Computations related to the Riemann Hypothesis

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Conjectures

Riemann Hypothesis (RH):
ρ a nontrivial zero of $\zeta(s) \Rightarrow \mathrm{Re}(\rho) = 1/2$.

Generalized/Extended RH (GRH/ERH):
Similar conjecture(s) for Dirichlet L-functions, other ζ and L-functions.

Pair correlation conjecture & friends: The zeros of $\zeta(s)$ and its generalizations are distributed like eigenvalues of random Hermitian matrices. More precisely, zeros of $\zeta(s)$ distributed like eigenvalues of random matrices taken from the Gaussian unitary ensemble (GUE). Other buzzwords: "random matrix theory", "spectral interpretation of zeros", "Montgomery-Odlyzko law", "quantum chaos". These conjectures imply RH/GRH/ERH.

Other(?):
All zeros of $\zeta(s)$ are simple.

Notation

$$s = \sigma + it$$

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad \sigma > 1$$

$$\chi(s) = \frac{\zeta(s)}{\zeta(1-s)}$$

$$= \pi^{s-1/2} \, \frac{\Gamma((1-s)/2)}{\Gamma(s/2)}$$

$N(T) := \#\{\rho : \ \zeta(\rho) = 0, 0 \le \mathrm{Re}(\rho) \le 1, 0 \le \mathrm{Im}(\rho) \le T\}$
counting zeros according to their multiplicity.

Twisting $\zeta(s)$

$$Z(t) := \frac{\zeta(1/2+it)}{\sqrt{\chi(1/2+it)}}$$

If $t \in \mathbb{R}$ then $Z(t) \in \mathbb{R}$ and $|Z(t)| = |\zeta(1/2+it)|$.

When $t \in \mathbb{R}$, an alternate formulation for $Z(t)$ is

$$\vartheta(t) := \operatorname{Im} \ln \Gamma\left(\frac{1}{4} + \frac{it}{2}\right) - \frac{t}{2}\ln(\pi)$$

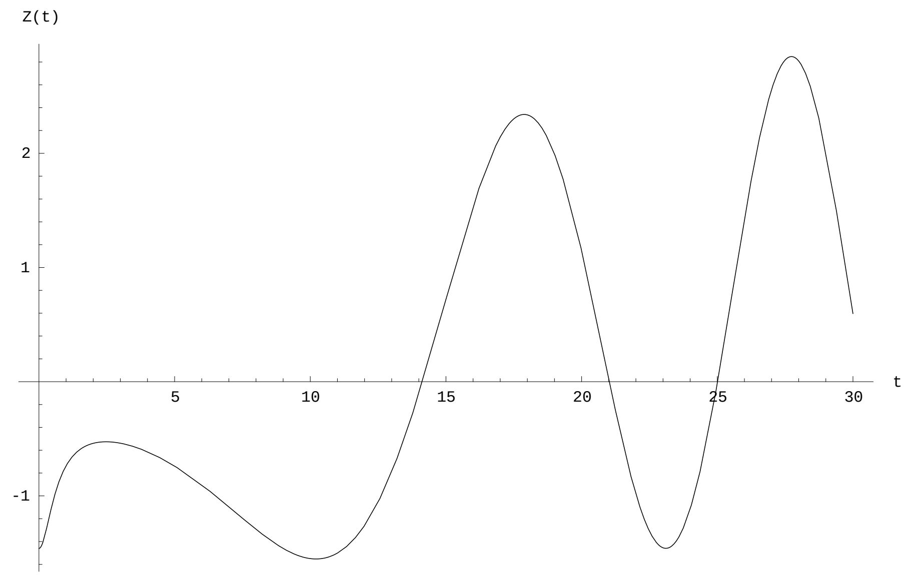
$$Z(t) = e^{i\vartheta(t)}\zeta(1/2+it)$$

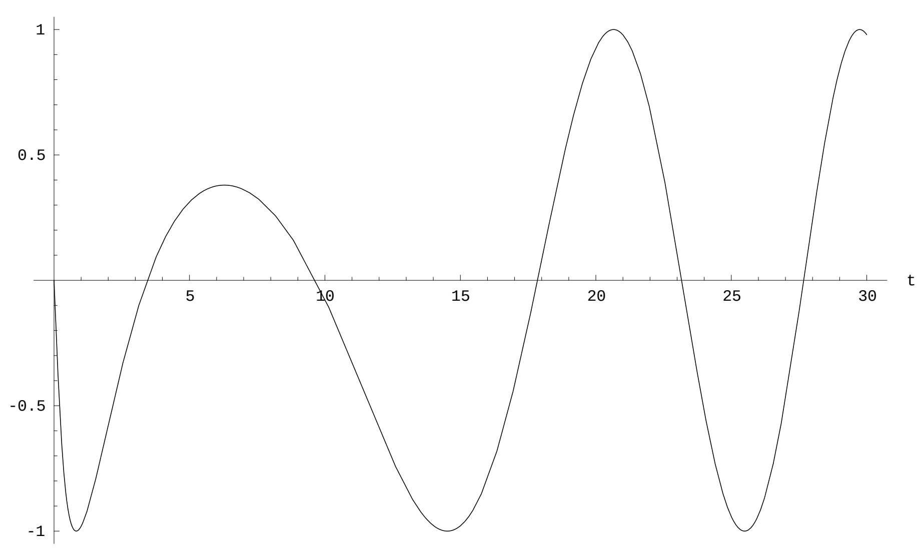
Stirling's formula gives a good approximation to $\vartheta(t)$:

