiating ϕ and ψ. The set of all compatibility conditions of $L_\succ(S)$ is generally infinite and is

$$\bigcup_{k=1}^{n} \bigcup_{\psi,\phi \in \mathcal{L}^k_\succ(S)} \bigcup_{c} \{comp(\phi,\psi,c) : c \in \mathbf{W}^n\}.$$

A fundamental problem is to obtain a finite subset of these conditions ensuring that all others are satisfied. The method used by Riquier [49] and Janet [29] to address this problem associates monomials $y^a = y_1^{a_1} y_2^{a_2} ... y_n^{a_n}$ with partial derivatives u_a^k. By a completion process for these sets of monomials they determine a finite set of points $c \in \mathbf{W}^n$ which determines a suitable finite set of compatibility conditions (see [53, 63] for computer implementations).

The idea behind our construction of *integ_condns*$(L_\succ(S))$ is similar to that underlying the Riquier-Janet method. In particular to identify the integrability conditions [43] we (implicitly) construct a system of *pde*s with its leading derivatives separated by exactly one differentiation $\frac{\partial}{\partial x_i}$ (in the Riquier-Janet approach the *pde*s are explicitly constructed). The set *integ_condns*$(L_\succ(S))$ generally contains fewer compatibility conditions than the set generated by the Riquier-Janet method.

It is natural to ask whether there is a simpler method than the Riquier-Janet method or our algorithm *integ_condns* for identifying an adequate finite set of compatibility conditions. One possibility is to replace the condition $c \in \mathbf{W}^n$ in the set of all compatibility conditions by $c_i = \max\{a_i, b_i\}$. This corresponds to the usual notion of computing integrability conditions by equating cross-partials using pairs of *pde*s. We know of several computer implementations of this approach. But we do not know of a proof that this approach ensures the satisfaction of all compatibility conditions. Another possibility is the set of minimal integrability conditions described in [43]. However we have found examples in which the satisfaction of these compatibility conditions was not enough to ensure satisfaction of the remaining conditions.

If $L_\succ(S)$ is a system of N *pde*s in one dependent variable and two independent variables then *integ_condns*$(L_\succ(S))$ contains only minimal integrability conditions (see Figure 3 of [43, p. 301]). There are $N-1$ such conditions versus $N(N-1)/2$ conditions if all pairs are considered. Non-obvious supplementary integrability conditions can arise for systems with more than two independent variables (see Figure 6 of [43, p. 316]).

2.6 The lex prolongation of a simplified orthonomic system

Consider an admissible system of *pde*s S which is in simplified orthonomic form with dependent variables $u^1, \ldots, u^m$ and recall that $L_\succ(S)$ is the set of *pde*s in S which are linear in their leading derivative. The prolongation $\rho^k(L_\succ(S))$ (see §2.1) generally contains many equations with the same leading derivative. To avoid this overlap we use the *lex prolongation* of S, denoted by $\rho_{\text{lex}}(S)$, which was implicitly defined in [43]. This prolongation has the property that each prolonged equation arises by differentiating exactly one leading linear equation. The *lex prolongation* results from an equivalence relation which partitions the set of derivatives of leading derivatives of $L_\succ(S)$ into equivalence classes with leading derivatives as representatives. To each derivative u_s^k in this set corresponds an equation with u_s^k as its leading derivative. This equation is obtained by appropriately differentiating the unique equation of $L_\succ(S)$ whose leading derivative is equivalent to u_s^k.

Define the lexicographic ordering $>_{\text{lex}}$ on the multi-indices of derivatives by $q >_{\text{lex}} r$ if the first nonzero entry of $(q_1 - r_1, q_2 - r_2, ..., q_n - r_n)$ is positive (otherwise $q \leq_{\text{lex}} r$). Let R_k be the set of leading derivatives of u^k in $L_\succ(S)$ and let $\hat{R}_k$ be the set of all derivatives of derivatives in R_k. Define $\Psi : \hat{R}_k \to R_k$ by $\Psi(u_s^k) = u_q^k$ where u_q^k is the lexicographically largest derivative in R_k with $q \leq_{\text{lex}} s$. Define an equivalence relation $\sim$ on $\hat{R}_k$ by $u_s^k \sim u_t^k$ if $\Psi(u_s^k) = \Psi(u_t^k)$. The equivalence class $[u_q^k]$ of a derivative u_q^k corresponds to the equivalence class $\mathcal{Z}(q)$ of [43] (see also Figure 5 of [43, p. 313]). Thus $\hat{R}_k = \bigcup_{u_q^k \in R_k} [u_q^k]$.

The *lex prolongation* of a simplified orthonomic system S is defined as the infinite set of equations

$$\rho_{\text{lex}}(S) := N_\succ(S) \cup M_\succ(S), \tag{2.11}$$

where

$$M_{\succ}(S) := \bigcup_{k=1}^{m} \bigcup_{u_q^k \in R_k} T(u_q^k), \tag{2.12}$$

and

$$T(u_q^k) := \{D^s\phi : u_{q+s}^k \in [u_q^k]\}. \tag{2.13}$$

The *pde* ϕ in $T(u_q^k)$ corresponds to $s=0$ and is the unique equation in $L_{\succ}(S)$ which has leading derivative u_q^k. $T(u_q^k)$ consists of all derivatives $D^s\phi$ whose leading derivative $u_{q+s}^k = D^s(u_q^k)$ is in the same class as u_q^k. The lex prolongation of a simplified orthonomic system S to order k, denoted by $\rho_{\text{lex}}^k(S)$, is the system of equations in $\rho_{\text{lex}}(S)$ of order less than or equal to k.

For example, for the system

$$S = N_{\succ}(S) \cup L_{\succ}(S) = \{u^2+v^2=0\} \cup \{u_{xx}=0, u_{xt}=v\} \tag{2.14}$$

with independent variables x, t and dependent variables u, v,

$$\begin{aligned} \rho_{\text{lex}}(S) &= \{u^2+v^2=0\} \cup \{u_{(2+k,l)}=0, u_{(1,1+l)}=v_{(0,l)}: k,l=0,1,2,\ldots\}, \\ \rho_{\text{lex}}^3(S) &= \{u^2+v^2=0\} \cup \{u_{xx}=0, u_{xxx}=0, u_{xxt}=0, u_{xt}=v, u_{xtt}=v_t\}, \end{aligned}$$

where $u_{(2,0)} = u_{xx}$, $u_{(2,1)} = u_{xxt}$, etc.

2.7 The locus of a system of pdes

Consider a q-th order admissible system of *pdes* F on $\mathcal{B} \subseteq J^q(\mathbb{F}^n, \mathbb{F}^m)$

$$F = \{f^i(x, u_c) = 0 : \; i = 1, \ldots, \#(F), (x, u_c) \in \mathcal{B}\}. \tag{2.15}$$

We define the *locus of* F (and the locus of f) by

$$\text{locus}_q(F) \equiv \text{locus}_q(f) := \{(x, u_c) : \; f^i(x, u_c) = 0, \; i = 1, \ldots, \#(F)\} \tag{2.16}$$

or equivalently

$$\text{locus}_q(F) \equiv \text{locus}_q(f) := f^{-1}(\mathbf{0}) \tag{2.17}$$

where $f : \mathcal{B} \to \mathbb{F}^{\#(F)}$ is defined by $f(x, u_c) \equiv (f^1(x, u_c), \ldots, f^{\#(F)}(x, u_c))$ and $\mathbf{0}$ is the zero vector in $\mathbb{F}^{\#(F)}$ ($\mathbb{F} = \mathbf{R}$ or $\mathbf{C}$). Given another q-th order admissible system of *pdes* G on $\mathcal{C} \subseteq J^q(\mathbb{F}^n, \mathbb{F}^m)$ then $F \cup G$ is a q-th order admissible system of *pdes* on $\mathcal{B} \cap \mathcal{C}$ and locus$(F \cup G)$ =locus$(F)\cap$locus(G). Note also that locus$(F) = \bigcap_{i=1}^{\#(F)}$locus$(f^i)$.

We do not make the usual identification of the locus of a q-th order *pde* system with its prolonged locus. Thus $\text{locus}_q(F) \supseteq \text{locus}_q(\rho^q(F))$, with inequality being the usual case. There is a natural embedding of $\text{locus}_q(F)$ into the higher order jet bundle $J^p(\mathbb{F}^n, \mathbb{F}^m)$, $p > q$. Specifically, given the natural projection π_q^p: $J^p(\mathbb{F}^n, \mathbb{F}^m) \to J^q(\mathbb{F}^n, \mathbb{F}^m)$, we define $\text{locus}_p(F) := (\pi_q^p)^{-1}\text{locus}_q(F)$. We shall write locus$(F)$ instead of $\text{locus}_p(F)$ with the understanding that p is the largest order derivative explicitly appearing in the specific calculations in which locus(F) arises.

2.8 Testing for geometrically new pdes

The *rif* algorithm when applied to an admissible system S keeps updating S by appending *pdes* to S resulting from differentiation and integrability conditions until these conditions do not represent 'geometrically new' *pdes* relative to S. We discuss how to test for such 'geometrically new' *pdes*.

In general a system of equations B is satisfied as a consequence of a system of equations A if every solution of A is also a solution of B, that is, locus$(B) \supseteq$ locus(A). In this case locus$(A \cup B)$= locus$(A)\cap$locus(B) = locus(A). If this is not the case then locus$(A \cup B) \subset$ locus(A) and there are genuinely new equations in B which, when appended to A, restrict locus(A).

We will say that an admissible system of *pdes* B is a *consequence* of a system of *pdes* S in simplified orthonomic form if

$$\mathrm{locus}(B) \supseteq \mathrm{locus}(\rho_{\mathrm{lex}}(S)). \tag{2.18}$$

Otherwise the system B contains 'geometrically new' *pde*s and the system $S \cup B$ is a genuinely new system.

To theoretically test (2.18) we will use $B \mid_{\rho_{\mathrm{lex}}(S)}$, defined by

$$B \mid_{\rho_{\mathrm{lex}}(S)} := (B \mid_{M_{\succ}(S)}) \mid_{N_{\succ}(S)}, \tag{2.19}$$

to represent a system in which all eliminations from $\rho_{\mathrm{lex}}(S) = N_{\succ}(S) \cup M_{\succ}(S)$ have been made in the following manner. First calculate

$$\tilde{B} = B \mid_{M_{\succ}(S)}, \tag{2.20}$$

by using $M_{\succ}(S)$ as a linear substitution list to eliminate all leading derivatives of $M_{\succ}(S)$ from B. This is a finite step linear process and in the terminology of [43] is equivalent to making all possible lexicographic substitutions from $L_{\succ}(S)$ into B.

At this stage testing (2.18) has been reduced to testing whether $\tilde{B}$ is a consequence of $N_{\succ}(S)$:

$$\mathrm{locus}(\tilde{B}) \supseteq \mathrm{locus}(N_{\succ}(S)). \tag{2.21}$$

It is a significant challenge to develop constructive implementations of such a test for various classes of *pde*s. In the special case where $N_{\succ}(S) = \emptyset$ then $\mathrm{locus}(N_{\succ}(S))$ is unrestricted (i.e. consists of all points in the relevant jet bundle) and testing (2.19) reduces to: if $\tilde{B} = \emptyset$ then $\tilde{B}$ is a consequence of $\rho_{\mathrm{lex}}(S)$, otherwise B contains genuinely new equations. Returning to the general case we use

$$\hat{B} = \tilde{B} \mid_{N_{\succ}(S)} \tag{2.22}$$

to represent a system $\hat{B}$ such that $\mathrm{locus}(\hat{B} \cup N_{\succ}(S)) = \mathrm{locus}(\tilde{B} \cup N_{\succ}(S))$ (i.e. $\mathrm{locus}(\hat{B}) \cap \mathrm{locus}(N_{\succ}(S))$ $=\mathrm{locus}(\tilde{B}) \cap \mathrm{locus}(N_{\succ}(S))$). Note that $\hat{B}$ is not necessarily unique. If B is a consequence of a system in simplified orthonomic form then we will always take $\hat{B} = B \mid_{\rho_{\mathrm{lex}}(S)} = \emptyset$. In cases where a nonlinear elimination algorithm is available (e.g. Buchberger's normal form algorithm for polynomials as in §5) then $\hat{B}$ can be taken as the simplified form of $\tilde{B}$ subject to $N_{\succ}(S)$ obtained by using the given elimination algorithm.

