Prym varieties of curves of genus 3

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Overview

- Non-hyperelliptic curves of genus 3 and their unramified double covers
- Some special divisors on curves of genus 5
- Prym varieties for genus 3
- Embedding a double cover in its Prym
- *Application*: Avoiding Chabauty on a curve of genus 3
- *Application*: Violation of the Hasse-principle in genus 3 and 5
- A word of warning: Don't believe small numbers

Curves of genus 3

Notation: Let K be a number field

Non-hyperelliptic curve of genus 3: Smooth plane quartic

Special form: Quadratic in quadratics in $\mathbb{P}^2$

$$C : Q_0(Q_1(u, v, w), Q_2(u, v, w), Q_3(u, v, w)) = 0$$

where $Q_1, Q_2, Q_3 \in K[u, v, w]$ are quadratic forms and $Q_0 \in K[q_1, q_2, q_3]$ is too.

Further reduction: If $Q_0(q_1, q_2, q_3) = 0$ has no solutions, then $C(K) = \emptyset$.

If it has, then (up to linear transformation) $Q_0 = q_1 q_3 - q_2^2$.

We limit attention to curves of the form

$$C : Q_1(u, v, w)Q_3(u, v, w) = Q_2(u, v, w)^2$$

Note: Any smooth plane quartic can be written in this form over an extension of K.

Unramified double covers

Given a genus 3 curve

$$C : Q_1(u,v,w)Q_3(u,v,w) = Q_2(u,v,w)^2$$

Consider the following projective curve in $\mathbb{P}^4$

$$D_\delta : \begin{cases} Q_1(u,v,w) & = & \delta\, r^2 \\ Q_2(u,v,w) & = & \delta\, rs \\ Q_3(u,v,w) & = & \delta\, s^2 \end{cases}$$

and the unramified double cover:

$$\begin{array}{llcl} \pi : & D & \longrightarrow & C \\ & (u:v:w:r:s) & \mapsto & (u:v:w). \end{array}$$

with the obvious $\sigma \in \mathrm{Aut}(D/C)$ defined by

$$\sigma : (u:v:w:r:s) \mapsto (u:v:w:-r:-s).$$

The curve D_δ is smooth and of genus 5.

The given model is canonical.

Covering collection (J.L. Wetherell)

Obviously,

$$\pi(D_\delta(K)) \subset C(K).$$

Conversely, there is a finite subset $\Delta \subset K^*$ such that

$$\bigcup_{\delta\in\Delta} \pi(D_\delta(K)) = C(K)$$

Sketch: For $K = \mathbb{Q}$:

- assume $Q_1, Q_2, Q_3 \in \mathbb{Z}[u, v, w]$
- if $P = (u_P : v_P : w_P) \in \mathbb{P}^2(\mathbb{Q})$ such that $Q_1(P)Q_3(P) = Q_2(P)^2$,

  WLOG: $u_P, v_P, w_P \in \mathbb{Z}$ and $\gcd(u_P, v_P, w_P) = 1$

- it follows

$$\gcd(Q_1(P), Q_2(P), Q_3(P)) \mid R = \mathrm{res}_u(\mathrm{res}_v(Q_1, Q_2), \mathrm{res}_v(Q_1, Q_3))$$

- Hence, if $Q_1(P) = \delta r_P^2$, $Q_3(P) = \delta s_P^2$, then

$$\delta \mid R$$

Description of the Jacobian of a curve of genus 2 (Cassels-Flynn)

Consider:

$$F : y^2 = f_0 + f_1 x + f_2 x^2 + \cdots + f_6 x^6$$

Canonical divisor class:

$$[\kappa_F] = [(x_0, y_0) + (x_0, -y_0)]$$

General point of $\mathrm{Jac}_F(\overline{K})$**:**

$$\mathfrak{g} = [(x_1, y_1) + (x_2, y_2) - \kappa_F], \text{ where } (x_1, y_1), (x_2, y_2) \in F(\overline{K}).$$

Kummer surface: $\mathrm{Kum}_F := \mathrm{Jac}_F / \langle -1 \rangle$

$$k(\mathfrak{g}) = (k_1 : k_2 : k_3 : k_4) = (1 : x_1 + x_2 : x_1 x_2 : \ldots)$$

Equation of Kum_F as a variety over K:

$$\mathrm{Kum}_F : (k_2^2 - 4 k_1 k_3)\, k_4^2 + (\ldots)\, k_4 + (\ldots) = 0$$

Embedding *D* in an abelian surface

Notation: We write Q_i for

- The quadratic form $Q_i \in K[u, v, w]$
- The symmetric matrix $Q_i \in K^{3\times 3}$.