$$\vartheta(t) = \frac{t}{2}\ln\left(\frac{t}{2\pi}\right) - \frac{t}{2} - \frac{\pi}{8} + \frac{1}{48t} + \frac{7}{5760t^3} + O(t^{-5})$$

$Z(t)$ **and** $\sin(\vartheta(t))$

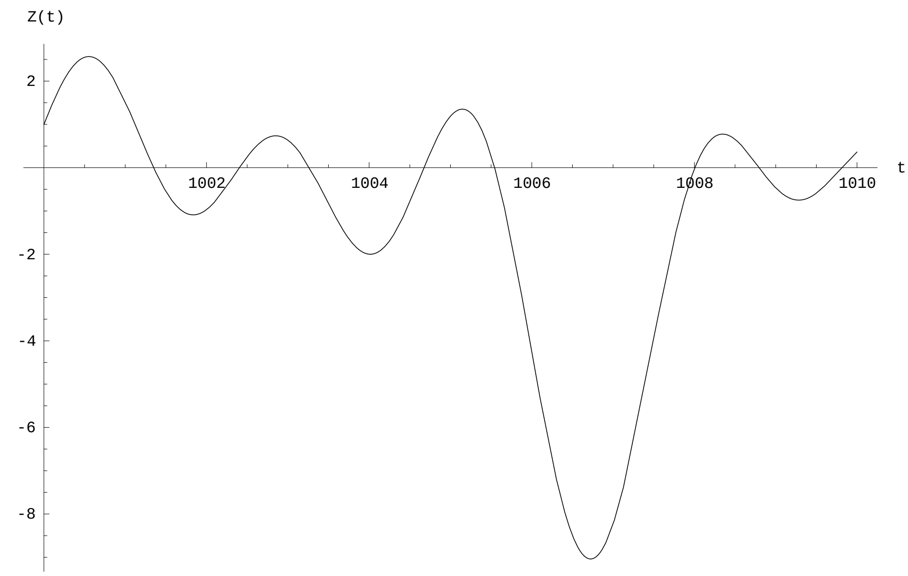
$$0 \le t \le 30$$

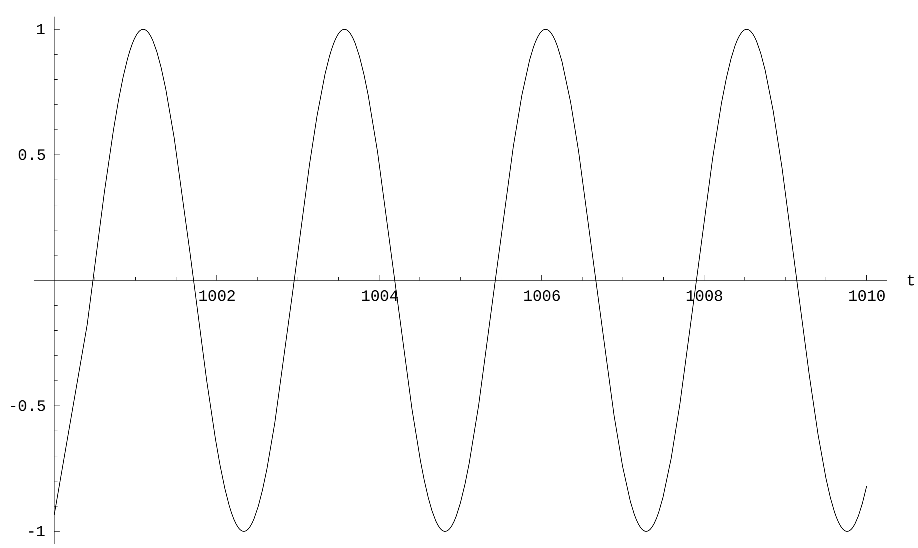




$Z(t)$ **and** $\sin(\vartheta(t))$

$$1000 \leq t \leq 1010$$





Checking the RH to height T

Basic approach:

- Try to find all sign changes of $Z(t)$, $0 \le t \le T$. Don't try too hard.
- Compare number of zeros found against $N(T)$. If counts agree then RH is true up to T.

Note, don't need to locate zeros very precisely.

Difficulties:

- How and where to compute $Z(t)$?
- How to compute $N(T)$?
- What if there is a multiple zero (or nearly multiple zero) of $Z(t)$?