2.9 Constant rank condition

Consider an admissible system S of *pde*s of order q on $\mathcal{B} \subseteq J^q(\mathbb{F}^n, \mathbb{F}^m)$. We will say that S is of *constant rank* in a neighbourhood $\mathcal{O}$ of $(x^0, u_c^0) \in \mathrm{locus}(S)$ if $\mathrm{rank}(\mathrm{Jac}(S)) = \mathrm{rank}(\widetilde{\mathrm{Jac}}(S)) =$ constant in $\mathcal{O}$. Here $\mathrm{Jac}(S)$ (respectively $\widetilde{\mathrm{Jac}}(S)$) is the Jacobian matrix of first order partial derivatives of S with respect to the variables x, u_c (respectively u_c) appearing in S.[1] It follows from [2] that in some neighbourhood $\mathcal{U}$ of (x^0, u_c^0), $\mathrm{locus}(S) \cap \mathcal{U}$ is an analytic submanifold of $J^q(\mathbb{F}^n, \mathbb{F}^m)$ which projects diffeomorphically onto a neighbourhood of x^0 in the space of independent variables (we will say that such a point (x^0, u_c^0) is a *Euclidean point*). Thus near (x^0, u_c^0), $\mathrm{locus}(S)$ can be diffeomorphically transformed into some finite dimensional Euclidean space.

Conversely let (x^0, u_c^0) be a Euclidean point of $\mathrm{locus}(S)$. Then there is a neighbourhood $\mathcal{U}$ of (x^0, u_c^0) for which $\mathcal{U} \cap \mathrm{locus}(S)$ is an analytic manifold which projects diffeomorphically onto a neighbourhood of x^0 in the space of independent variables and there exists [24] a system of equations R satisfying the constant rank condition with the same locus as S in $\mathcal{U}$ (i.e. $\mathrm{locus}(R) \cap \mathcal{U} = \mathrm{locus}(S) \cap \mathcal{U}$). Thus in a neighbourhood of Euclidean points of $\mathrm{locus}(S)$ constant rank can always be achieved. That is, in theory there is a procedure to locally construct a system of constant rank from S. We use the name *constant_rank* to denote such a procedure.

[1] The equality of the ranks of $\mathrm{Jac}(S)$ and $\widetilde{\mathrm{Jac}}(S)$ implies that no functional relation between the independent variables can be deduced from S in $\mathcal{O}$.

For ease of computation, in the example of §4, we will use the following easily verified equivalent characterization of constant rank: S is of *constant rank* in a neighbourhood $\mathcal{U}$ of $(x^0, u_c^0) \in \text{locus}(S)$ if and only if there is a subsystem T (i.e. a subset of equations T) of S with the same locus as S in a neighbourhood of (x^0, u_c^0), for which $\text{rank}(\text{Jac}(T)) = \#(T)$ at (x^0, u_c^0).

2.10 Definition of Reduced Involutive Form

Let S be an admissible system of *pdes* on $\mathcal{B} \subseteq J^q(\mathbb{F}^n, \mathbb{F}^m)$ and let $(x^0, u_c^0) \in \text{locus}(S)$. We say that S is in *reduced involutive form* at (x^0, u_c^0) if there exists a neighbourhood $\mathcal{O}$ of (x^0, u_c^0) in $J^q(\mathbb{F}^n, \mathbb{F}^m)$ such that:

(1) The system S is in simplified orthonomic form in $\mathcal{O} \cap \text{locus}(S)$.

(2) $\text{locus}(I) \supseteq \text{locus}(\rho_{\text{lex}}(S))$ in $\mathcal{O}$ where $I = \textit{integ\_condns}(S)$.

(3) $\text{locus}(J) \supseteq \text{locus}(\rho_{\text{lex}}(S))$ in $\mathcal{O}$ where $J = \bigcup_{j=1}^{n} D_{x_j} N_{\succ}(S)$.

(4) S is of constant rank in $\mathcal{O}$.

We have used the terminology $A \subseteq B$ in $\mathcal{O}$ to mean $A \cap \mathcal{O} \subseteq B \cap \mathcal{O}$. Condition (2) states that in $\mathcal{O}$, the integrability conditions are a consequence of S (i.e. $I|_{\rho_{\text{lex}}(S)} = \emptyset$ in $\mathcal{O}$). Condition (3) means that all first order total derivatives of any equation in $N_{\succ}(S)$ are a consequence of S in $\mathcal{O}$ (i.e. $J|_{\rho_{\text{lex}}(S)} = \emptyset$ in $\mathcal{O}$).

3. The Reduced Involutive Form Algorithm

The *rif* algorithm consists of three nested **repeat**...**until** loops: the constant rank loop, the spawning loop, and the integrability conditions loop. Each loop modifies the system S and the pivot set Λ until all the escape conditions of the **until** statements are satisfied. That is, there are no new integrability conditions, no new spawned equations, and the system is of constant rank (unless the system is inconsistent).

Pseudo-code for the reduced involutive form algorithm:

input: A q-th order admissible system S on $\mathcal{B} \subseteq J^q(\mathbb{F}^n, \mathbb{F}^m)$ and a finite list of pivots of the form $\Lambda = (\lambda^1(x, u_c) \neq 0, \lambda^2(x, u_c) \neq 0, \ldots)$.

output: System (S, Λ) of finite order (r say) which is in reduced involutive form on a subset of $J^r(\mathbb{F}^n, \mathbb{F}^m)$.

repeat $\{$constant rank loop until constant rank obtained$\}$
  repeat $\{$spawning loop until no new spawned linear *pdes*$\}$
    repeat $\{$integrab. condns loop until no new integrab. condns.$\}$
      $(S, \Lambda) \leftarrow \textit{orthsimp}(S, \Lambda)$
      $I \leftarrow \textit{integ\_condns}(L_{\succ}(S))$
      $K \leftarrow \rho_{\text{lex}}(S)$
      $I \leftarrow I\,|_K$
      $S \leftarrow S \cup I$
    until $I = \emptyset$
    $J \leftarrow \bigcup_{j=1}^{n} D_{x_j} N_{\succ}(S)$
    $J \leftarrow J\,|_K$
    $S \leftarrow S \cup J$
  until $J = \emptyset$
  $S_{\text{temp}} \leftarrow S$
  $(S, \mathcal{B}) \leftarrow \textit{constant\_rank}(S, \mathcal{B})$
until $S = S_{\text{temp}}$ or (S, Λ) inconsistent
output $S, \Lambda, \mathcal{B}$

In the integrability conditions loop the procedure *orthsimp* takes as its input the system S and the pivot list Λ. By a sequence of linear eliminations amongst leading linear *pdes* and substitutions into $N_{\succ}(S)$ it outputs a system in simplified orthonomic form and an updated pivot list. The system and pivots are also denoted by S, Λ respectively. The output system has the same solutions as the input system away from solutions on which any of the pivots vanish. See Appendix A for pseudo-code for a simple, but inefficient version of *orthsimp*. Integrability conditions between the leading linear *pde*s $L_\succ(S)$ of S are calculated by *integ_condns*, then simplified with respect to the lex prolongation of S, and appended to the system in the fifth step of the integrability conditions loop. The integrability conditions loop terminates when no further 'new' integrability conditions can be found. The spawning step $J \leftarrow \bigcup_{j=1}^n D_{x_j} N_\succ(S)$ spawns leading linear *pde*s J by forming the set of all first order total derivatives of each leading nonlinear *pde* of the system. In the step $J \leftarrow J\mid_K$ these *pde*s are simplified with respect to the lex prolongation of S and any 'new' spawned linear *pde*s are then appended to S. The spawning loop is repeated until there are no 'new' spawned *pde*s. The *constant_rank* procedure determines whether the Jacobian of the system is of constant rank at (i.e. in a neighbourhood of) points of the system's locus where none of the pivots vanish. The system is in *rif*-form at all Euclidean points of its locus which satisfy the constant rank condition. In particular if the constant rank condition is satisfied at all Euclidean points of this locus in its current domain of admissibility $\mathcal{B}$ then the *constant_rank* procedure does not alter the system, so $S_{\text{temp}} = S$ and the *rif* algorithm has terminated. If the constant rank condition is not satisfied at some Euclidean points of the system's locus then the Implicit Function Theorem is applied to a neighbourhood of one of these points. In particular a system of equations satisfying the constant rank condition is determined which has the same locus as $N_\succ(S)$ in this neighbourhood. This system replaces $N_\succ(S)$ and the domain of admissibility $\mathcal{B}$ (regarded as a subset of $J^\infty(\mathbb{F}^n, \mathbb{F}^m)$) is appropriately shrunk. Consequently $S \neq S_{\text{temp}}$ and the adjusted system is returned to the beginning of the integrability conditions loop. If there are no Euclidean points then the *rif* algorithm has terminated with the system being inconsistent in $\mathcal{B}$.

Since the equations $L_\succ(S)$ automatically satisfy the constant rank condition we assume that *constant_rank* only alters $N_\succ(S)$. The system is inconsistent if the *constant_rank* procedure determines that there is a nontrivial functional relationship between the system's independent variables or one of the (supposedly nonzero) pivots vanishes upon simplification subject to the system. Cases due to the vanishing or non-vanishing of the pivots have to be treated in different runs of the *rif* algorithm. Different parts of the system's domain of admissibility may also require separate treatment due to the locality of the *constant_rank* procedure. To decrease the number of these cases it is desirable to attempt to maximise the system's domain of admissibility and minimise the number of pivots resulting at each run of the *rif* algorithm. In §5 we will see that case-splitting on the domain of admissibility can be completely avoided for polynomially nonlinear systems of *pde*.

Proof of termination of the rif algorithm for orderly rankings

Suppose that the *rif* algorithm does not terminate when it is applied to a system S. Let $L^i_\succ(S)$, $N^i_\succ(S)$, L^i, $\mathcal{B}^i$ be the respective forms of $L_\succ(S)$, $N_\succ(S)$, the leading derivatives of $L_\succ(S)$, and the system's domain of admissibility after the i-th iteration of the integrability conditions loop. Then regarding the $\mathcal{B}^i$ as subsets of $J^\infty(\mathbb{F}^n, \mathbb{F}^m)$, $\mathcal{B} = \mathcal{B}^0 \supseteq \mathcal{B}^1 \supseteq \mathcal{B}^2 \supseteq \cdots$ with $\mathcal{B}^i \neq \emptyset$ for all i. By the argument for termination of *orthsimp* given in Appendix A there exists an integer i_0 such that $L^i = L^{i_0}$ for all $i \geq i_0$. As a consequence of the properties of *orthsimp* and *constant_rank*, $L^i_\succ(S) = L^{i_0}_\succ(S)$ for all $i \geq i_0$. Let $i \geq i_0$ and consider a *pde* $\psi = 0$ in $N^i_\succ(S)$. Let u^j_q be the leading derivative of $\psi = 0$. Then:

(a) u^j_q is not a derivative of any derivative in L^{i_0}.

(b) For each $l = 1, \ldots, n$, $D_{x_l} u^j_q = u^j_{q+1_l}$ is a derivative of some derivative in L^{i_0}.

(c) $\text{ord}(q) \leq \text{ord}(s)$ where s is the multi-index with $s_l = \max\{a_l : u^k_a \in L^{i_0}\}$, $l = 1, \ldots, n$.

Statement (a) follows from the fact that all derivatives of members of L^{i_0} have been eliminated from $N^i_\succ(S)$ by the step $S_{unsol} \leftarrow S_{unsol}\mid_{\rho^\infty(\{eq\})}$ of *orthsimp*.

Suppose that statement (b) is not true. Then for some l, $D_{x_l} u^j_q = u^j_{q+1_l}$ is not a derivative of any derivative in L^{i_0}. The spawning step $J \leftarrow \bigcup_{j=1}^n D_{x_j} N_\succ(S)$ generates the *pde* $D_{x_l}\psi = 0$ which, from property (2.7), has leading derivative $u^j_{q+1_l}$. This *pde* is also leading linear in $u^j_{q+1_l}$. At the next iteration of *orthsimp*, either the leading linear *pde* $D_{x_l}\psi = 0$ is placed in solved-form for $u^j_{q+1_l}$, or $u^j_{q+1_l}$ is eliminated using another leading linear *pde*. Both cases violate the condition $L^{i_0} = L^i$ for $i \geq i_0$, so statement (b) is true.