A curve of genus 2:

$$F_\delta : y^2 = -\delta \det(Q_1 + 2xQ_2 + x^2 Q_3)$$

Main tool:

$$\begin{array}{ccc} D_\delta & \hookrightarrow & \mathrm{Jac}_{F_\delta} \\ \downarrow \pi & & \downarrow k \\ C & \hookrightarrow & \mathrm{Kum}_{F_\delta} \end{array}$$

Intersection:

$$\pi(D_\delta(K)) \subset C(\overline{K}) \cap k(\mathrm{Jac}_{F_\delta}(K))$$

Prym varieties

Given unramified double cover $\pi : D \to C$:

$$\begin{array}{ccccc} D & & \mathrm{Jac}_D & \hookleftarrow & \mathrm{Prym}(D/C) \\ \pi \downarrow & & \pi_* \downarrow & & \downarrow \\ C & & \mathrm{Jac}_C & & 0 \end{array}$$

Definition: $\mathrm{Prym}(D/C)$ is the connected component of $\mathrm{Ker}(\pi_*)$ containing 0.

Properties:

- $\mathrm{Prym}(D/C)$ inherits a principal polarization from Jac_D
- $\mathrm{Prym}(D/C)$ is generally *not* a Jacobian
- $\dim\ \mathrm{Prym}(D/C) = \mathrm{genus}(C) - 1$

Theorem: With $\pi : D_\delta \to C$ as before,

$$\mathrm{Prym}(D_\delta/C) = \mathrm{Jac}_{F_\delta}$$

A closer study of D

Consider the quadratic forms / symmetric matrices:

$$M_1 = \left(\begin{array}{c|cc} Q_1 & & \\ \hline & -\delta & 0 \\ & 0 & 0 \end{array}\right) \quad M_2 = \left(\begin{array}{c|cc} Q_2 & & \\ \hline & 0 & -\frac{1}{2}\delta \\ & -\frac{1}{2}\delta & 0 \end{array}\right) \quad M_3 = \left(\begin{array}{c|cc} Q_3 & & \\ \hline & 0 & 0 \\ & 0 & -\delta \end{array}\right)$$

Their linear span:

$$\Lambda = \{\lambda_1 M_1 + \lambda_2 M_2 + \lambda_3 M_3\} \simeq \mathbb{P}^2; \; D_\delta = \mathrm{Var}(\Lambda)$$

The locus of singular quadrics:

$$\Gamma \;=\; \{M \in \Lambda : \mathrm{rk}(M) \leq 4\} \;:\; \det(\lambda_1 M_1 + \lambda_2 M_2 + \lambda_3 M_3) = 0$$

$$\Gamma \;=\; \underset{\det(\lambda_1 Q_1 + \lambda_2 Q_2 + \lambda_3 Q_3) = 0}{\Gamma^+} \;\cup\; \underset{4\lambda_1\lambda_3 - (\lambda_2)^2 = 0}{\Gamma^-}$$

- $\dim \Gamma = 1$
- $\{M \in \Lambda : \mathrm{rk}(M) \leq 3\} = \mathrm{Sing}(\Gamma)$
- $\{M \in \Gamma^- : \mathrm{rk}(M) \leq 3\} = \Gamma^+ \cap \Gamma^-$

Geometry of some special divisors on D

[Arbarello-Cornalba-Griffiths-Harris VI, Excercises F]

Variety of special divisor classes:

$$W^1_4 = \{\mathfrak{d} \in \mathrm{Pic}^4(D) : l(\mathfrak{d}) \geq 2\}$$

Residuality:

