p-ADIC HODGE THEORY

KIRAN KEDLAYA

Let C be a "smooth curve over $R = \mathbb{Z}[1/N]$", by which we mean a smooth projective curve over R minus a divisor that is finite étale over R. Let p be a prime not dividing N. Let J be "the Jacobian over C over R", which will be an extension of the Jacobian of the complete curve extended by a torus. Let O be a fixed element of $C(R)$. Chabauty's method is based on the following diagram:

$$\begin{array}{ccccccc} C(R) & \longrightarrow & C(\mathbb{Z}_p) & & & & \\ \downarrow{\scriptstyle \mathrm{Alb}} & & \downarrow{\scriptstyle \mathrm{Alb}} & \searrow^{P \mapsto (\omega \mapsto \int_O^P \omega)} & & & \\ J(R) \otimes \mathbb{Q} & \longrightarrow & J(\mathbb{Z}_p) \otimes \mathbb{Q} & \longrightarrow & \mathrm{Lie}\, J_{\mathbb{Q}_p} & = & H^0(J_{\mathbb{Q}_p}, \Omega^1)^\vee \simeq H^0(C_{\mathbb{Q}_p}, \Omega^1)^\vee \end{array}$$

Goal 1: Explain some p-adic Hodge theory. This is preparation for:

Goal 2: Redraw this diagram in a more Galois-cohomological way to allow for a nonabelian version.

1. p-adic Hodge theory

Let X be a smooth proper scheme over $\mathbb{Z}_p$ (or more generally, a smooth proper scheme over $\mathbb{Z}_p$ minus a relative normal crossings divisor). A normal crossings divisor is a divisor that étale locally looks like a union of transverse hyperplanes in affine space. A relative normal crossings divisor is the same, except over $\operatorname{Spec} \mathbb{Z}_p$.

The $\mathbb{Q}_p$-vector space $H^i_{\mathrm{et}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$ has an action of $G_p := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. The $\mathbb{Q}_p$-vector space $H^i_{\mathrm{dR}}(X_{\mathbb{Q}_p}, \mathbb{Q}_p)$ has a Hodge filtration, and is isomorphic to $H^i_{\mathrm{crys}}(X_{\mathbb{F}_p}, \mathbb{Z}_p) \otimes \mathbb{Q}_p$, which has a Frobenius action.

Fontaine asks: What do H^i_{et} and H^i_{dR} have to do with each other?

Let $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p$. Fontaine defines a "big ring" B_{crys}, a topological $\mathbb{Q}_p$-algebra with a continuous G_p-action, a decreasing filtration (commuting with the G_p-action), and an extra endomorphism ϕ (Frobenius).

Theorem 1.1 (Faltings, Tsuji)**.** *There is an isomorphism of $\mathbb{Q}_p$-vector spaces*

$$\left(H^i_{\mathrm{et}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}}\right)^{G_p} \simeq H^i_{\mathrm{dR}}(X_{\mathbb{Q}_p}, \mathbb{Q}_p)$$

respecting the filtration and Frobenius action. To go the other way: there is an isomorphism

$$\left(\mathrm{Fil}^0\left(H^i_{\mathrm{dR}}(X_{\mathbb{Q}_p}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}}\right)\right)^{\phi=1} \simeq H^i_{\mathrm{et}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$$

respecting the Galois action.

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Let $T = T_p(J) = \varprojlim_n J[p^n](\overline{\mathbb{Q}})$ be the p-adic Tate module. Let $V = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq H_1^{\mathrm{et}}(C_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$. Let G_R be the Galois group over $\mathbb{Q}$ of the maximal extension unramified outside Np; then G_R acts on V. For $\ell \neq p$, let $H^1_f(G_\ell, V) = \ker\left(H^1(G_\ell, V) \to H^1(I_\ell, V)\right)$. Define $H^1_f(G_p, V) = \ker\left(H^1(G_p, V) \to H^1(G_p, V \otimes B_{\mathrm{crys}})\right)$. Let $H^1_f(G_R, V)$ be the set of elements of $H^1(G_R, V)$ that map to $H^1_f(G_\ell, V)$ for each prime ℓ.