To establish statement (c) first note that from (b), for each $l = 1, \ldots, n$, there is a *pde* $\phi = 0$ in $L^{i_0}_{\succ}(S)$ such that $D_{x_l} u^j_q$ is a derivative of the leading derivative of $\phi = 0$. Let u^j_c be the leading derivative of $\phi = 0$. Statement (a) implies that u^j_q is not a derivative of u^j_c. Thus $q_k \geq c_k$, $k \neq l$, $q_l + 1 \geq c_l$ and $q_l < c_l$. From the definition of s, $c_l \leq s_l$, so $q_l \leq s_l$. Therefore ord(q) $\leq$ord(s) yielding statement (c).

As a consequence the only derivatives that appear in $N^i_{\succ}(S)$ for all $i \geq i_0$ are derivatives of order less than or equal to ord(s). There are only finitely many (say P) such derivatives. If P or more geometrically independent relations are found amongst these derivatives then the constant rank condition fails, there is a functional relationship between the system's independent variables, it is inconsistent and the algorithm terminates. Thus at most $P - 1$ geometrically independent relations can be found amongst these derivatives. For i sufficiently large ($i \geq i_1 \geq i_0$ say) no geometrically new leading nonlinear *pde*s can be found and the exit conditions of the integrability conditions and the spawning loops are satisfied.

At most one further iteration of the constant rank loop is required to adjust the system so that it has constant rank. Then the algorithm terminates, contrary to our assumption.

$\square$

4. Detailed example of the rif algorithm

We apply the *rif* algorithm to the fourth order nonlinear real analytic system of *pde*s S on $J^4(\mathbf{R}^2, \mathbf{R})$, with independent variables x, y and dependent variable u, given by

$$u_{yyyy} - u_{xyyy} = 0, \tag{4.1a}$$
$${u_{yyyy}}^2 - 2u_{yyyy}u_{xyyy} + {u_{xyyy}}^2 + u_{yy} - u_{xy} = 0, \tag{4.1b}$$
$${u_{yy}}^2 - {u_{xy}}^2 + {u_x}^p + u_x - u = 0. \tag{4.1c}$$

We suppose that $p \geq 4$ is an integer and that there are no initial pivot restrictions (i.e. $\Lambda = \emptyset$).

The *rif* algorithm will be applied to the above system with respect to the ranking $\succ$ graded first by total order of derivative and then lexicographically by derivative (so that ∂_x precedes ∂_y):

$$u \prec u_x \prec u_y \prec u_{xx} \prec u_{xy} \prec u_{yy} \prec u_{xxx} \prec \cdots \tag{4.2}$$

Note that the largest derivatives with respect to $\succ$ of (4.1) are u_{yyyy}, u_{yyyy} and u_{yy} respectively. The set of leading linear *pde*s is $L_{\succ}(S) = \{(4.1a)\}$, and the set of leading nonlinear *pde*s is $N_{\succ}(S) = \{(4.1b), (4.1c)\}$.

We apply the version of *orthsimp* given in Appendix A to the system. Initially the unsolved equations are all of S and there are no solved equations ($S_{unsol} = S$, $S_{sol} = \emptyset$). $L_{\succ}(S_{unsol})$ consists of the single equation (4.1a), so this equation is solved for its leading derivative u_{yyyy} yielding

$$u_{yyyy} = u_{xyyy}. \tag{4.3}$$

In this case solving involves dividing by the coefficient of u_{yyyy} in (4.1a) (that is 1) so 1 is added to the pivot list Λ. Elimination of u_{yyyy} from (4.1b) using the above equation yields

$$u_{yy} - u_{xy} = 0. \tag{4.4}$$

Since there are no implicit occurrences of u_{yyyy} (that is there are no derivatives of u_{yyyy} such as u_{xyyyy} etc) remaining in the system we have completely eliminated u_{yyyy} from the system. The system S is now equivalent to $S_{sol} \cup L_{\succ}(S_{unsol}) \cup N_{\succ}(S_{unsol})$ where S_{sol} consists of (4.3), $L_{\succ}(S_{unsol})$ consists of (4.4) and $N_{\succ}(S_{unsol})$ consists of (4.1c). We solve (4.4) for u_{yy} to obtain

$$u_{yy} = u_{xy}. \tag{4.5}$$

Explicit elimination of u_{yy} from (4.1c) yields

$$u_x{}^p + u_x - u = 0. \tag{4.6}$$

This equation describes a surface in $J^1(\mathbf{R}^2, \mathbf{R})$ (see Figure 1 for this surface in the case p=4). Repeated substitutions from (4.5) into (4.3) until no further substitutions are possible reduce it to an identity so this last equation can be discarded. The original system is now in simplified orthonomic form and is $S = L_{\succ}(S) \cup N_{\succ}(S)$ where

$$\begin{aligned} L_{\succ}(S) &= \{u_{yy} = u_{xy}\}, && (4.7a)\\ N_{\succ}(S) &= \{u_x{}^p + u_x - u = 0\}, && (4.7b)\\ \Lambda &= (1 \neq 0). && (4.7c) \end{aligned}$$

Since there is only one leading linear *pde* in (4.7) application of *integ_condns* yields no integrability conditions, S is not altered and we exit the integrability conditions loop.

In the spawning step, $J \leftarrow D_x N_{\succ}(S) \cup D_y N_{\succ}(S)$, leading linear *pde*s are spawned by differentiating the leading nonlinear *pde* (4.7b) once with respect to x and y. This yields

$$\begin{aligned} D_x(u_x{}^p + u_x - u) &= (pu_x{}^{p-1} + 1)u_{xx} - u_x = 0, && (4.10a)\\ D_y(u_x{}^p + u_x - u) &= (pu_x{}^{p-1} + 1)u_{xy} - u_y = 0, && (4.10b) \end{aligned}$$

which, as expected, are linear in their respective leading derivatives u_{xx} and u_{xy}.

The spawned equations J are not satisfied as a consequence of $K = \rho_{\text{lex}}(S)$ (i.e. simplifying them with respect to $L_{\succ}(S) = \{u_{yy} = u_{xy}\}$ and then subject to $N_{\succ}(S)$ given by (4.7b)). Note that we only had to consider $\rho_{\text{lex}}(S)$ to order 2. Consequently locus$(J) \not\supseteq$ locus(K) and we return to the first step of the spawning loop.

As the result of the application of *orthsimp* the system becomes

$$\begin{aligned} u_{yy} &= \frac{u_y}{pu_x{}^{p-1} + 1}, && (4.11a)\\ u_{xy} &= \frac{u_y}{pu_x{}^{p-1} + 1}, && (4.11b)\\ u_{xx} &= \frac{u_x}{pu_x{}^{p-1} + 1}, && (4.11c)\\ u_x{}^p + u_x - u &= 0, && (4.11d) \end{aligned}$$

where Λ is now $(1 \neq 0, pu_x{}^{p-1} + 1 \neq 0)$ since division by the pivot $pu_x{}^{p-1} + 1$ has occurred. For the purpose of retaining the system in polynomial form, we can alternatively, by doing fraction-free arithmetic [21], obtain (4.11) in the form

$$\begin{aligned} (pu_x{}^{p-1} + 1)u_{yy} &= u_y, && (4.12a)\\ (pu_x{}^{p-1} + 1)u_{xy} &= u_y, && (4.12b)\\ (pu_x{}^{p-1} + 1)u_{xx} &= u_x, && (4.12c)\\ \omega := u_x{}^p + u_x - u &= 0. && (4.12d) \end{aligned}$$

Integrability conditions are now sought across equations which are linear in their highest derivative. Using the method of [43] there are two integrability conditions corresponding to the points c given by $(1,2)$, $(2,1)$ (corresponding to u_{xyy} and u_{xxy} respectively). These integrability conditions are D_x (4.12a) - D_y(4.12b)= 0 and D_x (4.12b) - D_y(4.12c)= 0. The second of these is automatically satisfied since (4.12b) and (4.12c) are total derivatives of the same equation. The first condition is

$$p(p-1){u_x}^{p-2}u_{xx}u_{yy} - u_{xy} - p(p-1){u_x}^{p-2}{u_{xy}}^2 + u_{yy} = 0. \tag{4.13}$$

In the step $I \leftarrow I\,|_K$ the integrability condition (4.13) is simplified subject to $K = \rho_{\text{lex}}(S)$. In particular it is first simplified subject to $L_\succ(S)$ (i.e. subject to (4.12a,b,c)). To perform fraction-free elimination (4.13) is multiplied by $(p{u_x}^{p-1}+1)^2$ and then simplified subject to $L_\succ(S)$. It becomes

$$\begin{aligned} p(p-1){u_x}^{p-2}(u_x)(u_y) - (p{u_x}^{p-1}+1)(u_y) \\ - p(p-1){u_x}^{p-2}{u_y}^2 + (p{u_x}^{p-1}+1)(u_y) = 0 \end{aligned} \tag{4.14}$$

or equivalently

$$\psi := p(p-1){u_x}^{p-2}u_y(u_x - u_y) = 0. \tag{4.15}$$

To complete the step $I \leftarrow I\,|_K$, (4.15) is simplified subject to $N_\succ(S)$ given by (4.12d). Since (4.15) is not identically satisfied as a consequence of $\omega = 0$ (i.e. equation (4.12d)) the output of this step consists of I given by (4.15). Geometrically, the locus of $\omega = 0$ in the jet bundle with coordinates (u, u_x, u_y) has been restricted by the equation (4.15) so it is a geometrically new equation (see Figure 1 where the curves A, B and the u_y axis represent the current restricted jet bundle locus). I is appended to the system and S now consists of (4.12),(4.15). I is a new equation so we re-enter the integrability conditions loop.

The leading linear equations were not changed in the last application of the integrability conditions loop so the loop produces the same condition (4.13) in its next iteration. Consequently the system exits the integrability conditions loop in the form (4.12),(4.15).

At the spawning step the spawned leading linear *pdes* $D_x\psi = 0$ and $D_y\psi = 0$ are derived from ψ. In the step $J \leftarrow J\,|_K$ fraction-free elimination is used to simplify $D_x\psi = 0$ and $D_y\psi = 0$ subject to $L_\succ(S)$. For future computational convenience this is done for

$$\phi := \kappa {u_x}^\alpha {u_y}^\beta (u_x - u_y) = 0, \tag{4.16}$$

where α, β and κ are constants. Use of (4.12a,b,c) gives

$$(p{u_x}^{p-1}+1)D_x\phi = \kappa(\alpha+\beta+1){u_x}^\alpha {u_y}^\beta (u_x - u_y) = 0, \tag{4.17a}$$

$$\begin{aligned} (p{u_x}^{p-1}+1)D_y\phi &= \alpha\kappa {u_x}^{\alpha-1}{u_y}^{\beta+1}(u_x - u_y) \\ &\quad + \beta\kappa {u_x}^\alpha {u_y}^\beta (u_x - u_y) = 0, \end{aligned} \tag{4.17b}$$

so that

$$(p{u_x}^{p-1}+1)D_x\psi = p^2(p-1){u_x}^{p-2}u_y(u_x - u_y) = 0, \tag{4.18a}$$

$$\begin{aligned} (p{u_x}^{p-1}+1)D_y\psi &= p(p-1)(p-2){u_x}^{p-3}{u_y}^2(u_x - u_y) \\ &\quad + p(p-1){u_x}^{p-2}u_y(u_x - u_y) = 0, \end{aligned} \tag{4.18b}$$

are the spawned equations simplified subject to $L_\succ(S)$. Continuing this step $J = \{(4.18a), (4.18b)\}$ is simplified subject to $N_\succ(S) = \{\psi = 0, \omega = 0\}$. Since $p \geq 4$ is an integer (4.18a,b) are both consequences of $\psi = 0$. In other words the intersection of the loci of (4.18a) and (4.18b) contains the locus of ψ in u, u_x, u_y space (see Figure 1). So S is not altered and we escape the spawning loop.

In the step $S_{temp} \leftarrow S$ we store S as S_{temp} to later test the escape condition of the constant rank loop. The step $(S, \Lambda) \leftarrow$ *constant_rank*(S, Λ) involves the calculation of the Jacobian of S.