$$\mathfrak{d} \mapsto [\kappa_D] - \mathfrak{d} : W^1_4 \to W^1_4$$

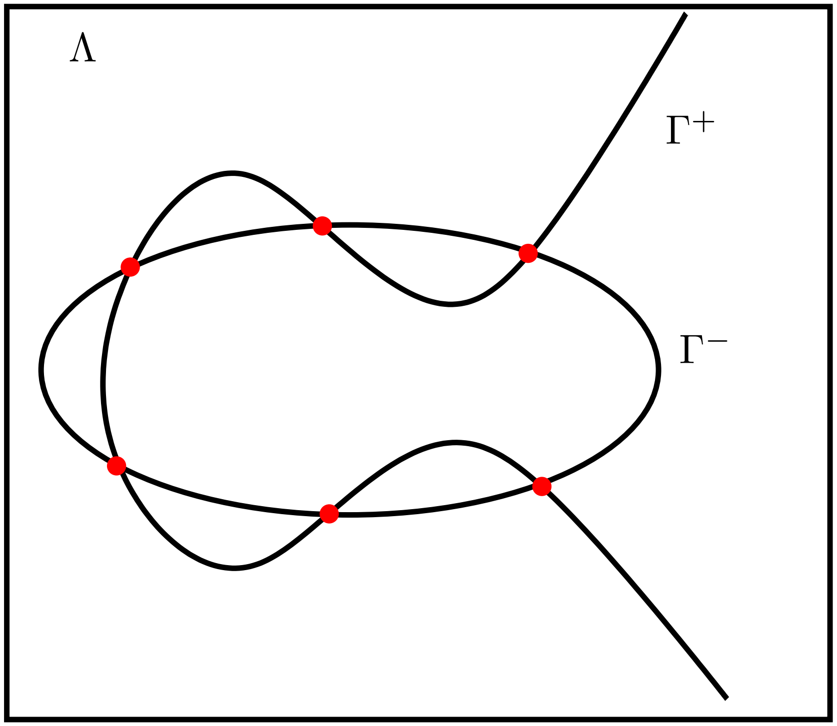
Characterisation: *Effective* divisors $[\sum_{i=1}^4 P_i] \in W^1_4$ are of the form

$V \cap D,$ where $V \subset \mathbb{P}^4$ is a plane.

- If $\deg(V \cap D) = 4$ then $\Lambda|_V$ is a pencil, i.e. $\exists M \in \Lambda : V \subset M$.
- A quadric in $\mathbb{P}^4$ containing a plane is singular: $W^1_4 \to \Gamma$.
- A quadric $M \subset \mathbb{P}^4$ of rank 4 has two systems of planes (It's a cone over a quadric in $\mathbb{P}^3$).
- If $V_1, V_2 \subset M$ belong to opposite systems, then $[(V_1 \cap D) + (V_2 \cap D)] = [\kappa_D]$.
- $W^1_4 \to \Gamma$ is a double cover, with $(\mathfrak{d} \mapsto \kappa_D - \mathfrak{d}) \in \mathrm{Aut}(W^1_4/\Gamma)$.

Decomposition of W_4^1

We have W_4^1 as a double cover of Γ:



Let F_δ be the component of W_4^1 over

$$\Gamma^- : 4\lambda_1\lambda_3 = (\lambda_2)^2; \text{ parametrically: } (\lambda_1 : \lambda_2 : \lambda_3) = (1 : 2x : x^2)$$

For some $\tilde{\delta}$:

$$F_\delta : y^2 = -\tilde{\delta}\det(Q_1 + 2x\,Q_2 + x^2 Q_3)$$

Description of $\mathrm{Prym}(D/C)$

Note: If $V \subset M \in \Gamma^-$ then $\pi(V)$ is a line. Hence, $\pi_*(V \cap D) \in \mathrm{Pic}_C$ is $[\kappa_C]$.

$$\begin{array}{rccc} \pi_* : & F_\delta & \longrightarrow & \mathrm{Pic}^4(C) \\ & \mathfrak{d} & \longmapsto & [\kappa_C] \end{array}$$

Embedding:

$$\begin{array}{ccc} \mathrm{Jac}_F & \hookrightarrow & \mathrm{Jac}_D \\ [p_1 + p_2 - \kappa_F] & \mapsto & p_1 + p_2 - [\kappa_D] \end{array}$$

It follows that $\pi_*(\mathrm{Jac}_F) = 0$, so

$$\mathrm{Jac}_F \hookrightarrow \mathrm{Prym}(D/C).$$

Since Jac_F is of the right dimension, equality must hold.