$$\begin{array}{ccccccc}
C(R) & \longrightarrow & C(\mathbb{Z}_p) & & & & \\
\downarrow & & \downarrow & & & & \\
H^1_f(G_R, V) & \longrightarrow & H^1_f(G_p, V) & = & H^0(G_p, B_{\mathrm{crys}} \otimes V)/\operatorname{Fil}^0 & & \\
\uparrow & & \uparrow & & \uparrow & & \\
J(R) \otimes \mathbb{Q} & \longrightarrow & J(\mathbb{Z}_p) \otimes \mathbb{Q} & \longrightarrow & \operatorname{Lie} J_{\mathbb{Q}_p} & = & H^0(J_{\mathbb{Q}_p}, \Omega^1)^\vee \simeq H^0(C_{\mathbb{Q}_p}, \Omega^1)^\vee
\end{array}$$

The comparison theorem relates $H^1_{\mathrm{dR}}(C_{\mathbb{Q}_p}, \mathbb{Q}_p)$ and $H^1_{\mathrm{et}}(\overline{C}_{\mathbb{Q}_p}, \mathbb{Q}_p) = V^*(1)$. In

$$0 \to H^0(C_{\mathbb{Q}_p}, \Omega^1) \to H^1_{\mathrm{dR}}(C_{\mathbb{Q}_p}, \mathbb{Q}_p) \to H^1(C_{\mathbb{Q}_p}, \mathcal{O}) \to 0,$$

the space on the left is Fil^1 in the Hodge filtration.

In the diagram above, the first upward map is an isomorphism if Ш is finite. The second upward map is an isomorphism. That the third upward map is an isomorphism follows from the comparison theorem.

The map

$$J(R) \otimes \mathbb{Q}_p \to H^1_f(G_R, V)$$

is a Kummer map. Taking Galois cohomology of

$$0 \to J(\overline{\mathbb{Q}})[p^n] \to J(\overline{\mathbb{Q}}) \xrightarrow{p^n} J(\overline{\mathbb{Q}}) \to 0$$

yields

$$\begin{array}{ccc}
\varprojlim_n \dfrac{J(R)}{p^n J(R)} & \longrightarrow & H^1(G_R, V) \\
& \searrow & \uparrow \\
& & H^1_f(G_R, V)
\end{array}$$

The arrow

$$H^1_f(G_p, V) \leftarrow H^0(G_p, B_{\mathrm{crys}} \otimes V)/\operatorname{Fil}^0$$

comes from the exact sequence

$$0 \to \mathbb{Q}_p \to B_{\mathrm{crys}}^{\phi=1} \to \frac{B_{\mathrm{crys}}}{B_{\mathrm{crys}}^+} \to 0$$

(where $\operatorname{Fil}^0 B_{\mathrm{crys}} = B^+_{\mathrm{crys}}$) by tensoring with V and taking invariants.

Let $U_n^{\mathrm{et}} = \pi_1^{\mathrm{unip}}(C, O)_{/\mathbb{Q}_p}/L^{n+1}$.

Then we have a diagram

$$
\begin{array}{ccccccc}
C(R) & \longrightarrow & C(\mathbb{Z}_p) & & & & \\
\downarrow & & \downarrow & & & & \\
H^1_f(G_R, U_n^{\mathrm{et}}) & \longrightarrow & H^1_f(G_p, U_n^{\mathrm{et}}) & = & H^0(G_p, B_{\mathrm{crys}} \otimes U_n^{\mathrm{et}})/\,\mathrm{Fil}^0 & \overset{\text{comparison theorem}}{=\!=\!=\!=} & U_n^{\mathrm{dR}} \\
\downarrow & & \downarrow & & \downarrow & & \\
H^1_f(G_R, V) & \longrightarrow & H^1_f(G_p, V) & = & H^0(G_p, B_{\mathrm{crys}} \otimes V)/\,\mathrm{Fil}^0 & & \\
\uparrow & & \uparrow & & \uparrow & & \\
J(R) \otimes \mathbb{Q} & \longrightarrow & J(\mathbb{Z}_p) \otimes \mathbb{Q} & \longrightarrow & \mathrm{Lie}\, J_{\mathbb{Q}_p} & = & H^0(J_{\mathbb{Q}_p}, \Omega^1)^\vee \simeq H^0(C_{\mathbb{Q}_p}, \Omega^1)^\vee
\end{array}
$$

Remark 1.2. The p-adic Tate module T is the maximal pro-p abelian quotient of $\pi_1^{\mathrm{et}}(C_{\overline{\mathbb{Q}}_p}, O)$.