The submatrix of the full Jacobian matrix of the system corresponding to $L_\succ(S)$ is guaranteed to be of constant rank so it is only necessary to check the rank of the submatrix corresponding to $N_\succ(S) = \{\omega = 0,\ \psi = 0\}$. This Jacobian matrix is

$$\begin{aligned} \mathrm{Jac}(N_\succ(S)) &= \begin{pmatrix} \omega_u & \omega_{u_x} & \omega_{u_y} \\ \psi_u & \psi_{u_x} & \psi_{u_y} \end{pmatrix} \\ &= \begin{pmatrix} -1 & p u_x^{p-1} + 1 & 0 \\ 0 & p(p-1)(p-2)u_x^{p-3}u_y(2u_x - u_y) & p(p-1)u_x^{p-2}(u_x - 2u_y) \end{pmatrix}. \end{aligned} \tag{4.19}$$

The rank of this Jacobian is required at points (x, y, u, u_x, u_y) satisfying $\psi = p(p-1)u_x^{p-2}u_y(u_x - u_y) = 0$ and $\omega = u_x^p + u_x - u = 0$. The conditions $\psi = 0, \omega = 0$ result in three cases:

$$u_x = 0, \quad u = 0, \tag{4.20a}$$
$$u_x \neq 0, \quad u_y = 0, \quad {u_x}^p + u_x - u = 0, \tag{4.20b}$$
$$u_x \neq 0, \quad u_y \neq 0, \quad u_x = u_y, \quad {u_y}^p + u_y - u = 0. \tag{4.20c}$$

In cases (4.20b,c) the constant rank condition is satisfied since

$$\mathrm{rank}(\mathrm{Jac}(N_\succ(S))) = 2 = \#(N_\succ(S)). \tag{4.21}$$

Consequently on those parts of the solution locus in the jet bundle the algorithm has terminated. In case (4.20a) the constant rank condition is not satisfied since

$$\mathrm{rank}(\mathrm{Jac}(N_\succ(S))) = 1 \neq 2 = \#(N_\succ(S)). \tag{4.22}$$

To modify the system using *constant_rank* so that the constant rank condition is satisfied, $\psi = 0$ is replaced with the condition

$$\hat{\psi} = p(p-1)u_x u_y(u_x - u_y) = 0 \tag{4.23}$$

which has the same locus as $\psi = 0$ (i.e. the term u_x^{p-2} in ψ is replaced by u_x). Then the constant rank condition is satisfied except at $u_x = u_y = u = 0$. The impossibility of satisfying the constant rank condition at $u_x = u_y = u = 0$ is clear from Figure 1. There can be no diffeomorphism which maps the three branches of locus(S) in a neighbourhood of $u_x = u_y = u = 0$ into a single curve or plane.

S has been altered by *constant_rank* so $S \neq S_{temp}$ and the constant rank loop is repeated. It is easily shown that the system escapes from the integrability conditions loop and the spawning loop unchanged. In the spawning step two new leading linear equations are obtained by differentiating $\hat{\psi}$ and then simplifying them subject to K. In particular using (4.17a,b) the equations are obtained in the form

$$3p(p-1)u_x u_y(u_x - u_y) = 0 \tag{4.24}$$
$$p(p-1)u_y^2(u_x - u_y) + p(p-1)u_x u_y(u_x - u_y) = 0 \tag{4.25}$$

which by virtue of $\hat{\psi} = 0,\ \omega = 0$ is equivalent to the single equation

$$\eta := p(p-1)u_y^2(u_x - u_y) = 0. \tag{4.26}$$

Since $J = \{\eta = 0\} \neq \emptyset$ is not a consequence of $\hat{\psi} = 0$ or $\omega = 0$ we re-enter the spawning loop.

Again it is easily shown that the system escapes the integrability condition and spawning loops. The system is of constant rank when $N_\succ(S) = \{\omega = 0, \hat{\psi} = 0, \eta = 0\}$ is replaced by $N_\succ(S) = \{\omega = 0, \hat{\eta} = 0\}$ where $\hat{\eta} := p(p-1)u_y(u_x - u_y) = 0$. Note that $\hat{\psi} = 0$ is a consequence of $\hat{\eta} = 0$, but not vice-versa. For the final loop through the pseudo-code the algorithm makes no more changes to (S, Λ) and all the conditions for termination are satisfied.

The output system is

$$\begin{aligned} L_{\succ}(S) &= \{(pu_x^{p-1}+1)u_{yy} = u_y,\ \ (pu_x^{p-1}+1)u_{xy} = u_y, \\ &\qquad\qquad\qquad\qquad (pu_x^{p-1}+1)u_{xx} = u_x\}, \\ N_{\succ}(S) &= \{u_x{}^p + u_x - u = 0,\ \ p(p-1)u_y(u_x - u_y) = 0\}, \\ \Lambda &= (1 \neq 0,\ \ pu_x^{p-1}+1 \neq 0), \end{aligned} \tag{4.27}$$

which is in reduced involutive form at all points of its jet bundle locus except at those with $u = u_x = u_y = 0$ where the constant rank condition fails.

The case where the pivot $pu_x^{p-1}+1$ vanishes has to be treated separately by re-running the algorithm with S given by (4.1) together with the additional equation

$$pu_x^{p-1} + 1 = 0. \tag{4.28}$$

Again the system is obtained in the form (4.7) but with the extra equation above. Differentiating the leading nonlinear *pde*s (4.7b) and (4.28) and simplification subject to (4.28) yields $u_x = 0, u_y = 0$. But then (4.28) yields the inconsistent equation $1 = 0$ and the system has no solutions in this case.

5. The Reduced Involutive Form Algorithm for polynomially nonlinear systems

Our approach is to develop geometric algorithms for general analytic *pde* systems and then attempt to give algebraic realizations of these algorithms for various classes of *pde*s. Specifically in this section we present the *gröbner_rif* algorithm, an algebraic realization of the *rif* algorithm for polynomially nonlinear systems of *pde*s.

For general analytic nonlinear systems of *pde*s the *rif* algorithm cannot be regarded as effective since:

(a) An effective algorithm has not been provided for testing whether one system of equations is a consequence of another (i.e. an effective realization of (2.19));

(b) We have not given an effective realization of *constant_rank*, the algorithm which transforms systems into constant rank form.

For polynomially nonlinear *pde* systems Buchberger's Gröbner basis and normal form algorithms will be used to provide an effective realization addressing (a). To address (b) in the polynomial case we will use algorithms for determining the radical of a polynomial ideal. Our method for addressing (b) will have the advantage that it will enable the local constant rank condition to be satisfied at all Euclidean points of the system's jet bundle locus.

In this section S denotes an r-th order system of *pde*s which is polynomial in its n independent and m dependent variables with coefficients from the field $\mathbb{F}$. Thus each equation of S belongs to the ring $\mathbb{F}[X]$ where X is the list of indeterminants (x, u_c),$0 \leq \mathrm{ord}(c) \leq r$ and S is an admissible system of *pde*s on $J^r(\mathbb{F}^n, \mathbb{F}^m)$. Let $< S >$ denote the polynomial ideal generated by S. We describe how the *gröbner_rif* algorithm is obtained from the *rif* algorithm by giving algebraic realizations of the steps of the *rif* algorithm.

5.1 Procedure orthsimp for polynomially nonlinear systems

Several alterations of *orthsimp* are required in the polynomial case. In the first of these the step

$$S_{unsol} \leftarrow simplify(S_{unsol}) \tag{5.1}$$

of *orthsimp* given in Appendix A is replaced by

$$S_{unsol} \leftarrow gbasis_{\succ}(S_{unsol})\ . \tag{5.2}$$

Here $gbasis_{\succcurlyeq}$ converts the equations of S_{unsol} into an algebraic Gröbner basis with respect to a ranking $\succcurlyeq$. The ranking $\succcurlyeq$ must be an allowable Gröbner basis ranking which should also be compatible with the ranking $\succ$ used in the *rif* algorithm in the following way. The ranking $\succcurlyeq$ restricted to the set of derivatives of dependent variables of the system must be the same as $\succ$ (see [66]).

The second modification involves the use of fraction-free elimination to carry out the steps

$$S_{sol} \leftarrow S_{sol} \mid_{\rho^{\infty}(\{eq\})} \tag{5.3}$$

and

$$S_{unsol} \leftarrow S_{unsol} \mid_{\rho^{\infty}(\{eq\})} \tag{5.4}$$

of *orthsimp*. We show how to eliminate eq from any equation $\psi = 0$ of S_{sol} or S_{unsol}. Now eq must be a leading linear *pde* of form $\phi := f u_a^k - g = 0$, with leading derivative u_a^k. Consider a term in $\psi = 0$ of the form $h \cdot (u_a^k)^l$ where h does not depend on u_a^k (i.e. $\psi = \cdots + h \cdot (u_a^k)^l + \cdots = 0$). Since $(f)^l (u_a^k)^l = (g)^l$, the term $h \cdot (u_a^k)^l$ in $\psi = 0$ can be eliminated by replacing $\psi = 0$ by the equation $(f)^l \psi - h \cdot \{(f)^l (u_a^k)^l - g^l\} = 0$ (i.e. replacing $(f)^l h \cdot (u_a^k)^l$ in $(f)^l \psi = 0$ by $h \cdot (g)^l$). The pivot condition $f \neq 0$ must be adjoined to the pivot list Λ for the above replacement to be valid and the case $f = 0$ has to be treated separately. The case where $\psi = 0$ contains a term of form $h \cdot (u_c^k)^l$ and u_c^k is a derivative of u_a^k is only slightly more complicated. For that case the set of equations $\rho^{\mathrm{ord}(c)}(\{\phi = 0\})$ will contain an equation of form $f \cdot (u_c^k) - \tilde{g} = 0$ where $\tilde{g}$ involves derivatives of u_a^k strictly lower in the ranking $\succ$ than u_c^k. Hence we can eliminate $h \cdot (u_c^k)^l$ from $\psi = 0$ in the same way as before. Further by a finite number of substitutions from $\rho^{\infty}(\{\phi = 0\})$ all derivatives of u_a^k appearing in $\psi = 0$ can be eliminated.

In summary, for polynomially nonlinear systems S the output of *orthsimp* is a system S such that $L_{\succ}(S)$ is in solved form and $N_{\succ}(S)$ is in the form of a Gröbner basis, together with an updated pivot list Λ.

5.2 Testing for algebraically and geometrically new pdes

Recall from §2.8 that a system of *pde*s B was a geometric consequence of a system S in simplified orthonomic form if $\mathrm{locus}(B) \supseteq \mathrm{locus}(\rho_{\mathrm{lex}}(S))$ where $\rho_{\mathrm{lex}}(S) = N_{\succ}(S) \cup M_{\succ}(S)$. Also recall that

$$B \mid_{\rho_{\mathrm{lex}}(S)} := (B \mid_{M_{\succ}(S)}) \mid_{N_{\succ}(S)} \tag{5.5}$$

and from §5.1 that the first step of this simplification, the evaluation of $\tilde{B} = B \mid_{M_{\succ}(S)}$, can be accomplished by a finite number of fraction-free linear substitutions from $M_{\succ}(S)$ into B. To give an effective realization of $\tilde{B} \mid_{N_{\succ}(S)}$ for polynomial systems we use Buchberger's normal form function to simplify $\tilde{B}$ subject to the polynomial relations $N_{\succ}(S)$:

$$\hat{B} = \tilde{B} \mid_{N_{\succ}(S)} := normal\_form_{\succcurlyeq}(\tilde{B}, N_{\succ}(S)) \ . \tag{5.6}$$

Then $\hat{B} = \emptyset$ if $\tilde{B}$ lies in the polynomial ideal generated by $N_{\succ}(S)$. For example suppose $\tilde{B} = \{u_x^3 = 0\}$. Then $N_{\succ}(S) = \{u_x^2 = 0\}$ implies that $\hat{B} = \emptyset$ and $N_{\succ}(S) = \{u_x^4 = 0\}$ implies that $\hat{B} = \{u_x^3 = 0\}$. In both cases $\{u_x^3 = 0\}$ is not a geometrically new equation relative to $N_{\succ}(S)$ since neither of the equations restrict the locus of $N_{\succ}(S)$. In the first case $\tilde{B} = \{u_x^3 = 0\}$ is not algebraically new in the sense that u_x^3 lies in the polynomial ideal generated by u_x^2. In the second case $\tilde{B} = \{u_x^3 = 0\}$ is algebraically new since u_x^3 does not lie in the polynomial ideal generated by u_x^4.

If $\hat{B} = \emptyset$ then the geometric condition (2.18) is also achieved. An argument similar to that given in [38] can be used to prove that the conditions $I = \emptyset$ and $J = \emptyset$ can be achieved in a finite number of iterations of the integrability and spawning loops.

5.3 Satisfying the constant rank condition

With the modifications discussed in §5.1, §5.2, the inner two loops of the *rif* algorithm are effective and preserve the admissibility of the system on all of the system's jet bundle locus. This part of the algorithm terminates with the satisfaction of the conditions (1), (2) and (3) for reduced involutive form given in §2.10. Only the constant rank condition (4) remains to be satisfied.