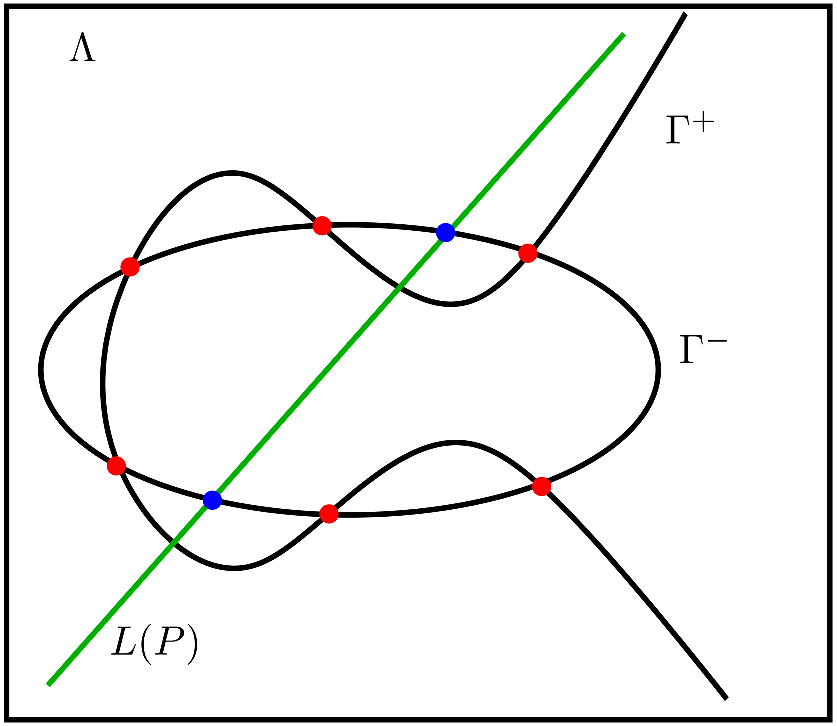
Embedding D in $\mathrm{Prym}(D/C)$

A simple mapping:

$$\begin{array}{rcl} D & \to & W^1_4 + W^1_4 \subset \mathrm{Pic}^8(D) \\ P & \mapsto & \sum\{[\mathfrak{d}] \in F : \mathfrak{d} \geq 2P\} \end{array}$$

In terms of Λ: Given P:

$$L(P) := \{M \in \Lambda : T_P(D) \subset M\}$$



Determining $C \subset \mathrm{Kum}_F$

Recall the diagram:

$$\begin{array}{ccc} D_\delta & \hookrightarrow & \mathrm{Jac}_{F_\delta} \\ \pi \downarrow & & \downarrow k \\ C & \hookrightarrow & \mathrm{Kum}_{F_\delta} \end{array}$$

Computation: Use interpolation.

- Take a quartic point $\overline{P}$ on C (intersect C with a line).
- Lift to an octic point P on D.
- Map over to Jac_F. ($[p_1 + p_2 - \kappa_F]$ with p_1, p_2 quadratic is easier here)
- Project down to Kum_F. This gives a quartic point again.
- Interpolate equations for C.

Observation: $C \subset \mathrm{Kum}_F$ is cut out by a quartic equation: 35 degrees of freedom.

Application: Chabauty

Consider:

$$C : (-4u^2 - 4vw + 4w^2)(2u^2 + 4uv + 4v^2) = (-2u^2 + 2uw - 4vw + 2w^2)^2$$

We have

$$\bigcup_{\delta \in \{\pm 1, \pm 2, \pm 5, \pm 10\}} \pi(D_\delta(\mathbb{Q})) = C(\mathbb{Q}).$$

Local considerations show $D_\delta(\mathbb{Q}) = \emptyset$ for $\delta \neq 1$. Component of W_4^1:

$$F : y^2 = x^5 + 8x^4 - 7x^3 - 7/2x^2 + 5x - 1$$

$$\mathrm{Jac}_F(\mathbb{Q}) = \langle \mathfrak{g} \rangle = \langle [(2\sqrt{2} - 2, 17\sqrt{2} - 25) + (-2\sqrt{2} - 2, -17\sqrt{2} - 25) - 2\infty] \rangle$$

$$\begin{aligned} \mathrm{Kum}_F : \quad & 11k_1^4 - 28k_1^3k_2 + 70k_1^3k_3 + 4k_1^3k_4 + 32k_1^2k_2^2 - 164k_1^2k_2k_3 - 10k_1^2k_2k_4 + 171k_1^2k_3^2 + \\ & 14k_1^2k_3k_4 + 4k_1k_2^3 - 20k_1k_2^2k_3 + 14k_1k_2k_3^2 + 14k_1k_2k_3k_4 + 14k_1k_3^3 - 32k_1k_3^2k_4 - \\ & 4k_1k_3k_4^2 + k_2^2k_4^2 - 2k_2k_3^2k_4 + k_3^4 = 0 \end{aligned}$$