Where to compute $Z(t)$

Define g_n, the nth Gram point, to be the solution to $\vartheta(g_n) = n\pi$. I.e., g_n is the nth zero of $\sin(\vartheta(t))$.

Gram's law: As a *rule of thumb*

- $(-1)^n Z(g_n) > 0$
- There is one zero of $Z(t)$ between g_n and g_{n+1}.
- $N(g_n) = n + 1$.

This suggests we start by computing $Z(g_n)$, and then find small (or zero) h_n such that $(-1)^n Z(g_n + h_n) > 0$ and $g_n + h_n < g_{n+1} + h_{n+1}$. Turing showed how knowledge about h_n can be used to compute $N(T)$ exactly.

How to compute $N(T)$ . . .

We have

$$N(T) = \frac{1}{\pi}\vartheta(T) + 1 + S(T)$$

where $S(T)$ can be given as a path integral. One can show that $S(T) = O(\ln(T))$ as $T \to \infty$.

This implies

$$N(T) = \frac{T}{2\pi}\ln\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + O(\ln(T)).$$

Backlund gave an explicit error bound for the approximation. This is a good start . . . .

. . . how to compute $N(T)$

A theorem of Littlewood shows that $S(T)$ goes to zero "on average":

$$\int_0^T S(t)\,dt \ll \ln(T)$$

so that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T S(t)\,dt = 0.$$

Turing gave an explicit bound on

$$\left| \int_{T_1}^{T_2} S(t)\,dt \right|$$

and showed how this can be used to compute $N(T)$ exactly (for certain T).

Turing's method

Suppose $h_m = 0$ for some m, and that h_n are "small" for n near m. Note that

$$S(g_m) = N(g_m) - m - 1 \in \mathbb{Z},$$

and, in fact, $S(g_m) \in 2\mathbb{Z}$ since any zeros off the critical line come in pairs, while the sign of $Z(g_m)$ gives parity of number of zeros on the critical line, and, by assumption, $(-1)^m Z(g_m) > 0$.

Thus, to show that $S(g_m) = 0$, i.e. $N(g_m) = m+1$, it suffices to show $-2 < S(g_m) < 2$. Assume otherwise. If h_n remains small for $n = m+1, m+2, \ldots, m+k$ then $S(t)$ cannot change by much over an interval of length k. This contradicts Turing's bound once k is large enough.

How to compute $Z(t)$

For small t, or high accuracy, can use Euler-Maclaurin summation to compute $\zeta(1/2+it)$ [CO92]. Then use $Z(t) = e^{i\vartheta t}\zeta(1/2+it)$. Requires $t^{1+\epsilon}$ operations.

Otherwise, use the Riemann-Siegel formula for $Z(t)$, $t \in \mathbb{R}$. This requires $t^{1/2+\epsilon}$ operations:

$$
\begin{aligned}
Z(t) = 2\sum_{n=1}^{N} \frac{\cos(\vartheta(t) - t\ln(n))}{\sqrt{n}} \\
&+ (-1)^{N-1}(2\pi/t)^{1/4}\sum_{k=0}^{K} C_k(z)(\sqrt{2\pi/t})^k \\
&+ R_K(t)
\end{aligned}
$$

where $N := \left\lfloor \sqrt{t/(2\pi)} \right\rfloor$ and $z := 1-2(\sqrt{t/(2\pi)}-N)$.

Gabcke gives series expansions for computing $C_k(t)$, and good bounds for $R_K(t)$ [Gab79].

In practice, $K \leq 2$ suffices. When $Z(t)$ is nearly zero, more accuracy might be needed, and one can fall-back on Euler-Maclaurin summation.

Recent Computations

- Rigorous computational proof of RH for first $1.5 \cdot 10^9$ zeros, by van de Lune et. al. [vdLtRW86].
- Ongoing networked computation coordinated by Sebastian Wedeniwski [Wed]. $30,592,710,000$ zeros and counting.
- "Spot checking" near the 10^{20}-th and 10^{21}-st zeros (near $t = 1.52 \cdot 10^{19}$ and $t = 1.44 \cdot 10^{20}$ respectively) by Andrew Odlyzko [Odl92, Odl98]. ". . . several billion high zeros . . . " computed.
- Computations to check GRH/ERH and related conjectures [Rum93, KS99, Rub98].

Further reading

- Edwards for historical background [Edw74].
- Odlyzko (& Schönhage) on computing $\zeta(\sigma + it)$ using t^ϵ operations for many values of t [Odl92, OS88].
- Rubinstein on a completely different approach to computing $\zeta(s)$, $L(s, \chi)$, etc.
- Borwein, Bradley and Crandall for survey of many methods for computing $\zeta(s)$ [BBC00].
- In addition to above, Montgomery [Mon73], Katz & Sarnak [KS99] on pair correlation conjecture etc.

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