The most difficult condition to obtain for general analytic nonlinear systems is the constant rank condition (4). To achieve the constant rank condition for polynomial systems the step

$$(S, \Lambda) \leftarrow constant\_rank(S, \Lambda) \tag{5.7}$$

is replaced by

$$S \leftarrow L_{\succ}(S) \cup gbasis_{\succ} \sqrt{N_{\succ}(S)} \tag{5.8}$$

where $\sqrt{N_{\succ}(S)}$ is the *radical* of the polynomial ideal[2] generated by the polynomials belonging to $N_{\succ}(S)$. The validity of this replacement follows from the proposition below.

The algorithms in [22] can effectively determine a finite set of generators for the radical of a polynomial ideal, provided that the coefficients belong to a domain for which the zero recognition problem can be solved (for example this is the case if the coefficients of the system are rational numbers, or some computable extension of the rational number field). These algorithms do not depend on factorization (which is in general impossible) and involve instead the use of Gröbner bases and square-free factorizations for which effective algorithms exist. In the case of a univariate polynomial $p(x)$ the radical of the ideal generated by the polynomial is generated by the square-free factorisation of $p(x)$, i.e. by $p(x)/\gcd(p(x), p'(x))$. The paper [22] shows that primary decomposition for a system of multivariate polynomials can be reduced to primary decomposition for the zero dimensional (univariate) case by an algorithm which is roughly an induction on dimension, and uses Gröbner bases techniques, ideal quotients and localization (pushing variables into the coefficient field). The same algorithm together with square-free factorization can be applied to the computation of radicals [22, §9].

The following proposition verifies that a polynomial ideal which is radical satisfies the constant rank condition at Euclidean points of its locus.

Proposition: *Let the map $\varrho : \mathbb{F}^N \to \mathbb{F}^{\#(\varrho)}$ denote a finite system of multivariate polynomials over the real or complex field. Suppose that the ideal generated by the polynomials is radical and let y^0 be a point on the locus of ϱ. Then y^0 is a Euclidean point if and only if ϱ satisfies the constant rank condition on some neighbourhood $\mathcal{O}$ of y^0.*

Proof: If there is a neighbourhood $\mathcal{O}$ of y^0 on which $\mathrm{Jac}(\varrho(y))$ has constant rank then by the Implicit-parametrization Theorem y^0 is a Euclidean point [2]. Conversely let y^0 be a Euclidean point on the locus of ϱ (i.e. $y^0 \in \varrho^{-1}(\mathbf{0})$). Then [24] there exists a neighbourhood $\mathcal{U}$ of y^0 and an analytic function $\varphi : \mathbb{F}^N \to \mathbb{F}^{\#(\varphi)}$ such that $\mathcal{U} \cap \varphi^{-1}(\mathbf{0}) = \mathcal{U} \cap \varrho^{-1}(\mathbf{0})$ and $\mathrm{Jac}(\varphi(y))$ has constant rank $\#(\varphi)$ on $\mathcal{U}$. The dimension of the Euclidean space at y^0 is $N - \#(\varphi)$ and so $\mathrm{rank}(\mathrm{Jac}(\varrho(y))) \leq \mathrm{rank}(\mathrm{Jac}(\varphi(y)) = \#(\varphi)$ for all $y \in \mathcal{U}$.

The crucial step of the proof is the observation that there exist matrices A, B analytic in y such that

$$\varrho(y) = A\varphi(y), \quad \varphi(y) = B\varrho(y), \tag{5.9}$$

for y in some neighbourhood $\mathcal{V}$ of y^0. The first equation follows from the analytic Nullstellensatz and the second from the analytic form of the Nullstellensatz for radical ideals [24]. The proof is easily completed by showing that $\mathrm{rank}(\mathrm{Jac}(\varrho(y))) \geq \#(\varphi)$ is a consequence of (5.9).

Without loss we can choose coordinates $\hat{y}_1, \ldots, \hat{y}_N$ such that $\varphi_i(y) = \hat{y}_i = f_i$, $i = 1, 2, \ldots, \#(\varphi)$, $y_i = \hat{y}_i$ for $i = \#(\varphi) + 1, \ldots N$, and $\hat{y}^0 = \mathbf{0}$ corresponds to y^0. Then we obtain

$$p = Af, \quad f = Bp, \tag{5.10}$$

[2] Here the radical of the polynomial ideal $< I >$ generated by I is defined as $\sqrt{I} := \{f \in \mathbb{F}[X] : f^l \in < I >$ for some non-negative integer $l\}$.

where $p = \varrho(y)$, for $\hat{y}$ in some neighbourhood $\mathcal{W}$ of $\hat{y}^0 = \mathbf{0}$. Differentiation of (5.10) with respect to the new coordinates $\hat{y}_i$ gives

$$\mathrm{Jac}(p) = A\ \mathrm{Jac}(f) + (\partial A)f, \quad \mathrm{Jac}(f) = B\ \mathrm{Jac}(p) + (\partial B)p, \tag{5.11}$$

where ∂A is a $\#(p) \times N \times \#(f)$ dimensional matrix whose entries are derivatives of entries of A. Similarly ∂B is a $\#(f) \times N \times \#(p)$ analytic matrix. Now

$$\mathrm{rank}(\mathrm{Jac}(p)) \geq \mathrm{rank}(B\ \mathrm{Jac}(p)) \overset{(5.11)}{=} \mathrm{rank}(BA\ \mathrm{Jac}(f) + B(\partial A)f). \tag{5.12}$$

From (5.10), $f = Bp = BAf$, and differentiation of this relation with respect to the $\hat{y}_i$ gives

$$BA\ \mathrm{Jac}(f) = \mathrm{Jac}(f) - \partial(BA)f. \tag{5.13}$$

So from (5.12),

$$\mathrm{rank}(\mathrm{Jac}(p)) \geq \mathrm{rank}(\mathrm{Jac}(f) + [B(\partial A) - \partial(BA)]f). \tag{5.14}$$

Since $f_i = \hat{y}_i$, and A and B are analytic in $\hat{y}$, the entries in the matrix $[B(\partial A) - \partial(BA)]f$ can be made as uniformly small as desired by making $\hat{y}$ sufficiently small. In addition $\mathrm{Jac}(f)$ is the $\#(f) \times N$ matrix whose (i,j)-th entry is δ_{ij}. So for sufficiently small values of $\hat{y}$ we have

$$\mathrm{rank}(\mathrm{Jac}(p)) \geq \mathrm{rank}(\mathrm{Jac}(f)) = \#(f) = \#(\varphi). \tag{5.15}$$

Thus $\mathrm{rank}(\mathrm{Jac}(\varrho(y))) = \#(\varphi) =$ constant in some sufficiently small neighbourhood $\mathcal{O}$ of y^0. Using the fact that $\mathrm{rank}(\widetilde{\mathrm{Jac}}(\varphi(y))) = \mathrm{rank}(\mathrm{Jac}(\varphi(y)))$ and repeating the above argument with Jac replaced by $\widetilde{\mathrm{Jac}}$ shows that $\mathrm{rank}(\widetilde{\mathrm{Jac}}(\varrho(y))) = \mathrm{rank}(\mathrm{Jac}(\varrho(y))) = \#(\varphi)$ in $\mathcal{O}$. □

6. Application to nonclassical analysis of cubic nonlinear Schrödinger equations

In this section we apply the *gröbner_rif* algorithm to a nonlinear system with 856 *pde*s, in 8 independent and 8 dependent variables. The resulting system was simple enough for its nontrivial explicit solution to be obtained. This is part of continuing work with P. Clarkson and E. Mansfield.

In particular we apply the *gröbner_rif* algorithm to the nonlinear determining equations which arise when the Nonclassical Method [3, 16] is applied to the coupled nonlinear Schrödinger system

$$i\Psi_t + \nabla^2\Psi + (|\Psi|^2 + |\Phi|^2)\Psi = 0, \tag{6.1a}$$
$$i\Phi_t + \nabla^2\Phi + (|\Psi|^2 + |\Phi|^2)\Phi = 0, \tag{6.1b}$$

where Ψ and Φ are complex-valued functions of x, y, z, t. Setting

$$\Psi = p + iq, \quad \Phi = r + is, \tag{6.2}$$

(6.1) is equivalent to four coupled *pde*s in the real-valued functions p, q, r, s.

The Nonclassical Method, which is a generalization of Lie's method for determining symmetries of differential equations, involves setting up a system of (nonclassical) determining equations for components of a vector field. These determining equations are in general nonlinear, in contrast to the linear homogeneous determining equations of Lie's method. For the application of the Nonclassical Method to (6.1) the vector field has the form

$$L = v_1\partial_x + v_2\partial_y + v_3\partial_z + v_4\partial_t + v_5\partial_p + v_6\partial_q + v_7\partial_r + v_8\partial_s, \tag{6.3}$$

where its components $v_1, v_2, v_3, \ldots, v_8$ are functions of x, y, z, t, p, q, r, s. If f is any analytic function of these variables and L is a nonclassical vector field of (6.1) then fL is also a nonclassical vector field of (6.1) [3]. Thus we can split the search for nonclassical vector fields into two cases: $v_4 = 1$ and $v_4 = 0$. We analyse the case $v_4 = 1$ in this paper.

The nonclassical determining equations for the case $v_4 = 1$ were automatically generated using the Maple program [27]. The resulting nonlinear system consisted of 856 *pdes* and was automatically simplified by the *gröbner_rif* algorithm to a system of 120 *pdes*. This system, which was not yet in *rif*-form, contained a simple subsystem of single term equations, with all second order partial derivatives of v_5, v_6, v_7, v_8 with respect to x, y, z, t, equal to zero. Thus by a simple integration

$$\begin{aligned} v_5 &= pv_9 + qv_{10} + rv_{11} + sv_{12} + v_{25} \\ v_6 &= pv_{13} + qv_{14} + rv_{15} + sv_{16} + v_{26} \\ v_7 &= pv_{17} + qv_{18} + rv_{19} + sv_{20} + v_{27} \\ v_8 &= pv_{21} + qv_{22} + rv_{23} + sv_{24} + v_{28} \end{aligned}$$

where $v_j = v_j(x, y, z, t)$, $j = 9, 10, 11, \ldots, 28$. After substituting the above equations into the system, it contained over a hundred equations. Application of the *gröbner_rif* algorithm to the system resulting from the substitution above gave only one case which led to a truly nonclassical vector field (i.e. only one case that was not equivalent to a classical Lie symmetry vector field). In that case the *rif*-form[3] of the determining system was:

$$\begin{aligned}
&\partial_{tt}v_1 = 4v_{24}\partial_t v_1, \;\; \partial_{tt}v_2 = 4v_{24}\partial_t v_2, \;\; \partial_{tt}v_3 = 4v_{24}\partial_t v_3, \\
&\partial_x v_1 = -v_{24}, \;\; \partial_x v_2 = 0, \;\; \partial_x v_3 = 0, \;\; \partial_x v_{21} = -v_1 v_{22}, \\
&\partial_x v_{22} = v_1 v_{21}, \;\; \partial_x v_{23} = \tfrac{1}{2}\partial_t v_1 - v_1 v_{24}, \;\; \partial_x v_{24} = 0, \\
&\partial_y v_1 = 0, \;\; \partial_y v_2 = -v_{24}, \;\; \partial_y v_3 = 0, \;\; \partial_y v_{21} = -v_2 v_{22}, \;\; \partial_y v_{22} = v_2 v_{21}, \\
&\partial_y v_{23} = \tfrac{1}{2}\partial_t v_2 - v_2 v_{24}, \;\; \partial_y v_{24} = 0, \\
&\partial_z v_1 = 0, \;\; \partial_z v_2 = 0, \;\; \partial_z v_3 = -v_{24}, \;\; \partial_z v_{21} = -v_3 v_{22}, \\
&\partial_z v_{22} = v_3 v_{21}, \;\; \partial_z v_{23} = \tfrac{1}{2}\partial_t v_3 - v_3 v_{24}, \;\; \partial_z v_{24} = 0, \\
&\partial_t v_{21} = (v_1^2 + v_2^2 + v_3^2 - 2v_{23})v_{22} + 3v_{21}v_{24}, \\
&\partial_t v_{22} = 3v_{22}v_{24} + (2v_{23} - v_1^2 - v_2^2 - v_3^2)v_{21}, \\
&\partial_t v_{23} = 2v_{23}v_{24}, \;\; \partial_t v_{24} = 2v_{24}^2, \\
&\partial_p v_1 = 0, \;\; \partial_p v_2 = 0, \;\; \partial_p v_3 = 0, \;\; \partial_p v_{21} = 0, \;\; \partial_p v_{22} = 0, \;\; \partial_p v_{23} = 0, \;\; \partial_p v_{24} = 0, \\
&\partial_q v_1 = 0, \;\; \partial_q v_2 = 0, \;\; \partial_q v_3 = 0, \;\; \partial_q v_{21} = 0, \;\; \partial_q v_{22} = 0, \;\; \partial_q v_{23} = 0, \;\; \partial_q v_{24} = 0, \\
&\partial_r v_1 = 0, \;\; \partial_r v_2 = 0, \;\; \partial_r v_3 = 0, \;\; \partial_r v_{21} = 0, \;\; \partial_r v_{22} = 0, \;\; \partial_r v_{23} = 0, \;\; \partial_r v_{24} = 0, \\
&\partial_s v_1 = 0, \;\; \partial_s v_2 = 0, \;\; \partial_s v_3 = 0, \;\; \partial_s v_{21} = 0, \;\; \partial_s v_{22} = 0, \;\; \partial_s v_{23} = 0, \;\; \partial_s v_{24} = 0, \\
&v_4 = 1, \;\; v_9 = v_{24}, \;\; v_{10} = -v_{23}, \;\; v_{11} = v_{22}, \;\; v_{12} = -v_{21}, \;\; v_{13} = v_{23}, \;\; v_{14} = v_{24}, \\
&v_{15} = -v_{21}, \;\; v_{16} = -v_{22}, \;\; v_{17} = -v_{22}, \;\; v_{18} = v_{21}, \;\; v_{19} = v_{24}, \;\; v_{20} = -v_{23}, \\
&v_{25} = 0, \;\; v_{26} = 0, \;\; v_{27} = 0, \;\; v_{28} = 0
\end{aligned}$$