Equation for $C \subset \mathrm{Kum}_F$:

$$\begin{aligned} \phi : \quad & 429136k_1^4 + 1330784k_1^3k_3 + 567232k_1^3k_4 - 159200k_1^2k_2^2 - 2866016k_1^2k_2k_3 + 33440k_1^2k_2k_4 + 4248768k_1^2k_3^2 + \\ & 27552k_1^2k_3k_4 + 881664k_1^2k_4^2 + 288072k_1k_2^3 - 777432k_1k_2^2k_3 - 256928k_1k_2^2k_4 + 244832k_1k_2k_3^2 + \\ & 907424k_1k_2k_3k_4 - 745472k_1k_2k_4^2 + 593152k_1k_3^3 - 991488k_1k_3^2k_4 + 357440k_1k_3k_4^2 + 573440k_1k_4^3 + 34895k_2^4 - \\ & 69720k_2^3k_3 + 1120k_2^3k_4 + 151704k_2^2k_3^2 - 364448k_2^2k_3k_4 + 226032k_2^2k_4^2 - 251552k_2k_3^3 + 569376k_2k_3^2k_4 + \\ & 10752k_2k_3k_4^2 - 315392k_2k_4^3 + 156704k_3^4 - 167552k_3^3k_4 - 283136k_3^2k_4^2 + 200704k_3k_4^3 + 114688k_4^4 = 0 \end{aligned}$$

Application: Chabauty (continued)

If $P \in D_1(\mathbb{Q}) \subset \mathrm{Jac}_F(\mathbb{Q})$, then

$$P = n\mathfrak{g} \text{ for some } n \in \mathbb{Z}$$

Base change to $\mathbb{F}_{13}$: If $k(n\mathfrak{g}) \in C \pmod{13}$ then $n = \pm 1 \pmod{10}$.

13-adically: $\phi(N) = \phi(k((1 + 10N)\mathfrak{g})) = \phi(k(\mathfrak{g} + \mathrm{Exp}(N \mathrm{Log}(10\mathfrak{g}))))$.

$$\phi(N) = \phi_0 + \phi_1 N + \phi_2 N^2 + \cdots \in \mathbb{Z}_{13}[\![N]\!] \text{ with } \nu_{13}(\phi_i) \geq i$$

Observation: $\phi(k(\mathfrak{g})), \phi(k(11\mathfrak{g})) \pmod{13^2}$ determine $\phi_0, \phi_1 \pmod{13^2}$.

Fact: $\phi(k(\mathfrak{g})) = 0$ and $\phi(k(11\mathfrak{g})) \neq 0 \pmod{13^2}$; therefore $\nu_{13}(\phi_1) = 1$.

Theorem (Straßmann): If $f(z) = \sum_n f_n z^n \in \mathbb{Z}_p[\![n]\!]$ is convergent on $\mathbb{Z}_p$ and $\nu_p(f_N) < \nu_p(f_n)$ for all $n > N$ then

$$\#\{z \in \mathbb{Z}_p : f(z) = 0\} \leq N.$$

Corollary: $D_1(\mathbb{Q}) = \{\mathfrak{g}, -\mathfrak{g}\}$ and $C(\mathbb{Q}) = \{(0 : 1 : 0)\}$.

Other application

Consider the everywhere locally soluble curve

$$C : (v^2 + vw - w^2)(uv + w^2) = (u^2 - v^2 - w^2)^2.$$

We have

$$\bigcup_{\delta \in \{\pm 1, \pm 2\}} \pi(D_\delta(\mathbb{Q})) = C(\mathbb{Q})$$

and only D_1 is everywhere locally soluble.