and the single leading nonlinear equation

$$v_{21}^2 + v_{22}^2 = 0$$

together with the pivot conditions

$$\Lambda = (v_{21} \neq 0, v_{24} \neq 0).$$

The ranking used in the reduction was graded first by total order of derivative, then lexicographically by derivative ($\partial_x \succ \partial_y \succ \partial_z \succ \partial_t \succ \partial_p \succ \partial_q \succ \partial_r \succ \partial_s$) and finally lexicographically on dependent variable ($v_1 \succ v_2 \succ v_3 \succ \cdots \succ v_{27} \succ v_{28}$).

[3] Application of the theorem in Appendix B shows that the prolongation to order 2 of this system is involutive.

Further use of the *rif*-form and exploitation of non-orderly lexicographic rankings to triangularize the system and isolate *ode*s [46] led to its exact solution:

$$
\begin{aligned}
v_1 &= (2b_1t - x - ib_6)/(b_5 - 2t), \\
v_2 &= (2b_2t - y - ib_7)/(b_5 - 2t), \\
v_3 &= (2b_3t - z - ib_8)/(b_5 - 2t), \\
v_4 &= 1, \\
v_5 &= \left(p - q(b_1x + b_2y + b_3z + b_4) + \ (r - is)(b_5 - 2t)\ \mathrm{e}^{iH}\right)/(b_5 - 2t), \\
v_6 &= \left(q + p(b_1x + b_2y + b_3z + b_4) - i(r - is)(b_5 - 2t)\ \mathrm{e}^{iH}\right)/(b_5 - 2t), \\
v_7 &= \left(r - s(b_1x + b_2y + b_3z + b_4) - \ (p - iq)(b_5 - 2t)\ \mathrm{e}^{iH}\right)/(b_5 - 2t), \\
v_8 &= \left(s + r(b_1x + b_2y + b_3z + b_4) + i(p - iq)(b_5 - 2t)\ \mathrm{e}^{iH}\right)/(b_5 - 2t),
\end{aligned}
$$

where

$$
\begin{aligned}
H &= \frac{xv_1 + yv_2 + zv_3 + \frac{1}{2}(x^2 + y^2 + z^2) - iG}{b_5 - 2t}, \\
G &= (b_5 - 2t)\left[A + \frac{B}{(b_5 - 2t)} - C\log(b_5 - 2t)\right], \\
A &= b_9 - i(b_1^2 + b_2^2 + b_3^2)t, \\
B &= \tfrac{1}{2}i\left((b_6 + ib_1b_5)^2 + (b_7 + ib_2b_5)^2 + (b_8 + ib_3b_5)^2\right), \\
C &= ib_4 + \tfrac{3}{2} + b_1(b_6 + ib_1b_5) + b_2(b_7 + ib_2b_5) + b_3(b_8 + ib_3b_5),
\end{aligned}
$$

and $B, C, b_1, b_2, ..., b_9$ are complex constants. We present the above solutions in a more suggestive form by multiplying the vector field by $(b_5 - 2t)$ and relabelling some of the constants. In addition we use the vector notation $\boldsymbol{r} = (x, y, z)$, $\boldsymbol{\nabla} = (\partial_x, \partial_y, \partial_z)$ and introduce the complex operators $\partial_\Psi, \partial_{\Psi^*}, \partial_\Phi, \partial_{\Phi^*}$ (where Ψ^*, Φ^* are the complex conjugates of Ψ, Φ respectively). In this notation the vector field is

$$
\begin{aligned}
L = \ & (2\boldsymbol{a}t - \boldsymbol{r} + \boldsymbol{b}) \cdot \boldsymbol{\nabla} + 2(\tau - t)\partial_t \\
& + (1 + i\boldsymbol{a} \cdot \boldsymbol{r} + ik)\ (\Psi\ \partial_\Psi\ + \Phi\ \partial_\Phi\ ) \\
& + (1 - i\boldsymbol{a} \cdot \boldsymbol{r} - ik)\ (\Psi^*\partial_{\Psi^*} + \Phi^*\partial_{\Phi^*}) \\
& \quad + 4(\tau - t)\ \mathrm{e}^{iH}\ (\Phi^*\partial_\Psi\ - \Psi^*\partial_\Phi\ )
\end{aligned}
$$

where

$$
\begin{aligned}
H &= \frac{\boldsymbol{r} \cdot \boldsymbol{v} + \frac{1}{2}\boldsymbol{v} \cdot \boldsymbol{v} - iG}{2(\tau - t)}, \\
\boldsymbol{v} &= 2\boldsymbol{a}t - \boldsymbol{r} + \boldsymbol{b}, \\
G &= 2(\tau - t)\left[A + \frac{B}{2(\tau - t)} - C\log 2(\tau - t)\right], \\
A &= l - i\boldsymbol{a} \cdot \boldsymbol{a}t, \\
B &= \tfrac{1}{2}i(2\tau\boldsymbol{a} + \boldsymbol{b}) \cdot (2\tau\boldsymbol{a} + \boldsymbol{b}), \\
C &= \tfrac{3}{2} + i(k + \boldsymbol{a} \cdot \boldsymbol{b} + 2\tau\boldsymbol{a} \cdot \boldsymbol{a}).
\end{aligned}
$$

Here k, l, τ are complex constants, $\boldsymbol{a}, \boldsymbol{b}$ are constant vectors in $\mathbf{C}^3$ and $\boldsymbol{a} \cdot \boldsymbol{b} := \Sigma a_j b_j$.

7. Discussion

The *rif* algorithm described in this paper transforms analytic systems of nonlinear *pde*s to what we call reduced involutive form. The purpose of this paper is to present the algorithm and give a nontrivial example of its application. In particular we applied a preliminary computer algebra implementation of our *gröbner_rif* algorithm to the determination of nonclassical vector fields of the coupled cubic nonlinear Schrödinger system in 3+1 dimensions. Very few such calculations have been carried out in 3+1 dimensions. The size, nonlinearity and complexity of the calculation, we believe, provide a reasonable initial test of the feasibility of our approach. A nontrivial result was obtained, and the resulting vector field was written in a form which suggests its generalization to $(N+1)$-dimensional coupled nonlinear Schrödinger systems.

We were motivated to develop an algorithm that would give computational access to formal features of nonlinear *pde* systems, and be efficient enough to be applied to systems of interest. Our approach here is to present an algorithm which is fairly general (at the cost of loss of effectiveness), and then to create effective versions of this algorithm for various classes of *pde*s.

A *pde* arising in the application of the *rif* algorithm is regarded as geometrically new if its jet bundle locus does not contain the jet bundle locus of the current system. However there is no effective algorithm for testing this condition for general analytic systems of *pde*. The realization of the *rif* algorithm given in this paper for polynomially nonlinear *pde* systems uses the method of algebraic (not differential) Gröbner bases for testing whether a *pde* is algebraically new relative to a given *pde* system. This leads to an effective realization of the above geometric condition for polynomial systems of *pde*.

Geometrical properties of systems of *pde* correspond to properties of the jet bundle locus of these systems. But from an algorithmic and computational viewpoint we have access only to the equations defining the locus and not to the locus itself. Conditions are needed to ensure that operations on the equations of the system (e.g. differentiations) correspond to geometrical operations on their loci. In this paper the constant rank condition plays that rôle. In the realization of the *rif* algorithm for polynomially nonlinear systems the step of taking the radical of a polynomial ideal effectively realizes the constant rank condition. To facilitate the attainment of the constant rank condition, algorithms for including integrability conditions in polynomially nonlinear *pde* systems should remove repeated factors in the *pde*s whenever possible.

After a change of coordinates, a system which is in *rif*-form will usually no longer be in *rif*-form. However as indicated in Appendix B the *rif*-form of a system can be simply prolonged (differentiated) to obtain the system in involutive form. The property of involutivity of a system is a geometric one which is preserved under changes of coordinates. Thus we gain computational access to valuable geometrical features of involutive systems of *pde*: their symbol, Cartan characters and other geometric features. For example a system of *pde*s which is involutive is also locally solvable [42]. Local solvability guarantees that the properties of the jet bundle locus of a *pde* system correspond to the properties of the solutions of the system. It would be interesting to develop methods for determining the symbol and Cartan characters of a system directly from its *rif*-form rather than its prolongation.

Practical implementation of the *rif* algorithm requires extensive use of update algorithms to avoid repeated calculations, and divide and conquer strategies such as those used in the *standard_form* algorithm [48, §8] or those in [67]. The split between leading linear and leading nonlinear *pde*s used by the *rif* algorithm is a natural one since the prolongation of any analytic system of *pde*s will always contain many equations that are linear in their leading derivatives. The *rif* algorithm takes advantage of this linearity by copiously applying the Gauss algorithm with the aim of minimising the size and order of the leading nonlinear *pde*s arising during its application. We adopted this approach in view of the high complexity of algorithms for simplifying nonlinear systems [6] versus the much greater efficiency of the Gauss algorithm.

The *rif* algorithm has a redundancy criterion for integrability conditions. For systems in one dependent variable, with N leading linear *pde*s and two independent variables, our criterion means only $N-1$ integrability conditions have to be considered, versus $N(N-1)/2$ conditions if all pairs of *pde*s are considered. The example of §4 illustrates that, even if all the integrability conditions of a nonlinear *pde* system resulting from equating cross partials across pairs of equations are satisfied modulo the system, there may be other integrability conditions of the system which are not satisfied. These other integrability conditions were uncovered through the use of the constant rank procedure.

Pivoting and the resulting proliferation of cases is an inherent and costly part of our algorithm. This phenomenon occurs in algorithms which use pseudo-reduction (such as the *rif* algorithm and that of Mansfield [37]). On the other hand algorithms which use reduction, such as Wittkopf's *diff_reduce* algorithm, have the advantage that they do not case-split, but also the disadvantage that they may not terminate. The Cartan-Kuranishi Theorem guarantees that, after a finite number of prolongations and projections, an analytic system satisfying the constant rank condition at each iteration will become involutive on a subset of its locus. Replacing the Gröbner basis step of *diff_reduce* with a step of constructing the Gröbner basis of the radical of a polynomial system ensures that the constant rank condition, necessary for the application of the Cartan-Kuranishi Theorem, is satisfied at Euclidean points of the system's jet bundle locus. This modification alone is not sufficient to always force termination of *diff_reduce*, but the Cartan-Kuranishi Theorem suggests the possibility of using the additional condition of involutivity as a termination criterion.

Rankings which are not orderly (i.e. not graded first by total order of derivative) can also be used by the *rif* algorithm. Such non-orderly rankings can lead to natural physical initial value problems in contrast to the initial value problems yielded by orderly rankings. For example, the heat equation for $u(x,t)$ in *rif*-form with respect to a non-orderly ranking is $u_t = u_{xx}$ with the associated physical initial condition $u(x,0) = f(x)$. In any orderly ranking it is $u_{xx} = u_t$ with corresponding non-physical initial conditions $u(0,t) = g(t), u_x(0,t) = h(t)$. Non-orderly rankings of lexicographic type can be of use in the explicit integration of *pde* systems [19, 38, 46]. Indeed the explicit integration of the system given in §6 was achieved by the use of such rankings. It would be interesting to develop existence-uniqueness theorems for, and geometric properties of, systems in *rif*-form with respect to such non-orderly rankings.

Acknowledgements

GJR gratefully acknowledges the assistance of a Natural Sciences and Engineering Research Council grant from the government of Canada. We acknowledge the support of N. Kamran and P. Winternitz during a visit to the C.R.M. at the Université de Montréal and the assistance of J. B. Carrell, P. Clarkson, C. Connell, S. R. Czapor, E. Mansfield, K. Y. Lam, I. G. Lisle, W. M. Seiler and B. M. Trager. We thank Diana Kwan for the diagrams.

Appendix A: Pseudo-code for procedure orthsimp

We provide simple pseudo-code for the procedure *orthsimp* whose properties were outlined in §2. The reader is warned that this is not meant to be a model for practical implementation of *orthsimp*.

input S, Λ
$S_{sol} \leftarrow solved(S), \quad S_{unsol} \leftarrow unsolved(S)$
repeat {until no unsolved leading linear *pde*s remain}
  while $L_{\succ}(S_{unsol}) \neq \emptyset$ **do**
    Pick $eq \in L_{\succ}(S_{unsol})$ and remove eq from S_{unsol}.
    Solve eq for its leading derivative, and update Λ.
    Transfer from S_{sol} to S_{unsol} all equations whose leading derivative is a derivative of the leading derivative of eq.
    $S_{sol} \leftarrow S_{sol}|_{\rho^\infty(\{eq\})}$
    $S_{unsol} \leftarrow S_{unsol}|_{\rho^\infty(\{eq\})}$
    $S_{sol} \leftarrow S_{sol} \cup \{eq\}$
  endloop
  $S_{unsol} \leftarrow simplify(S_{unsol})$
until $L_{\succ}(S_{unsol}) = \emptyset$
$S \leftarrow S_{sol} \cup S_{unsol}$
output S, Λ

Suppose that eq has form $u^k_a = f$ after it has been placed in solved-form. Note that the steps of form $A \leftarrow A \mid_{\rho^\infty(\{eq\})}$ in *orthsimp* can be executed in the following way. First we let q be the maximum order of derivative in A according to our chosen ranking $\succ$. Then it will only be necessary to calculate $\rho^q(\{eq\}) = \{u^k_{a+c} = D^c f : 0 \le \operatorname{ord}(c) \le q - \operatorname{ord}(a)\}$. A finite number of substitutions from $\rho^q(\{eq\})$ eliminates from A all derivatives of the leading derivative of eq.

Whenever a leading linear equation eq is solved for its leading derivative the pivot list Λ is updated by appending the leading coefficient of eq to the pivot list. A list of pivots $\hat{\Lambda}$ which has been simplified subject to the system can be produced by including $\hat{\Lambda} \leftarrow \Lambda|_{\rho_{\text{lex}}(S)}$ as the penultimate step of *orthsimp* and this can speed termination of the *rif* algorithm. In particular if any of the simplified pivots vanish then the algorithm should terminate. When applying the *rif* algorithm a binary tree of cases generally arises with nodes labelled by decisions based on whether a pivot is zero or nonzero. We warn the reader that care should be taken if the simplified pivot list is used in this tree since the information used to simplify a pivot may have originated from further down the tree.

The *rif* algorithm will still terminate if the simplification step given by $S_{unsol} \leftarrow simplify(S_{unsol})$ is omitted. Any function which does not alter the locus of the *pde* system and preserves the constant rank condition can be used for the function *simplify*. The choice of this function can be tailored to the class of *pde*s to which the *rif* algorithm is being applied. For example for polynomially nonlinear *pde* systems Buchberger's Gröbner basis function can be used (see §5).

Proof of termination of orthsimp

Suppose that *orthsimp* does not terminate. The steps of form $A \leftarrow A|_{\rho^\infty(\{eq\})}$ can be accomplished using a finite number of substitutions so the only way for *orthsimp* not to terminate is for the step of picking eq in $L_\succ(S_{unsol})$ to be repeated an infinite number of times. Suppose that this step has been executed j times. After each of these steps, the step $S_{unsol} \leftarrow S_{unsol}|_{\rho^\infty(\{eq\})}$ eliminates all derivatives of the leading derivative of eq from S_{unsol}. So the leading derivative of the equation eq chosen in the $(j+1)$-th iteration will not be a derivative of the leading derivative of any eq chosen in a previous iteration. Thus we obtain an infinite sequence of leading derivatives, where each derivative is not a derivative of those that precede it in the sequence. Associating each of these leading derivatives u^k_q with a corresponding monomial $(u^k)(x_1)^{q_1}(x_2)^{q_2}...(x_n)^{q_n}$ as in [61] gives an infinite sequence of distinct monomials, where each monomial is not a multiple of any monomial that precedes it. But this is in contradiction to Theorem 1 of [61] (also see [64, 29]) which states that any such sequence can only have a finite number of distinct monomials. $\square$

Appendix B: Prolongation of rif-form to involutive form

Let S be an admissible system of *pde*s of order r on $J^r(\mathbb{F}^n, \mathbb{F}^m)$ which is in *rif*-form with respect to an orderly ranking $\succ$ in a neighbourhood $\mathcal{O}$ of $(x^0, u^0_c) \in$ locus(S).

For a non-negative integer M, let $\mathcal{P}_{M-1}$ denote the set of derivatives of dependent variables of S of order less than or equal to $M-1$ which are not derivatives of any leading derivative of $L_\succ(S)$. Define

$$Cartan\_form(S, M) = \{dv - \sum_{i=1}^{n} D_{x_i} v \mid_{\rho_{\text{lex}}(S)} dx_i : v \in \mathcal{P}_{M-1}\} \cup N_\succ(S).$$

Then for each non-negative integer M, *Cartan_form*(S, M) is an exterior differential system associated with S which is a union of the set $N_\succ(S)$ (regarded as a set of zero forms) with a set of 1-forms. Here we regard the 0-th order jet bundle as having coordinates $x_1, \ldots, x_n$, $\mathcal{P}_M$. That is, the x_i are regarded as independent variables and the variables in $\mathcal{P}_M$ are regarded as dependent variables. When $M \ge r$ this system has the same solutions as S.

The theorem below, whose proof will be given elsewhere, indicates that this exterior system is involutive for appropriately chosen M and is an analogue of the result [37] that Mansfield obtained for polynomially nonlinear *pde*s systems in the form of a Differential Gröbner basis (also see [10]).

Theorem (Prolongation to involutive form)
Let S be an admissible system of pdes *of order r on $J^r(\mathbb{F}^n, \mathbb{F}^m)$ which is in rif-form with respect to an orderly ranking $\succ$ in a neighbourhood $\mathcal{O}$ of $(x^0, u_c^0) \in$ locus(S). Let $A = \{\mathrm{ord}(c) - 1 \colon c \in \mathcal{C}\}$, $M = \max A \cup \{r\}$, where $\mathcal{C}$ is the set of all points c in $\mathbf{W}^n$ which arise in Step 6.3.2 (iii) of the calculation of integ_condns$(L_{\succ}(S))$ using Algorithm 6 of [43, p. 316]. Then:*
(a) *Cartan_form(S, M) is involutive according to Cartan's criteria.*
(b) *The prolongation of S to order M is involutive.*

The system of exterior forms *Cartan_form(S, M)* can be written as

$$\begin{aligned} \omega^i &= dv^i - f_a^i(x, v, w) dx^a, \\ \theta^\lambda &= g^\lambda(x, v), \end{aligned}$$

where the dependent variables v belong to $\mathcal{P}_{M-1}$ and the dependent variables w belong to $\mathcal{P}_M - \mathcal{P}_{M-1}$. Condition (3) of *rif*-form implies that g is independent of w and that the relations $G_c^\lambda = D_{x^c} g^\lambda = \partial_{x^c} g^\lambda + f_c^j \partial_{v^j} g^\lambda = 0$ are a consequence of $g = 0$. Conditions (1) and (2) imply that any relation $\psi(x, u, v) = 0$ derived by elimination of the derivatives $w_{x^c}^p$ in the relations $H_{a,b}^i = D_{x^b} f_a^i - D_{x^a} f_b^i = \partial_{x^b} f_a^i + f_b^j \partial_{v^j} f_a^i + w_{x^b}^p \partial_{w^p} f_a^i - \partial_{x^a} f_b^i - f_a^j \partial_{v^j} f_b^i - w_{x^a}^p \partial_{w^p} f_b^i = 0$ is a consequence of $g = 0$.
Example
Consider the pair of *pdes* S for $u = u(x, y)$:

$$u_{(r,0)} = 0, u_{(0,r)} = 0.$$

(e.g. $u_{xx} = 0, u_{yy} = 0$ for $r = 2$). Here r is a nonzero positive integer.

The system is of order r and is already in *rif*-form with respect to an orderly ranking. Here the set of leading derivatives is $\{u_{(r,0)}, u_{(0,r)}\}$. The set of intersection points corresponding to the system's integrability conditions is $\mathcal{C} = \{(r, r)\}$ so $A = \{2r - 1\}$ and $M = \max\{2r - 1, r\} = 2r - 1$ (see Figure 2). From the theorem above the prolongation to order $M = 2r - 1$ of the system is involutive and consists of the $r^2 + r$ equations

$$u_{(r+k,p-k)} = 0, \quad u_{(p-k,r+k)} = 0, \quad k = 0, \ldots, p, \; p = 0, \ldots, r-1.$$

We now express S in terms of the involutive exterior system $Cartan\_form(S, M)$. Here $\mathcal{P}_{M-1} = \mathcal{P}_{2r-2} = \{u_{(i,j)} \; : \; 0 \le i, j \le r-1\} = \mathcal{P}_M$. In this case there are no zero forms and $Cartan\_form(S, M)$ consists of the r^2 one forms

$$\begin{aligned} & du_{(k,l)} - u_{(k+1,l)} dx - u_{(k,l+1)} dy, \\ & du_{(k,r-1)} - u_{(k+1,r-1)} dx, \\ & du_{(r-1,l)} - u_{(r-1,l+1)} dy, \\ & du_{(r-1,r-1)}, \end{aligned}$$

where $0 \le k, l < r - 2$ (also see [5, Example 1.2, p. 241–243] where the methods of Cartan are also used to bring the system of our example to involutive form).

Although a system S can be brought to *rif*-form and then prolonged to involutive form, there may be other equivalent involutive forms of S. There may even be an equivalent involutive system which is of lower order than the *rif*-form of S (e.g. see [58]).

Appendix C: Geometric involutive form algorithm

We give pseudo-code for a geometric involutive form algorithm. This is essentially the algorithm presented in [52] but with the addition of the constant rank step.
input: A q-th order admissible system S on $\mathcal{B} \subseteq J^q(\mathbb{F}^n, \mathbb{F}^m)$ and a finite list of pivots of the form $\Lambda = (\lambda^1(x, u_c) \neq 0, \lambda^2(x, u_c) \neq 0, \ldots)$.

$S \leftarrow \rho^q(S)$
repeat
  repeat
    while Symbol(S) not involutive **do**
      $(S, \Lambda) \leftarrow (\rho^{q+1}(S), \Lambda)$
      $q \quad \leftarrow q + 1$
    endwhile
    $S_{\text{temp}} \leftarrow S$
    $(S, \Lambda) \leftarrow (\pi_q^{q+1}(\rho^{q+1}(S)), \Lambda)$
  until $S = S_{\text{temp}}$
  $(S, \mathcal{B}) \leftarrow$ *constant_rank*$(S, \mathcal{B})$
until $S = S_{\text{temp}}$ or (S, Λ) inconsistent
output $S, \Lambda, \mathcal{B}$

To test the involutivity of the symbol we can use the methods in [5] or [42, 52]. In the case where S is a system of polynomially nonlinear *pde*s, the projection π_q^{q+1} is carried out using the fraction-free elimination of §5 to eliminate the linearly appearing $(q+1)$-th order derivatives and obtain a projected system of order q. To avoid excessive case-splitting this elimination could also be carried out using an elimination ideal based on ranking the $(q+1)$-th order derivatives higher than the q-th order derivatives in a lexicographic ranking. To test whether the projected q-th order system is an *algebraic* consequence of S we use the Gröbner basis function *normal_form*$_{\succ}$ as was the case with *gröbner_rif*. To execute the *constant_rank* step we replace *constant_rank*$(S, \mathcal{B})$ by *gbasis*$_{\succ}\sqrt{S}$.