We have

$$F : y^2 = x^6 + 2x^5 + 15x^4 + 40x^3 - 10x$$

and

$$\mathrm{Jac}_F(\mathbb{Q}) = \langle \mathfrak{g}_1, \mathfrak{g}_2 \rangle = \langle [\infty^+ - \infty^-], [((4\sqrt{41} - 41)/205, \ldots) + \cdots] \rangle$$

Using congruence information

We find

p	$\mathrm{Jac}_F(\mathbb{Q}) \pmod p$	relations
7	$\mathbb{Z}/55\mathbb{Z}$	$55\mathfrak{g}_1 \equiv 0,\ \mathfrak{g}_2 \equiv 15\mathfrak{g}_1$
11	$\mathbb{Z}/93\mathbb{Z}$	$93\mathfrak{g}_1 \equiv 0,\ \mathfrak{g}_2 \equiv 47\mathfrak{g}_1$

Intersecting $C(\mathbb{F}_p)$ **with** $k(\mathrm{Jac}_F(\mathbb{Q}) \pmod p)$**:**

$$\begin{array}{rclcl} D(\mathbb{Q}) & \subset & \{\pm 9\mathfrak{g}_1, \pm 22\mathfrak{g}_1, \pm 23\mathfrak{g}_1\} & + & \langle 55\mathfrak{g}_1, \mathfrak{g}_2 - 15\mathfrak{g}_1\rangle \\ D(\mathbb{Q}) & \subset & \{\pm 33\mathfrak{g}_1\} & + & \langle 93\mathfrak{g}_1, \mathfrak{g}_2 + 46\mathfrak{g}_1\rangle \end{array}$$

Deeper information at 11**:** gives $11 \cdot 2$ congruence classes modulo:

$$\langle 11 \cdot 93\mathfrak{g}_1, 11 \cdot (\mathfrak{g}_2 + 46\mathfrak{g}_1)\rangle$$

Intersection:

$$\langle 55\mathfrak{g}_1, \mathfrak{g}_2 - 15\mathfrak{g}_1, 11 \cdot 93\mathfrak{g}_1, 11 \cdot (\mathfrak{g}_2 + 46\mathfrak{g}_1)\rangle = \langle 11\mathfrak{g}_1, \mathfrak{g}_2 - 4\mathfrak{g}_1\rangle$$

Combining the information:

$$\begin{array}{lclclcl} \text{from } 7 & : & D(\mathbb{Q}) & \subset & \{0, \pm\mathfrak{g}_1, \pm 2\mathfrak{g}_1\} & + & \langle 11\mathfrak{g}_1, \mathfrak{g}_2 - 4\mathfrak{g}_1\rangle \\ \text{from } 11^2 & : & D(\mathbb{Q}) & \subset & \{\pm 4\mathfrak{g}_1\} & + & \langle 11\mathfrak{g}_1, \mathfrak{g}_2 - 4\mathfrak{g}_1\rangle \end{array}$$

Corollary: $D(\mathbb{Q}) = \emptyset$ and hence $C(\mathbb{Q}) = \emptyset$.

Empirical observation for small numbers

Observation: The relevant twists of Jac_F tend to have high rank:

Take quadratic forms $Q_1, Q_2, Q_3 \in \mathbb{Q}[u, v, w]$ and $(u_0 : v_0 : w_0) \in \mathbb{P}^2(\mathbb{Q})$ such that $Q_1Q_3 - Q_2^2 = 0$ is a smooth plane quartic containing $(u_0 : v_0 : w_0)$ and such that $Q_1(u_0, v_0, w_0)$ and $Q_2(u_0, v_0, w_0)$ are squares, then

$$y^2 = -\det(Q_1 + 2xQ_2 + x^2Q_3)$$

has a Jacobian with rank probably at least 2.

The other way around

Debunking:

- Take $\mathrm{Jac}_F(\mathbb{Q})$ of rank 1.
- Take $\mathfrak{g} \in \mathrm{Jac}_F(\mathbb{Q})$.
- Choose a plane $V \subset \mathbb{P}^3$ through $k(\mathfrak{g})$. Generically, $C := V \cap \mathrm{Kum}_F$ is a smooth plane quartic.
- Write $C : Q_1 Q_3 = Q_2^2$ (using $\mathrm{Jac}_F/\mathrm{Kum}_F$)
- Obtain new embedding $\iota : C \to \mathrm{Kum}_F$ via method sketched before.
- We find: $\iota(k(\mathfrak{g})) = k(2\mathfrak{g})$.

Some corollaries

- Any square-free sextic or quintic polynomial is of the form

$$\det(Q_1 + 2xQ_2 + x^2 Q_3)$$

- Any genus-2 Jacobian Jac_F over K occurs as a Prym over K

- Over $\overline{K}$, the fibre of $(D, C) \mapsto \mathrm{Prym}(D/C)$ of Jac_F is given by the plane sections of Kum_F

- Over K, this is not the case: Quartic sections do the trick