References

[1] Araĭs, E.A., V.P. Šapeev and N.N. Janenko. 1974. *Realization of Cartan's method of exterior differential forms on an electronic computer*, Sov. Math. Dokl. **15**(1) 203–205.

[2] Auslander, L. and R. E. MacKenzie. 1963. Introduction to Differentiable Manifolds. (McGraw–Hill, New York).

[3] Bluman, G.W. and J.D. Cole. 1969. *The general similarity solution of the heat equation*, J. Math. Mech. **18** 1025–1042.

[4] Bluman, G.W. and S. Kumei. 1989. Symmetries and Differential Equations. (Springer Verlag, New York).

[5] Bryant, R.L., S.S. Chern, R.B. Gardner, H.L. Goldschmidt and P.A. Griffiths. 1991. Exterior Differential Systems, Mathematical Sciences Research Institute Publications **18** (Springer Verlag, New York).

[6] Becker, T., V. Weispfenning. 1993. Gröbner bases: a computational approach to commutative algebra. (Springer Verlag, New York)

[7] Bocharov, A.V. and M.L. Bronstein. 1989. *Efficiently implementing two methods of the geometrical theory of differential equations: an experience in algorithm and software design*, Acta Appl. Math. **16** 143–166.

[8] Buchberger, B. 1965. An Algorithm for Finding a Basis for the Residue Class Ring of a Zero-Dimensional Polynomial Ideal (German), PhD. Thesis, Univ. of Innsbruck, Math. Inst.

[9] Carrà-Ferro, G. 1987. *Gröbner Bases and Differential Algebra,* Lecture Notes in Comp. Sci. **356** 128–140.

[10] Carrà-Ferro, G. and S.V. Duzhin. 1993. Differential-algebraic and differential-geometric approach to the study of involutive symbols. In *Modern Group Analysis: Advanced Analytical and Computational Methods in Mathematical Physics*. Edited by N.H. Ibragimov, M. Torrisi and A. Valenti. 93–99. (Kluwer, Dordrecht).

[11] Carrà Ferro, G. and W. Y. Sit. 1993. *On Term-Orderings and Rankings*, Lecture Notes in Pure and Applied Math., Computational Algebra, Dekker, **151**.

[12] Carminati, J. and R.G. McLenaghan. 1987. *An explicit determination of the space-times on which the conformally invariant scalar wave equation satisfies Huygens' principle. — Part II: Petrov type D space-times*. Ann. Inst. Henri Poincaré. **47** (4) 337–354.

[13] Cartan, É.J. 1904. Sur la structure des groupes infinis de transformations. *Oeuvres Complètes* Part II, Vol. 2, 571–714. (Gauthier-Villars, Paris).

[14] Cartan, É.J. 1937. Les problèmes d'équivalence. *Oeuvres Complètes* Part II, Vol. 2, 1311–1334. (Gauthier-Villars, Paris).

[15] Cartan E. 1946. *Les Systèmes Différentiels Extérieur et leurs Applications Géometrique* (Hermann, Paris).

[16] Clarkson, P.A. and M.D. Kruskal. 1989. *New similarity solutions of the Boussinesq equation*, J. Math. Phys. **30** 2201–2213.

[17] Clarkson, P.A. and E.L. Mansfield. 1993. *Algorithms for Nonclassical Symmetries*, Preprint, Department of Mathematics. (University of Exeter, Exeter, United Kingdom).

[18] Clarkson, P.A. and E.L. Mansfield. 1994. *On a shallow water wave equation*, Nonlinearity. **7** 975–1000.

[19] Czichowski, G. and M. Thiede. 1992 *Gröbner Bases, Standard Forms of Differential Equations and Symmetry Computation*, Seminar Sophus Lie Darmstadt-Erlangen-Greifswald-Leipzig.

[20] Ganža, V.G., S.V. Meleško, F.A. Murzin, V.P. Šapeev and N.N. Janenko. 1981. *Realization on a computer of an algorithm for studying the consistency of systems of partial differential equations*, Sov. Math. Dokl. **24**(1) 638–640.

[21] Geddes, K. O., S. R. Czapor, and G. Labahn. 1992. Algorithms for computer algebra. (Kluwer Academic Publishers).

[22] Gianni, P., B. Trager and G. Zacharias. 1989. Gröbner Bases and Primary Decomposition of Polynomial Ideals, In *Computational Aspects of Commutative Algebra*. Edited by L. Robbiano. 15–33. (Academic Press, New York).

[23] Goldschmidt, H. 1967. *Integrability Criteria for Systems of Partial Differential Equations*, J. Diff. Geom. **1** 269–307.

[24] Gunning, R.C. and H. Rossi. 1965. Analytic Functions of Several Complex Variables. (Prentice–Hall, London).

[25] Hartley, D. and R. W. Tucker. 1991. *A Constructive Implementation of the Cartan-Kähler Theory of Exterior Differential Systems*, J. Symb. Comp. **12** 655–667.

[26] Hereman, W. 1994. *Review of symbolic software for the computation of Lie symmetries of differential equations.* Euromath Bull., **1** 45–79.

[27] Hickman, M. 1993. *The Use of Maple in the Search for Symmetries*, Research Report no. 77, Department of Mathematics (University of Canterbury, Christchurch, New Zealand).

[28] Hudson, A. 1987. *Symbolic Computation of Involutivity of PDES*, Masters Thesis, University of Sydney.

[29] Janet, M. 1920. *Sur les systèmes d'équations aux dérivées partielles*, J. de Math **3** 65–151.

[30] Kähler, E. 1934. Einführung in die Theorie der Systeme von Differentialgleichungen. (B. G. Teubner, Leipzig).

[31] Kolchin, E.R. 1973. Differential Algebra and Algebraic Groups. (Academic Press, New York).

[32] Kovalevskaya, S. 1875. *Zur Theorie der Partiellen Differentialgleichungen*, J. Reine Agnew. Math. **80** 1–32.

[33] Kramer, D., H. Stephani, H. MacCallum, and E. Herlt. 1980. *Exact solutions of Einstein's field equations*, Deutscher Verl. d. Wiss. (Berlin).

[34] Kuranishi, M. 1957. *On E. Cartan's prolongation theorem of exterior differential systems*, Amer. J. Math., **79** 1–47.

[35] Lewy, H. 1957. *An example of a smooth linear partial differential equation without solution*, Ann. Math. **66** 155–158.

[36] Mansfield, E. 1991. *Differential Gröbner Bases*, Ph.D Thesis, University of Sydney.

[37] Mansfield, E. 1992. *A simple criterion for involutivity*, Preprint M94/16.

[38] Mansfield, E.L. and E.D. Fackerell. 1993 *Differential Gröbner Bases*, Preprint 92–108 School of Mathematics, Physics, Computer Science, and Electronics (Macquarie University, Sydney, Australia, 1992).

[39] Oaku, T. 1994. *Algorithms for finding the structure of solutions of linear partial differential equations*, In Proc. ISSAC '94.

[40] Ollivier, F. 1991. 1991. *Standard bases of differential ideals*, Lecture Notes in Comp. Sci., **508** 304–321.

[41] Olver, P.J. 1986. Application of Lie groups to differential equations. (Springer Verlag, New York).

[42] Pommaret, J. F. 1978. Systems of Partial Differential Equations and Lie Pseudogroups. (Gordon and Breach science publishers, Inc.)

[43] Reid, G.J. 1991. *Algorithms for reducing a system of PDEs to standard form, determining the dimension of its solution space and calculating its Taylor series solution*, Euro. J. Appl. Math. **2** 293–318.

[44] Reid, G.J. 1991. *Finding abstract Lie symmetry algebras of differential equations without integrating determining equations*, Euro. J. Appl. Math. **2** 319–340.

[45] Reid, G.J., I.G. Lisle, A. Boulton and A. D. Wittkopf. 1992. *Algorithmic determination of commutation relations for Lie symmetry algebras of PDEs*, in: Proc. ISSAC '92, Berkeley, California, Ed.: P.S. Wang (ACM Press, New York, 1992) 63-68.

[46] Reid, G.J., and D.K. McKinnon. 1992. *Solving systems of linear PDEs in their coefficient field by recursively decoupling and solving ODEs*, Preprint, Department of Mathematics, (University of British Columbia, Vancouver, Canada).

[47] Reid, G.J., D.T. Weih and A.D. Wittkopf. 1993. A Point symmetry group of a differential equation which cannot be found using infinitesimal methods. In *Modern Group Analysis: Advanced Analytical and Computational Methods in Mathematical Physics*. Edited by N.H. Ibragimov, M. Torrisi and A. Valenti. 93–99. (Kluwer, Dordrecht).

[48] Reid, G.J. and A. D. Wittkopf. 1993. The long guide to the Standard Form package. Programs and documentation available by anonymous ftp to `math.ubc.ca` in directory `/pub/reid/standardform.`

[49] Riquier, C. 1910. Les Systèmes d'Équations aux Dérivées Partielles. (Gauthier-Villars, Paris).

[50] Ritt, J.F. 1950. *Differential Algebra*, Amer. Math. Soc. Colloq. Publns. **13** (A.M.S., New York).

[51] Rust, C. 1993. *On The Classification of Rankings of Partial Derivatives*, Preprint.

[52] Schü, J., W.M. Seiler, and J. Calmet. 1992. *Algorithmic Methods for Lie Pseudogroups*, In Proc. Modern Group Analysis. Acireale.

[53] Schwarz, F. 1984. *The Riquier-Janet theory and its application to nonlinear evolution equations*, Physica **D11** 243–251.

[54] Schwarz, F. 1985. *Automatically determining symmetries of differential equations*, Computing. **34** 91–106.

[55] Schwarz, F. 1992. *An Algorithm for Determining the Size of Symmetry Groups*, Computing **49** 95–115.

[56] Schwarz, F. 1992. *Reduction and completion algorithms for Partial Differential Equations*, In Proc. ISSAC '92. 49–56. (ACM Press, Berkeley).

[57] Sherring, J. and G. Prince. 1992. *DIMSYM - Symmetry Determination and Linear Differential Equations Package*, Preprint, Department of Mathematics (LaTrobe University, Bundoora, Australia).

[58] Seiler, W. M., P. J. Vasiliou and C. Rogers. *Formal Analysis of the general Cauchy problem for the system:* $u_t = u_{xx}$, $u_y = u_{xx}$, Preprint.

[59] Spencer, D. 1969. *Overdetermined Systems of Linear Differential Equations*, Bull. A.M.S. **75** 179–239.

[60] Šurygin, V.A. and N.N. Janenko. 1961. *On the realization on electronic computing machines of algebraico-differential algorithms*, Problemy Kibernetika **6** 33–43 (in Russian).

[61] Thomas, J.M. 1929. *Riquier's existence theorems*, Annals of Math. **30** 285–310.

[62] Thomas, J. M. 1931. *Riquier's Existence Theorems*, Annals of Math. **35**, 306-311.

[63] Topunov, L. 1989. *Reducing systems of linear differential equations to a passive form*, Acta Appl. Math. **16** 191–206.

[64] Tresse, A. 1894. *Sur les invariants différentiels des groupes continus de transformations*, Acta Mathematica **18** 1–88. (English translation I. Lisle 1989, available from the author).

[65] Vessiot, E. 1924. *Sur une théorie nouvelle des problèmes généraux d'integration*, Bull. Soc. Math. Fr. **52** 336–395.

[66] Weispfenning, V. 1993. *Differential-Term Orders*, In Proc. of ISAAC '93, (Kiev, ACM press).

[67] Wolf, T. 1987. *A package for the analytic investigation and exact solutions of differential equations*, in: Proc. EUROCAL '87, Leipzig, GDR, Ed.: J.H. Davenport, Lecture Notes in Computer Science **378** (Springer Verlag, Berlin, 1989) 479–491.

List of Captions for Figures

Figure 1: Jetspace for the example of §4 with $p = 4$. The curves A, B, C are the loci of the intersection of $S_1 : u_x^4 + u_x - u = 0$ with $u_y = u_x$, $u_y = 0$ and $S_2 : 4u_x^3 + 1 = 0$ respectively.

Figure 2: Diagram showing order of involutivity for the *pde* system $\partial^r u/\partial x^r = 0$, $\partial^r u/\partial y^r = 0$ of the example in Appendix B.

Send your email request for copies of the figures to: reid@math.ubc.ca